Universal enveloping TROs of continuous JBW*-triples and structure of W*-TROs

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[1] **Bunce, Leslie J.; Feely, Brian; Timoney, Richard M.** Operator space structure of JC*-triples and TROs, I. Math. Z. 270 (2012), no. 3–4, 961–982.

Bunce, Leslie J.; Timoney, Richard M. On the operator space structure of Hilbert spaces. Bull. Lond. Math. Soc. 43 (2011), no. 6, 1205–1218.

Bunce, Leslie J.; Timoney, Richard M. On the universal TRO of a JC*-triple, ideals and tensor products. Q. J. Math. 64 (2013), no. 2, 327–340.

[2] **Bunce, Leslie J.; Timoney, Richard M.** Universally reversible JC*-triples and operator spaces. J. Lond. Math. Soc. (2) 88 (2013), no. 1, 271–293.

[1'] **Bohle, Dennis; Werner, Wend** The universal enveloping ternary ring of operators of a JB*-triple system. Proc. Edinb. Math. Soc. (2) 57 (2014), no. 2, 347–366.

24 Sep 2007

Dear Bernie,

I hope you are well.

Before the summer I gave a seminar about some of your work with Neal dealing with how operator space structures interact with JC*-triple structures.

There is another way that gives some of your results in the tube domain case (JC*-algebras), based on the notion of universal C*-algebra. **I am still hoping to extend this approach to the triple case but so far this has not worked out.** Perhaps though it is time to let you know about this, even if I have not really got the thing properly generalised yet. I attch a pdf file with some rough edges.

Yours, Richard

16 Feb 2010

Dear Bernie and Matt,

After what seems an interminable time, I think we are almost ready with a first installment of our contribution to the topic of operator space structures of JC*-triples. I'm not sure, but I think the attached might be in final form and I think it is in good enough shape to send you it, even if we have some further thoughts about it before we submit it.

Yours, Richard

7 Jan 2011

Dear Bernie and Matt,
I thought you might be interested in this, a second installment of my work with Les Bunce, and related very directly to your work. I can imagine that you are thinking about other problems now, but any comments you have would be welcome. I hope all is well with you and best wishes for 2011!
Yours, Richard

7 Oct, 2011

Dear Bernie,
I hope you are keeping well and that it is not too monotonous having all that sunshine. I have been asked as subject editor for Functional Analysis of the Proceedings of the Edinburgh Mathematical Society to look after the submission The universal enveloping TRO of a JB*-triple system
Yours, Richard

9 Oct 2011

Dear Richard,
I agree to take on the task of evaluating this paper. Let me first make a few personal remarks.
When you sent me your first version of your paper (in Sept 2007) with Bunce and Feely, I had the intention of reducing my research activity upon completion of the last paper with Matt Neal (which occured in February 2008). I refrained from serious mathematical activity for two years (by choice). That is why I never really responded to your original paper and the followup paper.
That is the bad news. The good news is that since March of 2010, I have resumed a research program, thanks primarily to the encouragement of **Antonio Peralta**.
So now I am definitely interested in your papers and the one you are asking me to referee.
Sincerely, Bernie Russo

10 Oct 2011

Dear Bernie,
Thank you for your up-beat messages **and for the homwork you set me - to read your papers.** Yours, Richard

19 Dec 2011

Dear Richard,
I believe the project to study K-theory of Jordan triples initiated by Bohle and Werner will be a substantial contribution to the area. I believe their approach will have more applications, for example, in the study of cohomology of JB*-triples. I therefore believe that paper 11142 is definitely worthy of publication in Proc EMS as it provides tools to be used in the above projects. **Of course, the results in section 3 on Cartan factors are special cases of results of Bunce-Feely-Timoney and Bunce-Timoney but the approach, via grids, is different.** The Gelfand-Naimark result (Corollary 2.6) and the thorough study of the radical (section 4) more than compensate for this overlap.
With best wishes, Bernie

28 Sep 2012

Dear Bernie,
I hope you are continuing to be in good form and that things are all well in Irvine. I thought you would like to know Les and I have (finally) got another preprint out:
Title: Universally reversible JC*-triples and operator spaces
Yours, Richard

9 Aug 2016

Dear Bernie,

Thank you very much for the paper. Reading through it, I have been distacted by the book [2] of Ayupov et al, which I have not previously studied (and possibly did not see before now). There seems to be lots of relevant stuff in that book. Your exposition reads well, though I have not read every detail as yet. **I guess the Horn-Neher stuff is pretty tough going, for example, but you manage to sweep all the trouble under the carpet!** I'm a bit puzzled by the appeal to Dixmier on page 7 (proof of Lemma 3.1). I suppose that $\ell^\infty = C(K)$ for some big compact Hausdorff $K = \beta\mathbb{N}$ but the point evaluations at points of $\beta\mathbb{N} \setminus \mathbb{N}$ are *-homs but not normal? Your eWe can't like this example but I'm missing something. Yours, Richard

10 Aug 2016

Richard,

Many many thanks for your comments. I think I have fixed the proof of Lemma 3.1 but I had to assume separability to use Takesaki's theorem V 5.1 in his book volume I. Tomorrow I will double check everything and circulate the new file.

Again I appreciate your comment, which was right on.

Regards, Bernie

Outline

0. JC*-triples and TROs

1. Universal Enveloping TROs

2. Continuous JBW*-triples

3. Structure of W*-TROs via JC*triples

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4. Universal Enveloping W*-TROs

5. Alternate proofs. The Bunce-Timoney techniques

0. JC*-triples and TROs

A JC*-triple is a norm closed complex subspace of $B(K,H)$ which contain $xy^*z + zy^*x$ whenever it contains x, y, z.

Harris, Lawrence A. Bounded symmetric homogeneous domains in infinite dimensional spaces. Proceedings on Infinite Dimensional Holomorphy (Internat. Conf., Univ. Kentucky, Lexington, Ky., 1973), pp. 13–40. Lecture Notes in Math., Vol. 364, Springer, Berlin, 1974.

Harris, Lawrence A. A generalization of C*-algebras. Proc. London Math. Soc. (3) 42 (1981), no. 2, 331–361.

A ternary ring of operators (hereafter TRO) is a norm closed complex subspace of $B(K,H)$ which contains xy^*z whenever it contains x, y, z, where K and H are complex Hilbert spaces.

