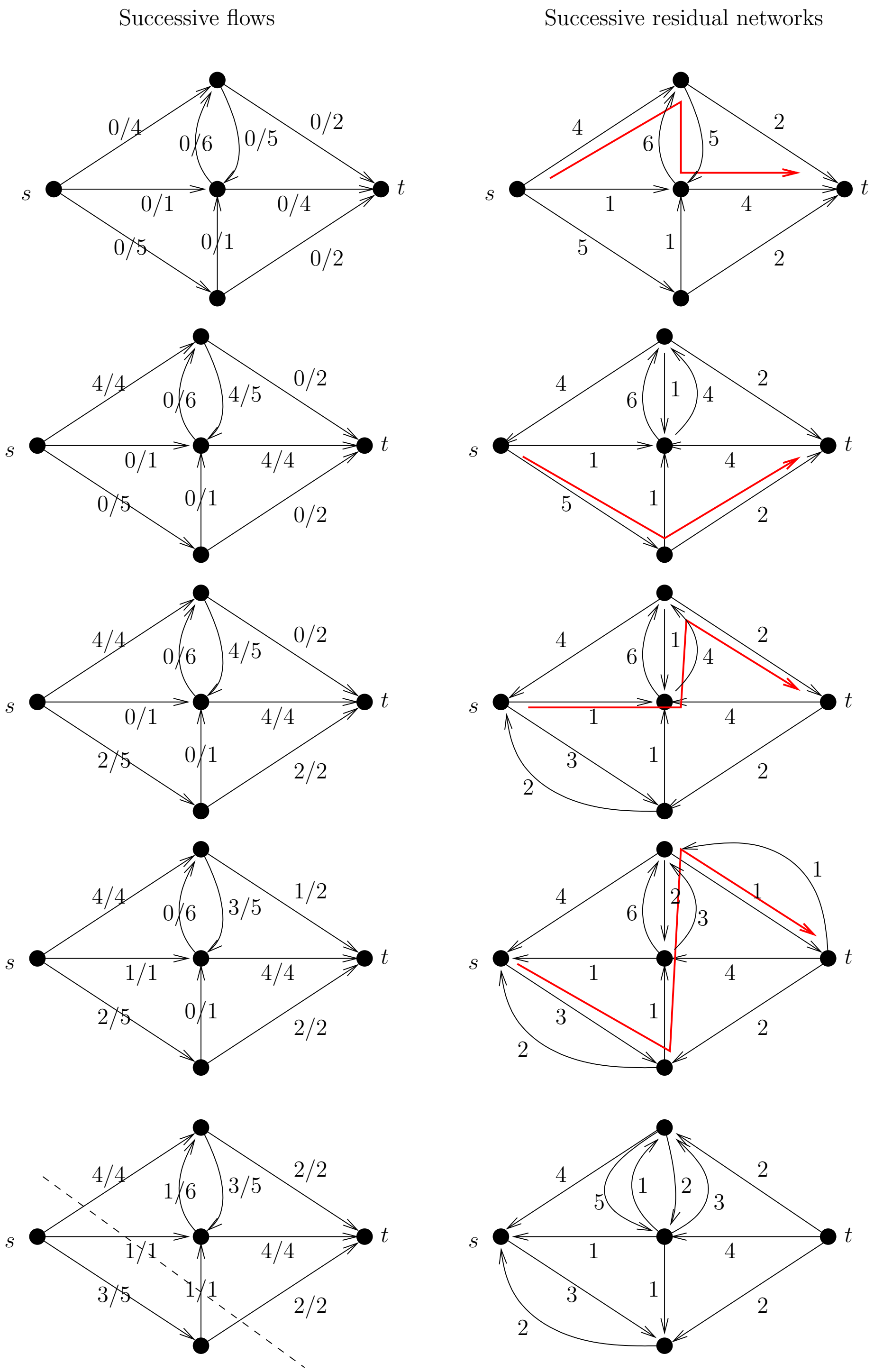
Graphs and Networks (MATH20150)

Problem sheet 11

1. See next page (it does not fit on this one).



In the final two steps, when computing the new flow between the vertex in the middle and the vertex on the top: You have the choice to add to the flow in the arc with capacity 6 or to remove from the flow in the arc with capacity 5. Both are fine, but give slightly different residual networks.

A minimal cut is indicated on the final flow. You can find it by using the remark we made after the Ford-Fulkerson algorithm (best), or by trying to "guess" (usually much less efficient). And you check it by checking that the capacity of the cut is equal to the value of the flow. (This is how you can justify that it is indeed a minimal cut.)

2. The network is built in such a way that a maximum flow will give a solution:

   The arcs in the middle model the months during which it is possible to work on each project. The capacity of each arc is 6 to indicate that at most 6 workers can work on any given project during each month. So if a flow indicates how to assign the workers, it will not be possible to assign more than 6 to each project each month.

   The arcs between the source and the months have capacity 8 to model the fact that the company can only employ 8 people each month on these project. Again, a flow will have value at most 8 on each of these arcs, so at most 8 people will be emplyed each month.

   The arcs between the projects and the sink have capacities 8, 10 and 12 to indicate the amount of man-hours needed to complete each project. If we find a maximum flow f with value at least 30, each one of these 3 arcs will be f-saturated. In the case of P_1 (for instance) it means that 8 is the out-flow in P_1, so the in-flow in P_1 is also 8, which means that 8 man-months have been affected to P_1, which is then completed.

   Using the Ford-Fulkerson algorithm, we do find a maximum flow with value 30, so the problem has a solution.

3. (a) If G is connected it has a spanning tree, which then has $n-1$ edges, impossible.

   (b) Let T be a spanning tree of G. We know that T has $|V|-1$ edges. If $|E| \leq |V|-1$ then we must have $|E| = |V|-1$ (if G has less edges that T, then G is not connected: removing an edge from a tree disconnects it) and $G = T$, so G contains no cycle.

   If $|E| > |V|-1$ then G is obtained by adding edges to T, which forms at least one cycle (seen in class).

   (c) Induction step: Let C be a cycle in G (we know there is one by the previous question). Remove an edge e from C, the resulting graph

G' has one cycle less. Since G' has $|E|-1$ edges, by induction G' has at least $|E|-1-|V|+1$ cycles, so (putting e back) G has at least $|E|-|V|+1$ cycles.