Spaceability, algebrability and residuality on some sets of analytic functions.

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The origin of the concept of lineability is due to Gurariy in 1966 [10], that showed that there exists an infinite dimensional linear space contained in the set of nowhere differentiable functions on $[0, 1]$.

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And in the same article, they proved that the space of the functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ that are *everywhere surjective* is lineable.

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In 2006 [5], Aron, Pérez-García and Seoane-Sepúlveda showed that the set of continuous functions whose Fourier series expansion diverges is algebrable.

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One of the main results of the paper asserts that if X is a Fréchet space and $Y \subset X$ is a closed linear subspace, then the complement $X \setminus Y$ is spaceable if and only if Y has infinite codimension.

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Also, if X and Y are Fréchet spaces and $T : Z \longrightarrow X$ a continuous linear operator with range $Y = T(Z)$ not closed, then the complement $X \setminus Y$ is spaceable.

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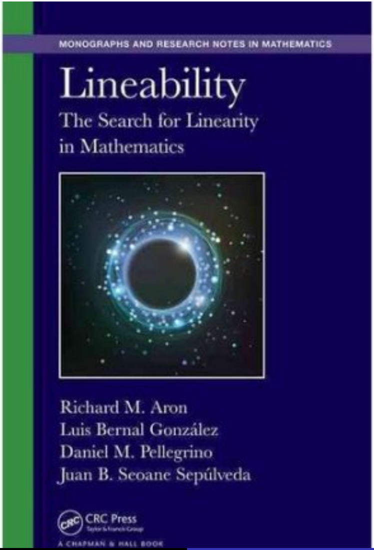

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if $Y \setminus A = \cup_{n=1}^{\infty} F_n$, with $\overset{\circ}{\overline{F_n}} = \emptyset$.

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Let $(M_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers such that $M_0 = 1$, and for each $n \geq 1$, $\dfrac{M_n}{M_k \, M_{n-k}} \geq \binom{n}{k}$, $(0 \leq k \leq n)$.

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Lemma (M. L. Lourenço, D. M. V., 2016)

Let $w = 2e^{i\theta}$, where $0 \leq \theta < 2\pi$. Let $f(z) = \dfrac{1}{z-w}$, for all $z \in \overline{D}$. Then $f \in \mathcal{H}$.

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Lemma (M. L. Lourenço, D. M. V., 2019)

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Consequently $\mathcal{H}$ is maximal lineable. Spaceability of $\mathcal{H}$ follows from Kitson-Timoney's result on operator ranges.

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$$\lim_{h \to 0} \frac{\|f(\omega + h) - f(\omega) - \zeta \cdot h\|}{\|h\|} = 0.$$

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Let $F : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $F(z, w) = (w, z)$. Then F is analytic but it is not (L)-analytic.

Thus the set $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is not empty and $\mathcal{G}$ is not a vector space.

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$$f \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2) \text{ if and only if } f_\alpha \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2).$$

Thus the set $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is not empty and $\mathcal{G}$ is not a vector space.

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Let $f \in \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$. For each $\alpha \imath \mathbb{C}$, consider

$$f_\alpha(z, w) := f(\alpha(z, w)), \text{ for all } (z, w) \in \mathbb{C}^2.$$

Then, for every $\alpha \in \mathbb{C}$, $\alpha \neq 0$:

$$f \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2) \text{ if and only if } f_\alpha \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2).$$

This fact allows us to exhibit more elements of $\mathcal{G}$.

Proposition

Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ *be given by* $f(z, w) = (e^w, e^z)$. *Then* $\{f_\alpha : \alpha > 0\}$ *is a linearly independent set in* $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ *and* $[f_\alpha : \alpha > 0] \subset \mathcal{G} \cup \{0\}$.

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Proposition

$\mathcal{G}$ *is spaceable.*

(Sketch of the proof:) Let $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ be given by $f(z,w) = (e^w, e^z)$ and consider the set of classes $\mathcal{C} = \{\widehat{f_\alpha} : \alpha > 0\}$ contained in the quotient space $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)/\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$.

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Suppose that $\sum_{k=1}^{n} \beta_k \widehat{f_{\alpha_k}} = \widehat{0}$, where $\beta_k \in \mathbb{C}$, for $k = 1, \cdots, n$.

This implies that $g = \sum_{k=1}^{n} \beta_k f_{\alpha_k} \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ and $g \equiv 0$.

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Applying Proposition 0.6 we have that the family $\mathcal{C}$ is linearly independent, so $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)/\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is infinite dimensional.

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Since $(\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2), \tau_b)$ is closed in $(\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2), \tau_b)$ ([17, Proposition 2.4] and $\tau_0 = \tau_b$ in $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$, it follows by Kitson-Timoney Theorem that $\mathcal{G}$ is spaceable.

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for some $m \in \mathbb{N}$, some $a_1, \cdots, a_m \in \mathbb{C} \setminus \{0\}$ and some distincts $b_1, \cdots, b_m \in \mathbb{C} \setminus \{0\}$.

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Hoppe´s Formula:

$$(g \circ f)^{(n)}(z) = \sum_{r=1}^{n} \frac{g^{(r)}(f(z))}{r!} \sum_{s=0}^{r} \binom{r}{s} (-f(z))^{r-s} (f^s)^{(n)}(z).$$

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We will denote by $\mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$ the set of all two-variable exponential like functions $\varphi : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$.

In [16], we proved a two-variable version of the Criterion for algebrability, and then using Proposition below, we also get the strong algebrability of $\mathcal{G}$.

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Proposition (M. L. Lourenço, D. M. V., 2019)

For each $\varphi \in \mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$, *then* $\varphi \circ F \in \mathcal{G}$, *where* $F(z, w) = (w, z)$, *for all* $(z, w) \in \mathbb{C}^2$.

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Regarding **Residuality**, we show that $\mathcal{H}$ is residual in $\mathcal{D}^\infty$ but $\mathcal{G}$ is not residual in $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$.

THANK YOU!

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