Hestenes, Magnus R. A ternary algebra with applications to matrices and linear transformations. Arch. Rational Mech. Anal. 11 1962 138–194.

Zettl, Heinrich A characterization of ternary rings of operators. Adv. in Math. 48 (1983), no. 2, 117–143.

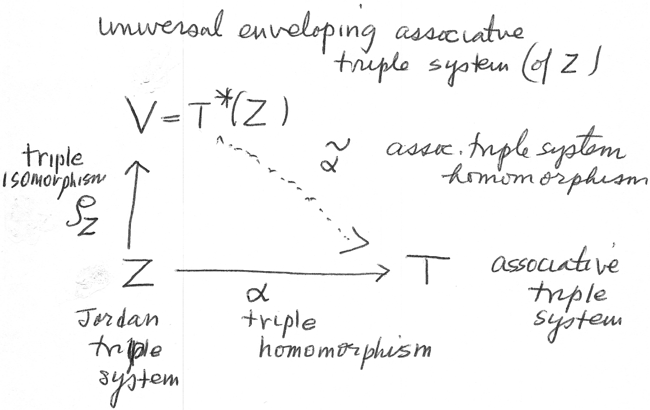
1. Universal Enveloping TROs

[1] **Bunce, Leslie J.; Feely, Brian; Timoney, Richard M.** Operator space structure of JC*-triples and TROs, I. Math. Z. 270 (2012), no. 3–4, 961–982.
[1'] **Bohle, Dennis; Werner, Wend** The universal enveloping ternary ring of operators of a JB*-triple system. Proc. Edinb. Math. Soc. (2) 57 (2014), no. 2, 347–366.

If E is a JC*-triple, denote by $C^*(E)$ and $T^*(E)$ the universal C*-algebra and the universal TRO of E respectively [1,Theorem 3.1,Corollary 3.2, Definition 3.3].

Recall that the former means that $C^*(E)$ is a C*-algebra, there is an injective JC*-homomorphism $\alpha_E :\to C^*(E)$ with the properties that $\alpha_E(E)$ generates $C^*(E)$ as a C*-algebra and for each JC*-homomorphism $\pi : E \to A$, where A is a C*-algebra, there is a unique *-homomorphism $\tilde{\pi} : C^*(E) \to A$ such that $\tilde{\pi} \circ \alpha_E = \pi$.

The latter means that $T^*(E)$ is a TRO, there is an injective TRO-homomorphism $\alpha_E :\to T^*(E)$ with the properties that $\alpha_E(E)$ generates $T^*(E)$ as a TRO and for each JC*-homomorphism $\pi : E \to T$, where T is a TRO, there is a unique TRO-homomorphism $\tilde{\pi} : T^*(E) \to T$ such that $\tilde{\pi} \circ \alpha_E = \pi$.



universal enveloping associative triple system (of Z)
V = T*(Z)
triple isomorphism
ρ_Z
α̃
assoc. triple system homomorphism
Z
T
α
triple homomorphism
Jordan triple system
associative triple system

Examples: Cartan Factors

$Z = M_{n,m}(\mathbb{C}),\ m, n \geq 2$, $T^*(Z) = M_{n,m}(\mathbb{C}) \oplus M_{m,n}(\mathbb{C})$

$Z = M_{n,1}(\mathbb{C})$ or $M_{1,n}(\mathbb{C})$, $T^*(Z) = \oplus_{k=1}^{n} M_{p_k,q_k}(\mathbb{C}),\ p_k = \binom{n}{k}, q_k = \binom{n}{k-1}$

$Z = A_n \subset M_n(\mathbb{C}),\ x^t = -x$, $T^*(Z) = M_n(\mathbb{C})$

$Z = S_n \subset M_n(\mathbb{C}),\ x^t = x$, $T^*(Z) = M_n(\mathbb{C})$

Z=spin factor, dimension $2n$, $T^*(Z) = M_{2^{n-1}}(\mathbb{C}) \oplus M_{2^{n-1}}(\mathbb{C})$
Z=spin factor, dimension $2n+1$, $T^*(Z) = M_{2^n}(\mathbb{C})$

spin system: $S = \{I, s_1, \ldots, s_n\} \subset M_m(\mathbb{C}),\ n \geq 2,\ s_i^* = s_i,\ s_i s_j + s_j s_i = 2\delta_{ij}$
spin factor $Z \subset M_m(\mathbb{C})$ is the linear span of S

Application

[1"] **Bohle, Dennis; Werner, Wend** A K-theoretic approach to the classification of symmetric spaces. J. Pure Appl. Algebra 219 (2015), no. 10, 4295–4321.

"*K*-theory" for associative algebras can be used to classify certain classes of operator algebras (C*-algebras).

Using the linking algebra of an associative triple system, one can obtain a *K*-theory for associative triple systems.

Using the universal enveloping associative triple system of a Jordan triple system, one can obtain a *K*-theory for Jordan triple systems, and hence a classification of a certain class of Jordan triple systems.

[1"'] **Bohle, Dennis; Werner, Wend** Inductive limits of finite dimensional hermitian symmetric spaces and K-theory. ArXiv, 2016.

2. Continuous JBW*-triples

[8] **Horn, Günther** Classification of JBW*-triples of type I. Math. Z. 196 (1987), no. 2, 271–291.

[9] **Horn, G.; Neher, E.** Classification of continuous JBW*-triples. Trans. Amer. Math. Soc. 306 (1988), no. 2, 553–578.

Let us recall the structure of all JBW*-triples U from [8,9]: there is a surjective linear isometric triple isomorphism

$$U \mapsto \oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha) \oplus pM \oplus H(N,\beta), \tag{1}$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, and β is a *-antiautomorphism of N of order 2 with fixed points $H(N,\beta)$.

Theorem 1 Let $W \subset B(H)$ be a continuous and σ-finite von Neumann algebra, and let e be a projection in W. Then $T^*(eW) = eW \oplus W^t e^t$, where x^t be any transposition on $B(H)$.

Theorem 2 If N is a continuous von Neumann algebra, then $T^*(H(N,\beta)) = N$.

2A. Continuous JBW*-triples. Theorem 1

[2] **Bunce, Leslie J.; Timoney, Richard M.** Universally reversible JC*-triples and operator spaces. J. Lond. Math. Soc. (2) 88 (2013), no. 1, 271–293.

