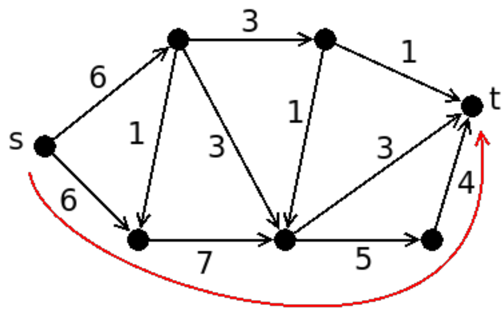
Graphs and Networks (MATH20150)

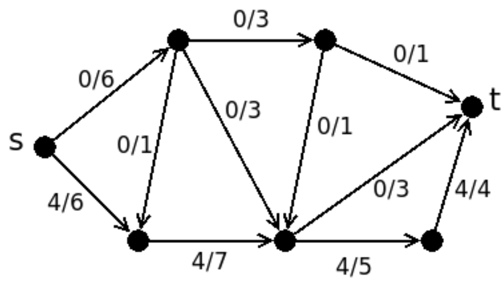
Problem sheet 10

1. We apply the Ford-Fulkerson algorithm, starting with the zero flow. Recall that you need to make choices when applying the algorithm (for instance, which path to take in the residual network), so your use of the algorithm may not have the same steps, and may not even produce the same flow. But it will be a maximal flow in all cases.

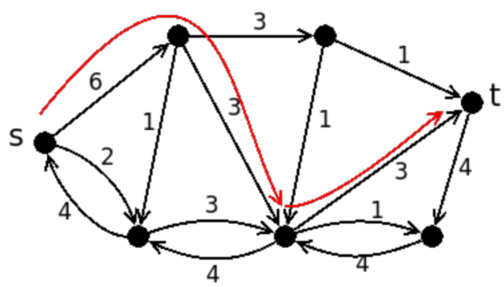
   (a) Since we start with the zero flow, the first residual network is the same as the original network, and we take the path indicated in red. We have $d = 4$



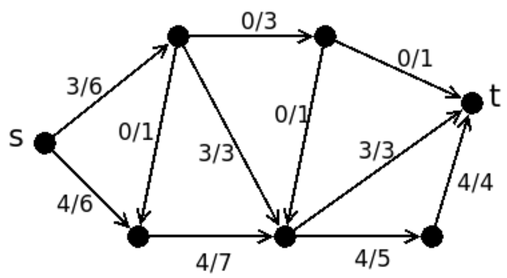
   (b) The new flow is then (with capacities indicated):



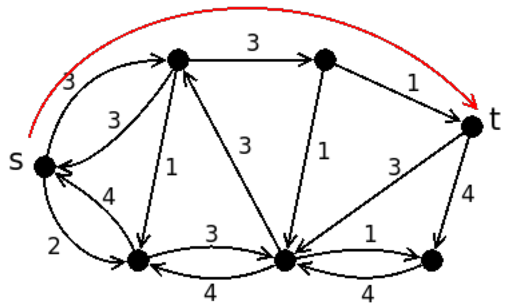
   (c) The residue network is then as follows, and we take the path indicated in red. We have $d = 3$.



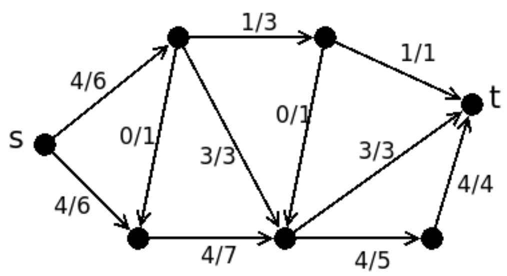
(d) The new flow is then (with capacities indicated):



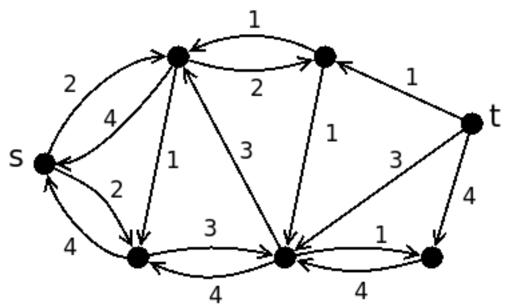
(e) The residue network is then as follows, and we take the path indicated in red. We have $d = 1$.



(f) The new flow is then (with capacities indicated):



(g) The residue network is then as follows. It is not possible to find a directed path from source to sink, so the previous flow was a maximum flow (with value 8).



You can observe that it would be possible to find different maximum flows (there are places where we can make different choices in the algorithm).

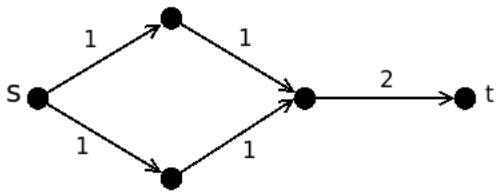
A minimum cut is given (for instance) by taking $V_T = \{t\}$.

2. The first two statements are false. This network gives a counter-example:

(The maximal flow in the previous question also provides a counter-example to the first statement.)

The third statement is true, by definition of capacity of a cut: If (V_S, V_T) is a cut with capacity $c(V_S, V_T)$, the capacity of the new cut is $\alpha c(V_S, V_T)$. And multiplying numbers by α preserves inequalitites.

The fourth statement if false. The following network is a counter-example (take $\alpha = 2$):



If we take $V_S = \{s\}$ (and the rest for V_T), then it is a minimal cut. But after adding 2 to all capacities, the cut is not minimal anymore (it is the one given by $V_T = \{t\}$ and $V_S =$ the rest).

4. Let $t = t(G)$. Then $G = G_1 \cup \cdots \cup G_t$ with $G_1, \ldots, G_t$ planar. In particular, $|E_i| \leq 3|V_i| - 6$ (by Corollary 8.15). Adding all these we get

$$|E| \leq 3(|V_1| + \cdots + |V_t|) - 6t$$

and since $|V_i| \leq |V|$ we get $|E| \leq 3|V|t - 6t$ and the result follows.

5. (a) Suppose $V(u) \neq V$. Since G is irreducible, it means that if we take $V_1 = V(u)$ and $V_2 = V \setminus V(u)$, the definition of reducible will not be satisfied. So there is $w \in V_2$ (i.e., $w \notin V(u)$) and an edge from $v \in V(u)$ to w. But then $w \in V(u)$ by definition of $V(u)$, contradiction

   (b) "$\Rightarrow$" The previous question gives us that for every $u \in V$, $V(u) = V$, which shows that G is strongly connected.

   "$\Leftarrow$" We assume that G is strongly connected. If G is not irreducible, then V can be partitioned into V_1 and V_2 as indicated above. Take $u \in V_1$ and $v \in V_2$. Then there is a directed path from v to u, and at some point along this directed path there will be an edge from V_2 to V_1, contradiction.