The property of being universally reversible will be important for some of the proofs to follow.

A JC-algebra $A \subset B(H)_{sa}$ is called *reversible* if

$$a_1, \ldots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

A is *universally reversible* if $\pi(A)$ is reversible for each representation (=Jordan homomorphism) $\pi : A \to B(K)_{sa}$.

A JC*-algebra $A \subset B(H)$ is called *reversible* if

$$a_1, \ldots, a_n \in A \Rightarrow a_1 \cdots a_n + a_n \cdots a_1 \in A.$$

and A is *universally reversible* if $\pi(A)$ is reversible for each representation (=Jordan *-homomorphism) $\pi : A \to B(K)$.

Since JC-algebras are exactly the self-adjoint parts of JC*-algebras, a JC*-algebra A is reversible (respectively, universally reversible) if and only if the JC-algebra A_{sa} is reversible (respectively, universally reversible).

A JC*-triple $A \subset B(H,K)$ is called *reversible* if $a_1, \ldots, a_{2n+1} \in A \Rightarrow$

$$a_1 a_2^* a_3 \cdots a_{2n-1} a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* a_{2n-1} \cdots a_3 a_2^* a_1 \in A.$$

and A is *universally reversible* if $\pi(A)$ is reversible for each representation (=triple homomorphism) $\pi : A \to B(H', K')$.

It is easy to check that if a JC*-algebra is universally reversible as a JC*-triple, then it is universally reversible as a JC*-algebra.

[2,Proposition 3.9] Let e be a projection in a von Neumann algebra W such that (a) e is properly infinite or (b) W is continuous. Then eW is a universally reversible JC*-triple.

[2,Theorem 4.11] Let T be a TRO. Then the following are equivalent:
(a) T is universally reversible (as a JC*-triple);
(b) T has no triple homomorphisms onto a Hilbert space of dimension at least 3;
(c) T has no TRO homomorphisms onto a Hilbert space of dimension at least 3.

Corresponding to an orthonormal basis of a complex Hilbert space H, let J be the unique conjugate linear isometry which fixes that basis elementwise. The transpose $x^t \in B(H)$ of an element $x \in B(H)$ is then defined by $x^t = Jx^*J$

[2,Theorem 5.4] Let $T \subset B(H)$ be a TRO with no nonzero representations onto a Hilbert space of any dimension other than (possibly) 2. Suppose that $x \mapsto x^t$ is a transposition of B(H). Then
(a) $T^*(T) = T \oplus T^t$ with $\alpha_T(x) = x \oplus x^t$
(b) T has at least three distinct JC-operator space structures.

Theorem 1

Let $W \subset B(H)$ be a continuous and σ-finite von Neumann algebra, and let e be a projection in W. Then $T^*(eW) = eW \oplus W^t e^t$, where x^t be any transposition on $B(H)$.

Proof.

By [2, Proposition 3.9], eW is universally reversible and so by [2, Theorem 4.11], it does not admit a TRO homomorphism onto a Hilbert space of dimension greater than 2. The proof is completed by applying the lemma and [2, Theorem 5.4]. □

Lemma Let W be a continuous and σ-finite von Neumann algebra, and let e be a projection in W. Then the TRO eW does not admit a nonzero TRO homomorphism onto $\mathbb{C}$.

[5] **Plymen, R. J.**,Dispersion-free normal states, Il Nuovo Cimento. A, LIV, 1968, no. 4, 862–870

Lemma Let W be a continuous and σ-finite von Neumann algebra, and let e be a projection in W. Then the TRO eW does not admit a nonzero TRO homomorphism onto $\mathbb{C}$.

Proof Suppose, by way of contradiction, that f is a nonzero TRO homomorphism of eW onto $\mathbb{C}$. Since $f(e) = f(ee^*e) = f(e)|f(e)|^2$, either $f(e) = 0$ or $|f(e)| = 1$. The former case can be ruled out since for $x \in W$, $f(ex) = f((e1)(e1)^*(ex)) = |f(e)|^2 f(ex)$ and f would be zero. If then $f(e) = \lambda$ with $|\lambda| = 1$, then replacing f by $\overline{\lambda} f$ it can be assumed that $f(e) = 1$.

For $x, y \in W$, $f((exe)(eye)) = f(exee^*eye) = f(exe)\overline{f(e)}f(eye) = f(eye)f(eye)$ and $f((exe)^*) = f(ex^*e) = f(e(exe)^*e) = \overline{f(exe)}$ so that $f|eWe$ is a *-homomorphism onto $\mathbb{C}$ and since $f(e) = 1 = \|f\|$, $f|eWe$ is a state of eWe. Since eWe is σ-finite, by [Takesaki, Theorem 5.1, page 352], $f|eWe$ is normal. Now apply the theorem of Plymen [5] to the effect that a continuous von Neumann algebra admits no dispersion-free normal state. (A state is dispersion-free if it preserves squares of self-adjoint elements.)

2B. Continuous JBW*-triples. Theorem 2

[3] **Hanche-Olsen, Harald** On the structure and tensor products of JC-algebras. Canad. J. Math. 35 (1983), no. 6, 10591074.

Given a JC-algebra A , there is a universal C*-algebra B of A, analogous to the definition of $C^*(E)$ given above for JC*-triples E, with the following properties: there is a Jordan homomorphism π from A into B_{sa} such that B is the C*-algebra generated by $\pi(A)$ and for every Jordan homomorphism π_1 from A into C_{sa} for some C*-algebra C, there is a *-homomorphism $\pi_2 : B \to C$ such that $\pi_1 = \pi_2 \circ \pi$.

It is clear that $B = C^*(E)$ where E is the complexification of A.

For the convenience of the reader, the following theorem is stated. It will be used in the proof of Theorem 2.

[3, Theorem 4.4]

Let A be a universally reversible JC-algebra, B a C*-algebra , and $\theta : A \to B_{sa}$ an injective Jordan homomorphism such that B is the C*-algebra generated by $\theta(A)$. If B admits an antiautomorphism φ such that $\varphi \circ \theta = \theta$, then θ extends to a *-isomorphism of $C^*(A)$ onto B.

Let E be a JC*-algebra. Similar to the construction of $C^*(E)$ when E is considered as a JC*-triple, there is a C*-algebra $C_J^*(E)$ and a Jordan *-homomorphism $\beta_E : E \to C_J^*(E)$ such that $C_J^*(E)$ is the C*-algebra generated by $\beta_E(E)$ and every Jordan *-homomorphism $\pi : E \to B$, where B is a C*-algebra, extends to a *-homomorphism of $C_J^*(E)$ into B. (see [1, Remark 3.4]) By [1, Proposition 3.7], $T^*(E) = C_J^*(E)$. This fact is used in the following theorem.

[2, Proposition 2.2] Let M be a JW*-algebra such that M is continuous or of type I_∞ or of type I_n with $3 \leq n\infty$. Then M is a universally reversible JC*-triple.

Theorem 2

If N is a continuous von Neumann algebra, then $T^*(H(N,\beta)) = N$.

Proof.

Let $E = H(N,\beta)$. By [2, Proposition 2.2], E is universally reversible. In [3, Theorem 4.4], stated above, let $A = E_{sa}$, $B = N$, $\varphi = \beta$ and $\theta(x) = x$ for $x \in A$. By [4, Corollary 2.9], N is the C*-algebra generated by $\theta(A)$, so that [3, Theorem 4.4] applies to conclude that $N = C_J^*(E) = T^*(E)$. □

[4] **Gasemyr, Jørund** Involutory antiautomorphisms of von Neumann and C*-algebras. Math. Scand. 67 (1990), no. 1, 87–96.
[6] **Ayupov, Shavkat**, JW-factors and anti-automorphisms of von Neumann algebras, Math. USSR-Investiya, 26, 1986, 201–209
[7] **Ayupov, Shavkat, Rakhimov, Abdugafur, Usmanov, Shukhrat**, Jordan, real and Lie structures in operator algebras, Mathematics and its Applications, 418, Kluwer Academic Publishers Group, Dordrecht, 1997, x+225 pp

Remark

[4, Corollary 2.9], which was used in the proof of the theorem, is a corollary to [4, Theorem 2.8], which states that if N is a von Neumann algebra admitting a *-antiautomorphism α and if $H(N,\alpha)_{sa}$ has no type I_1 part, then N is generated as a von Neumann algebra by $H(N,\alpha)_{sa}$. The author of [4] was apparently unaware that [4, Corollary 2.9] was proved in the case of a continuous factor by Ayupov in 1985 [6], and the result in this case appeared as Theorem 1.5.2 in the book [7] in 1997. It is interesting to note that in that case, N is generated algebraically by $H(N,\alpha)_{sa}$.

Remark

If E is a JC-algebra, then $C_J^*(E)$ is *-isomorphic to a quotient of $C^*(E)$. Indeed, by definition of $C^*(E)$, there exists a surjective *-homomorphism $\widetilde{\beta}_E : C^*(E) \to C_J^*(E)$ such that $\widetilde{\beta}_E \circ \alpha_E = \beta_E$. In particular, if $C^*(E)$ is simple, then $C_J^*(E)$ is *-isomorphic to $C^*(E)$.*

Remark

If $E = H(N, \beta)$ and N is a factor of type II_1 or a countably decomposable factor of type III, then since such N is simple [Kadison-Ringrose, 6.8.4,6.8.5]), we have, by a previous remark, $N = T^*(E) = C^*(E) = C_J^*(E)$.

In connection with a previous remark, we ask

Question

Does there exist a JC-algebra E such that $C_J^*(E)$ is not *-isomorphic to $C^*(E)$?*

3. Structure of W*-TROs via JC*-triples

[2] **Bunce, Leslie J.; Timoney, Richard M.** Universally reversible JC*-triples and operator spaces. J. Lond. Math. Soc. (2) 88 (2013), no. 1, 271–293.
[10] **Kaneda, Masayoshi** Ideal decompositions of a ternary ring of operators with predual. Pacific J. Math. 266 (2013), no. 2, 297–303.

Theorem

(a) **[2, Lemma 5.17]** *A W^*-TRO is TRO-isomorphic to the direct sum $eW \oplus Wf$, where W is a von Neumann algebra and e, f are centrally orthogonal projections in W.*

(b) **[10, Theorem]** *A W^*-TRO X can be decomposed into the direct sum of TROs X_L, X_R, X_T, and there is a complete isometry of X into a von Neumann algebra M which maps X_L (resp. X_R, X_T) into a weak*-closed left ideal (resp. right ideal, two-sided ideal)*

[16] Ruan, Zhong-Jin Type decomposition and the rectangular AFD property for W*-TRO's. Canad. J. Math. 56 (2004), no. 4, 843–870.

Ruan's main representation theorems from [16] are summarized in the following theorem, in which $S\overline{\otimes}T$ denotes the weak operator closure of the algebraic tensor product of subspaces S, T of operators on Hilbert spaces.

Theorem

Let V be a W^*-TRO.

i *If V is a W*-TRO of type I, then V is TRO-isomorphic to* $\oplus_\alpha L^\infty(\Omega_\alpha, B(K_\alpha, H_\alpha))$. ([16, Theorem 4.1])

ii If V is a W*-TRO of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, acting on a separable Hilbert space, then V is a stable W*-TRO, and hence TRO-isomorphic to a von Neumann algebra. ([16, Corollary 4.3])

iii If V is a W*-TRO of type $II_{1,\infty}$ (respectively $II_{\infty,1}$), then V is TRO-isomorphic to $B(H,\mathbb{C})\overline{\otimes}M$ (respectively $B(\mathbb{C},H)\overline{\otimes}N$), where M (respectively N) is a von Neumann algebra of type II_1. ([16, Theorem 4.4])

[8] **Horn, Günther** Classification of JBW*-triples of type I. Math. Z. 196 (1987), no. 2, 271–291.
[9] **Horn, G.; Neher, E.** Classification of continuous JBW*-triples. Trans. Amer. Math. Soc. 306 (1988), no. 2, 553–578.

Theorem 3 Let V be a W*-TRO acting on a separable Hilbert space. Then there is a continuous von Neumann algebra A such that V is TRO-isomorphic to

$$\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha)) \oplus eA \oplus Af$$

Consider the space V with the JC*-triple structure given by $\{xyz\} = (xy^*z + zy^*x)/2$, so that V becomes a JBW*-triple. As noted in (1), there is a surjective linear isometry

$$V \mapsto \oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha) \oplus pM \oplus H(N, \beta), \tag{2}$$

where each C_α is a Cartan factor, M and N are continuous von Neumann algebras, p is a projection in M, β is a *-antiautomorphism of N of order 2 with fixed points $H(N, \beta)$.

Proof of Theorem 3

For any W*-TRO, by (2), write $V = V_1 \oplus V_2 \oplus V_3$, where V_i are weak*-closed orthogonal triple ideals of V with V_1 triple isomorphic to a JBW*-triple $\oplus_\alpha L^\infty(\Omega_\alpha, C_\alpha)$ of type I, V_2 triple isomorphic to a right ideal pM in a continuous von Neumann algebra M, and V_3 triple isomorphic to $H(N, \beta)$ for some continuous von Neumann algebra N admitting a *-antiautomorphism β of order 2

Since the triple ideals coincide with the TRO ideals ([1, Lemma 2.1]), in particular each V_i is a sub-W*-TRO of V.

Consider first V_2. By [2, Lemma 5.17] or [10, Theorem], V_2 is TRO-isomorphic to $eA \oplus Af$, for some von Neumann algebra A. In particular, V_2 is triple isomorphic to $eA \oplus f^tA^t = (e \oplus f^t)(A \oplus A^t)$ and to pM, so by [9], $A \oplus A^t$ has the same type as M. It follows that A is a continuous von Neumann algebra.

[11] **Isidro, José M.; Stacho, Lászlo L.** On the Jordan structure of ternary rings of operators. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 46 (2003), 149–156 (2004).

Next it is shown that $V_3 = 0$. V_3 is triple isomorphic to $H(N, \beta)$ and TRO-isomorphic to $eA \oplus Af$, for a von Neumann algebra A. Thus the continuous JBW*-triple $H(N, \beta)$ of hermitian type is triple isomorphic to the JBW*-triple $(e \oplus f^t)(A \oplus A^t)$, which is necessarily continuous and hence of associative type. By the uniqueness of the representation theorem for continuous JBW*-triples ([9,Section 4]), $H(N, \beta) = 0$.

Finally, consider V_1. There are weak*-closed TRO ideals V_α such that $V_1 = \oplus_\alpha V_\alpha$ with V_α triple isomorphic to $L^\infty(\Omega_\alpha, C_\alpha)$ provided that $V_\alpha \neq 0$. It is shown in [11, Lemma 2.4 and Proof of Theorem 1.1] that no Cartan factor of type 2,3,4,5,6 can be isometric to a TRO. It follows easily that $L^\infty(\Omega_\alpha, C_\alpha)$ cannot be isometric to a TRO unless C_α is a Cartan factor of type 1. Therefore each C_α is a Cartan factor of type 1, and therefore V_α is triple isomorphic to $L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$ for suitable Hilbert spaces H_α and K_α.

Next consider V_α for a fixed α. To simplify notation let U denote V_α and W denote $L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$.

[12] **Chu, Cho-Ho; Neal, Matthew; Russo, Bernard** Normal contractive projections preserve type. J. Operator Theory 51 (2004), no. 2, 281–301.
[13] **Bunce, Leslie J.; Peralta, Antonio M.** Images of contractive projections on operator algebras. J. Math. Anal. Appl. 272 (2002), no. 1, 55–66.

By [2, Lemma 5.17] or [10, Theorem], U is TRO-isomorphic to $eA \oplus Af$, for some von Neumann algebra A, and therefore

$$eA \oplus Af \overset{TRO}{\simeq} U \overset{triple}{\simeq} W = L^\infty(\Omega, B(H, K)). \tag{3}$$

The right side of (3) is a JBW*-triple of type I and thus by [12, Theorem 5.2] or [13, Theorem 4.2], eA is a JBW*-triple of type I, which implies that A is a von Neumann algebra of type I.

Summarizing up to this point, V acts on a separable Hilbert space but is otherwise arbitrary, and $V = V_1 \oplus V_2 \oplus V_3$, where

$$V_1 \overset{TRO}{\simeq} \oplus_\alpha e_\alpha A_\alpha \oplus A_\alpha f_\alpha \quad , \quad V_2 \overset{TRO}{\simeq} eA \oplus Af \quad , \quad V_3 = 0, \tag{4}$$

where each A_α is a von Neumann algebra of type I, and A is a continuous von Neumann algebra.

Let us now consider V_1, and focus on a component on the right side of (4) for a fixed α, which is denoted, again for notation's sake, by $eB \oplus Bf$ where B is a von Neumann algebra of type I.

Write $B = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma))$, $e = \oplus_\gamma e_\gamma$, and $f = \oplus_\gamma f_\gamma$ so that

$$eB = \oplus_{\gamma\in\Gamma} e_\gamma L^\infty(\Sigma_\gamma, B(H_\gamma)),$$

$$Bf = \oplus_{\gamma\in\Gamma} L^\infty(\Sigma_\gamma, B(H_\gamma)) f_\gamma.$$

Since B acts on a separable Hilbert space, the reduction theory of von Neumann algebras ([Dixmier book, Part II]) can now be used to conclude this proof.

For a fixed $\gamma \in \Gamma$,

$$L^\infty(\Sigma_\gamma, B(H_\gamma)) = \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$L^2(\Sigma_\gamma, H_\gamma) = \int_{\Sigma_\gamma}^{\oplus} H_\gamma\, d\mu_\gamma(\sigma_\gamma),$$

$$B = \sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

$$e_\gamma = \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma)\, d\mu_\gamma(\sigma_\gamma),$$

and

$$eB = \sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma) B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma).$$

For notation's sake, for a fixed $\gamma \in \Gamma$, let $\sigma = \sigma_\gamma$, $\mu = \mu_\gamma$, $e = e_\gamma$, $\Sigma = \Sigma_\gamma$, $H = H_\gamma$, and suppose H is a separable Hilbert space. For each $n \le \aleph_0$, let $\Sigma_n = \{\sigma \in \Sigma : e(\sigma) \text{ has rank } n\}$, $e_n = e|_{\Sigma_n}$, and let K_n be a Hilbert space of dimension n.

Then

$$\int_{\Sigma}^{\oplus} e(\sigma) B(H)\, d\mu(\sigma) = \sum_{n\le\aleph_0}^{\oplus} \int_{\Sigma_n}^{\oplus} e_n(\sigma) B(H)\, d\mu(\sigma).$$

For each $\sigma \in \Sigma_n$, let $G_\sigma = \{\text{all unitaries } U : e_n(\sigma)H \to K_n\}$, let $G = \cup_{\sigma\in\Sigma_n} G_\sigma$, and then set

$$E = \{(\sigma, U) \in \Sigma_n \times G : U \in G_\sigma\}.$$

By the measurable selection theorem ([Dixmier book,Appendix V]), there exists a μ-measurable subset $\Sigma_n' \subset \Sigma_n$ of full measure and a μ-measurable mapping η of Σ_n' into G, such that $\eta(\sigma) \in G_\sigma$ for every $\sigma \in \Sigma_n'$.

It is easy to verify that for each $\sigma \in \Sigma_n'$, $T_{n,\sigma} : e_n(\sigma)x \mapsto \eta(\sigma)e_n(\sigma)x$ is a TRO-isomorphism of $e_n(\sigma)B(H)$ onto $B(H, K_n)$ and that $\{T_{n,\sigma} : \sigma \in \Sigma_n'\}$ is a μ-measurable field of TRO-isomorphisms.

Hence $\int_{\Sigma_n}^{\oplus} T_{n,\sigma}\, d\mu(\sigma)$ is a TRO-isomorphism of $\int_{\Sigma_n}^{\oplus} e_n(\sigma)B(H)\, d\mu(\sigma)$ onto $\int_{\Sigma_n}^{\oplus} B(H, K_n)\, d\mu(\sigma)$, that is

$$\int_{\Sigma_n}^{\oplus} e_n(\sigma)B(H)\, d\mu(\sigma) \overset{TRO}{\simeq} L^\infty(\Sigma_n, B(H, K_n)).$$

Going back to the earlier notation, since

$$eB = \sum_{\gamma\in\Gamma}^{\oplus} \int_{\Sigma_\gamma}^{\oplus} e_\gamma(\sigma_\gamma) B(H_\gamma)\, d\mu_\gamma(\sigma_\gamma).$$

it follows that

$$eB \overset{TRO}{\simeq} \sum_{\gamma\in\Gamma}^{\oplus} \sum_{n\leq\aleph_0} L^\infty(\Sigma_{\gamma,n}, B(H_\gamma, K_n)).$$

By the same arguments, it is clear that also

$$Bf \overset{TRO}{\simeq} \sum_{\gamma\in\Gamma'}^{\oplus} \sum_{n\leq\aleph_0} L^\infty(\Sigma'_{\gamma,n}, B(K_n, H'_\gamma)).$$

Since B was one of the A_α in (4), this completes the proof.

Corollary 1

Let V be a W*-TRO acting on a separable Hilbert space. If V has no type I part, then it is TRO-isomorphic to $eA \oplus Af$, where A is a continuous von Neumann algebra.

Corollary 2

A W*-TRO of type I, acting on a separable Hilbert space, is TRO-isomorphic to $\oplus_\alpha L^\infty(\Omega_\alpha, B(H_\alpha, K_\alpha))$.

Corollary 3

A W*-TRO of type $II_{1,1}$ which acts on a separable Hilbert space is TRO-isomorphic to $eA \oplus Af$, where e, f are centrally orthogonal projections in a von Neumann algebra A of type II_1.

4. Universal Enveloping W*-TROs

[14] **Hanche-Olsen, Harald; Størmer, Erling** Jordan operator algebras. Monographs and Studies in Mathematics, 21. Pitman (Advanced Publishing Program), Boston, MA, 1984. viii+183 pp.

In view of the success of the universal enveloping TRO of a JC*-triple, it should be fruitful to have a theory of universal enveloping W^*-TROs of JW^*-triples. Here we propose a definition modeled after the case of JBW-algebras in [14,7.1.8] and state four test questions.

Let E be a JW*-triple and consider

$$E \xrightarrow{\alpha_E} C^*(E) \xrightarrow{i} C^*(E)^{**},$$

where i is the canonical embedding of $C^*(E)$ in its enveloping von Neumann algebra $C^*(E)^{**}$. Let $\{e_\alpha\}$ be a maximal family of orthogonal central projections in $C^*(E)^{**}$ such that

$$E \to e_\alpha C^*(E)^{**}, \quad a \mapsto e_\alpha(i \circ \alpha_E(a))$$

is normal, and let $e = \sum_\alpha e_\alpha$.

Then e is the maximal central projection with this property. Define

$$W^*(E) = eC^*(E)^{**}$$

and let $\psi_E : E \to \mathrm{W}^*(E)$ be the map $a \mapsto e(i \circ \alpha_E(a))$.

For each normal triple homomorphism $\phi : E \to N$, where N is a von Neumann algebra, there is a *-homomorphism $\phi_1 : C^*(E) \to N$ such that $\phi_1 \circ \alpha_E = \phi$. The map ϕ_1 extends to a normal *-homomorphism

$$\overline{\phi} : C^*(E)^{**} \to N.$$

Let f be the support projection of $\overline{\phi}$ in $C^*(E)^{**}$, so that $\overline{\phi}(1 - f) = 0$ and $\overline{\phi}|_{fC^*(E)^{**}}$ is injective. The map

$$E \to fC^*(E)^{**}, \quad a \mapsto f(i \circ \alpha_E(a))$$

decomposes as

$$a \mapsto \phi(a) \to (\overline{\phi}|_{fC^*(E)^{**}})^{-1}(\phi(a))$$

which is a composition of normal maps, hence is normal, and thus $f \leq e$.

Moreover, $W^*(E)$ is the von Neumann algebra generated by $\psi_E(E)$. This proves the following theorem, which is the analog of [1, Theorem 3.1].

Theorem 4

If E is a JW^*-triple, then there exists a unique pair $(W^*(E), \psi_E)$, where $W^*(E)$ is a von Neumann algebra, ψ_E is an injective normal triple homomorphism from E into $W^*(E)$ such that
(a) $\psi_E(E)$ generates $W^*(E)$ as a von Neumann algebra, and
(b) if $\pi : E \to A$ is a normal triple homomorphism into a von Neumann algebra A, then there exists a unique normal *-homomorphism $\tilde{\pi} : W^*(E) \to A$ such that $\tilde{\pi} \circ \psi_E = \pi$.

By letting $WT^*(E)$ denote the W^*-TRO generated in $W^*(E)$, we obtain the following, which is the analog of [1, Corollary 3.2].

Corollary

If E is a JW^*-triple, then there exists a unique pair $(WT^*(E), \psi_E)$, where $WT^*(E)$ is W^*-TRO, ψ_E is an injective normal triple homomorphism from E into $WT^*(E)$ such that
(a) $\psi_E(E)$ generates $WT^*(E)$ as a W^*-TRO, and
(b) if $\pi : E \to T$ is a normal triple homomorphism into a W^*-TRO T, then there exists a unique normal TRO-homomorphism $\tilde{\pi} : WT^*(E) \to T$ such that $\tilde{\pi} \circ \psi_E = \pi$.

We now state our four test questions.

In [1,Theorem 5.4], it is proved that if $B(H,K)$ is of rank at least 2, then $T^*(B(H,K)) = B(H,K) \oplus B(K,H)$.

Question

1. What is $WT^*(B(H,K)$ in this case?

In [1, Proposition 3.6], it is proved that if the JC*-triple E is the sum of orthogonal ideals I, J, then $T^*(E) = T^*(I) \oplus T^*(J)$.

Question

2. If the W^*-TRO E is the sum of weak-closed orthogonal ideals I, J, does it follow that $WT^*(E) = WT^*(I) \oplus WT^*(J)$?*

In [1, Theorem 5.1], it is proved that if H is a Hilbert space, then $T^*(H)$ is the TRO generated by the annihilation operators in the CAR algebra of H.

Question

3. What is $WT^*(H)$?

In [2, Theorem 4.9], it is proved that if A is an abelian von Neumann algebra and E is a JC*-triple, then $T^*(A\widehat{\otimes}_\epsilon E) = A\widehat{\otimes}_\epsilon T^*(E)$, where $\widehat{\otimes}_\epsilon$ is the injective tensor product of Banach spaces

If a JC*-triple E is contained in a C*-algebra B, then $A\widehat{\otimes}_\epsilon E$ is a JC*-subtriple of $A\widehat{\otimes}_\epsilon B = A \otimes_{\min} B$. Similarly, if A is an abelian von Neumann algebra and $E \subset B(H)$ is a JW^*-triple, then the spatial tensor product $A \otimes E$ is a subtriple of $A \otimes B(H)$, and $A\overline{\otimes}E$ denotes the weak closure of $A \otimes E$.

Question

4. Do we have $WT^*(A\overline{\otimes}E) = A\overline{\otimes}WT^*(E)$, at least for $E =$ Hilbert space and $E = B(H,K)$ of rank at least 2?

5. Alternate proofs. The Bunce-Timoney techniques

Presented here are alternate approaches to the proofs of the assertions concerning V_2 and V_3 in the proof of Theorem 4, along the lines of the proof of the assertion concerning V_1. The purpose for doing this is that, despite the fact that these proofs are longer, they further illustrate the power of the techniques used from [1] and [2]. In addition, use is made of Theorems 1 and 2 on continuous JBW*-triples.

In what follows, it is assumed that V acts on a separable Hilbert space

Consider first V_2. Recall that V_2 is triple isomorphic to a right ideal pM in a continuous von Neumann algebra M, which is σ-finite. For notation's sake, denote V_2 by V and pM by W. By Theorem 1, $T^*(W) = W \oplus W^t$ and $\alpha_W(x) = x \oplus x^t$. By [2, Proposition 3.9 and Theorem 4.11] W does not admit a triple homomorphism onto a Hilbert space of dimension greater than 2, and therefore the same holds for V.

Next it is shown that W does not admit a triple homomorphism onto $\mathbb{C}$, and it follows that V does not admit a triple homomorphism, and *a priori*, a TRO-homomorphism onto $\mathbb{C}$, thus guaranteeing, by [2, Theorem 5.4], that $T^*(V) = V \oplus V^t$ and $\alpha_V(x) = x \oplus x^t$.

Suppose then, that $f : pM \to \mathbb{C}$ is a nonzero triple homomorphism, that is, for $x, y, z \in M$,

$$f\{px, py, pz\} = f\left(\frac{pxy^*pz + pzy^*px}{2}\right) = f(px)\overline{f(py)}f(pz). \tag{5}$$

Putting $x = y = z = 1$ in (5) yields $f(p) = |f(p)|^2 f(p)$, so $f(p) = 0$ or $|f(p)| = 1$.

Suppose $f(p) = 0$. Then setting $y = 1$ in (5) yields

$$f(pxpz + pzpx) = 0, \quad (x, z \in M)$$

and setting $z = 1$ in (5) yields

$$f(pxy^*p + py^*px) = 0, \quad (x, y \in M),$$

which implies

$$f(pxp + px) = 0 \quad (x \in M).$$

Thus

$$0 = f(pxy^*p) + f(py^*px) = -f(pxy^*) - f(pxy^*p)$$

and in particular

$$0 = f(py^*p) + f(py^*p) = -f(py^*) - f(py^*p)$$

so that $f(py^*) = 0$ for $y \in M$, that is, $f = 0$.

[15] **Effros, Edward G.; Ozawa, Narutaka; Ruan, Zhong-Jin** On injectivity and nuclearity for operator spaces. Duke Math. J. 110 (2001), no. 3, 489–521.

[15, Proposition 2.4]

A TRO-isomorphism between dual TROs is automatically weak*-continuous.

Assume now without loss of generality, that $f(p) = 1$. Writing $(pxp)(pyp) = (pxp)p^*(pyp)$, then for $x, y \in M$,

$$f((pxp) \circ (pyp)) = f\{pxp, p, pyp\} = f(pxp)f(pyp)$$

so that f is a Jordan *-homomorphism of pMp onto $\mathbb{C}$. It follows that f is a normal dispersion-free state on a continuous von Neumann algebra, and hence must be zero (see the proof of the Lemma in the proof of Theorem 1).

Thus $T^*(V) = V \oplus V^t$, $\alpha_V(x) = x \oplus x^t$ and there is a weak*-continuous TRO-isomorphism of $T^*(V)$ onto $T^*(W)$, by [15, Proposition 2.4].

Thus V is TRO-isomorphic to a weak*-closed ideal I in $W \oplus W^t$. Writing $I = (I \cap W) \oplus (I \cap W^t)$, then $I \cap W$ is a weak*-closed ideal in W, let's call it I_1, and $I \cap W^t$ is a weak*-closed ideal in W^t, let's call it I_2. As noted in [9], there are projections $p_1 \leq p, p_2 \leq p^t$ such that $I_1 = p_1 M$ and $I_2 = M^t p_2$.

More precisely,

$$I = I_1 \oplus I_2 = (p_1 \oplus 0)(M \oplus M^t) \oplus (M \oplus M^t)(0 \oplus p_2) = eA \oplus Af, \qquad (6)$$

where $A = M \oplus M^t$ is a continuous von Neumann algebra, $e = p_1 \oplus 0$ and $f = 0 \oplus p_2$.

In the rest of this section we shall refer to the classification scheme of Ruan.

Ruan Classification Scheme

If R is a von Neumann algebra and e is a projection in R, then $V := eR(1-e)$ is a W*-TRO. Conversely if $V \subset B(K,H)$ is a W*-TRO, then with $V^* = \{x^* : x \in V\} \subset B(H,K)$, $M(V) = \overline{XX^*}^{sot} \subset B(H)$, $N(V) = \overline{X^*X}^{sot} \subset B(K)$, let

$$R_V = \begin{bmatrix} M(V) & V \\ V^* & N(V) \end{bmatrix} \subset B(H \oplus K)$$

denote the linking von Neumann algebra of V. Then there is a SOT-continuous TRO-isomorphism $V \simeq eRe^{\perp}$, where $e = \left[\begin{smallmatrix} 1_H & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e^{\perp} = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_K \end{smallmatrix}\right]$.

In particular, if $V = pM$ where p is a projection in a von Neumann algebra M, then

$$R_V = \begin{bmatrix} pMp & pM \\ Mp & c(p)M \end{bmatrix} \subset B(H \oplus H),$$

where $c(p)$ denotes the central support of p (see [2, p. 965]).

A W*-TRO V is of type I,II, or III according as R_V is a von Neumann algebra of the corresponding type. A W*-TRO of type II is said to be of type $II_{\epsilon,\delta}$, where $\epsilon, \delta \in \{1, \infty\}$, if $M(V)$ is of type II_ϵ and $N(V)$ is of type II_δ.

With the Ruan classification in hand, we can now prove Corollaries 1 and 3.

Corollary 1

Let V be a W*-TRO acting on a separable Hilbert space. If V has no type I part, then it is TRO-isomorphic to $eA \oplus Af$, where A is a continuous von Neumann algebra.

Proof.

Suppose that V has no type I part. Then $M(V)$ has no type I part and the same holds for $M(V_\alpha)$ by (4). But $M(V_\alpha)$ is *-isomorphic to $e_\alpha A_\alpha e_\alpha \oplus c(f_\alpha)A_\alpha$, which is a von Neumann algebra of type I, hence $V_\alpha = 0$. □

Corollary 3 A W*-TRO of type $II_{1,1}$ which acts on a separable Hilbert space is TRO-isomorphic to $eA \oplus Af$, where e, f are centrally orthogonal projections in a von Neumann algebra A of type II_1.

With regard to Corollary 3, suppose now that V is of type $II_{1,1}$. It will be shown that A can be chosen to be of type II_1. Since

$$R_V \stackrel{*-iso.}{\simeq} R_{I_1} \oplus R_{I_2} = \begin{bmatrix} eAe & eA \\ Ae & c(e)A \end{bmatrix} \oplus \begin{bmatrix} c(f)A & Af \\ fA & fAf \end{bmatrix},$$

it follows that $c(f)A$ and $c(e)A$ are each of type II_1.

Since $p_1 M = p_1(c(p_1)M)$ and $M^t p_2 = (M^t c(p_2))p_2$, if $A = M \oplus M^t$ is replaced by $\tilde{A} = c(p_1)M \oplus c(p_2)M^t$, then $\tilde{A}$ is a continous von Neumann algebra, $eA \oplus Af = e\tilde{A} \oplus \tilde{A}f$, and

$$R_V \stackrel{*-iso.}{\simeq} \begin{bmatrix} e\tilde{A}e & e\tilde{A} \\ \tilde{A}e & \tilde{A} \end{bmatrix} \oplus \begin{bmatrix} \tilde{A} & \tilde{A}f \\ f\tilde{A} & f\tilde{A}f \end{bmatrix},$$

so that $\tilde{A}$ is of type II_1.

Consider next V_3. V_3 is triple isomorphic to $H(N, \beta)$ for some continuous von Neumann algebra N which admits a *-anti-automorphism β of order 2. For notation's sake, denote V_3 by V and $H(N, \beta)$ by W.

Note first that V is a universally reversible TRO. This follows by the same arguments which were used in the discussion of V_2 in this section. Indeed, by [2, Proposition 2.2] and the paragraph preceding it, W is a universally reversible JC*-triple, and therefore so is V. As before, V does not admit a triple homomorphism onto a Hilbert space of dimension different from 2.

On the other hand, V has no nonzero TRO-homomorphism onto $\mathbb{C}$, since such a homomorphism would extend to a *-homomorphism of the linking von Neumann algebra R_V of V onto $M_2(\mathbb{C})$, whose restriction ρ to the upper left corner of R_V would be a dispersion-free normal state on a continuous von Neumann algebra, and hence cannot exist (see the proof of the Lemma in the proof of Theorem 1).

So $T^*(V) = V \oplus V^t$, $\alpha_V(x) = x \oplus x^t$, and $V \oplus V^t$ is TRO-isomorphic to $T^*(W) = N$, by Theorem 2.

By [15, Proposition 2.4], the TRO-isomorphism is weak*-continuous.

Hence the weak*-closed TRO ideal V in $V \oplus V^t$ is mapped onto a weak*-closed TRO ideal in N, which is necessarily a two-sided ideal in N, say zN for some central projection z in N.

From (6) it follows that $V_2 \oplus V_3$ is TRO-isomorphic to

$$[(e \oplus 0)(A \oplus N)] \oplus [(A \oplus N)(f \oplus 0) \oplus [(0 \oplus z)(A \oplus N)], \tag{7}$$

so that $V_2 \oplus V_3$ is the direct sum of a weakly closed left ideal and a weakly closed right ideal in a continuous von Neumann algebra, which is tantamount to proving that $V_3 = 0$.