Strong Pseudoconvexity in Banach Spaces

Sofia Ortega Castillo

CIMAT
Guanajuato, Gto., Mexico

AGA
Trinity College Dublin

9th of May 2019

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

A domain Ω in a Banach space X is called a domain of holomorphy if its boundary consists of singular points for $\mathscr{H}(\Omega)$.

Example

The following are domains of holomorphy:

- Any domain in the complex plane
- Domains of convergence of multivariable power series
- Any convex domain in a Banach space

A counterexample: Hartogs' domain.

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0,\ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}$.

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0,\ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0, \ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}.$

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0, \ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0,\ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}.$

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0,\ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0,\ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}$.

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0,\ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0,\ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}.$

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p)t_j\overline{t_k} \geq 0,\ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Domains of holomorphy

Are there commonalities in the complex geometry of domains of holomorphy?

Remark

If $\Omega \subset \mathbb{C}^n$ is a convex domain with C^2 boundary, and if r is a defining function of $b\Omega$, then for all $p \in b\Omega$,
$\sum_{j,k=1}^{2n} \frac{\partial^2 r}{\partial x_j \partial x_k}(p)\zeta_j\zeta_k \geq 0,\ \forall \zeta \in T_p(b\Omega) := \{\xi \in \mathbb{R}^{2n} : \sum_{j=1}^{2n} \frac{\partial r}{\partial x_j}(p)\xi_j = 0\}$.

Proposition (Levi, 1910-1911)

Every domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is Levi pseudoconvex i.e. when r is a defining function of $b\Omega$, for all $p \in b\Omega$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0,\ \forall t \in T_p^{\mathbb{C}}(b\Omega) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}. \quad (1)$$

Strong pseudoconvexity

A domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly (Levi) pseudoconvex when moreover we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} > 0, \; \forall t \neq 0 \in T_p^{\mathbb{C}}(bU) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}, \quad (2)$$

for r a defining function of $b\Omega$ and all $p \in b\Omega$.

Proposition (*HFIRSCV* of Range, 1986)

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is strongly pseudoconvex iff there exists a defining function of $b\Omega$, $r : U \supset b\Omega \to \mathbb{C}$, such that there exist constants $L, M > 0$ satisfying that for all $p \in U$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2$,
2. $|\nabla r(p)| \geq M$.

Strong pseudoconvexity

A domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly (Levi) pseudoconvex when moreover we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} > 0, \ \forall t \neq 0 \in T_p^{\mathbb{C}}(bU) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}, \quad (2)$$

for r a defining function of $b\Omega$ and all $p \in b\Omega$.

Proposition (*HFIRSCV* of Range, 1986)

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is strongly pseudoconvex iff there exists a defining function of $b\Omega$, $r : U \supset b\Omega \to \mathbb{C}$, such that there exist constants $L, M > 0$ satisfying that for all $p \in U$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L \|t\|^2$,
2. $|\nabla r(p)| \geq M$.

Strong pseudoconvexity

A domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly (Levi) pseudoconvex when moreover we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} > 0, \ \forall t \neq 0 \in T_p^{\mathbb{C}}(bU) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}, \quad (2)$$

for r a defining function of $b\Omega$ and all $p \in b\Omega$.

Proposition (*HFIRSCV* of Range, 1986)

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is strongly pseudoconvex iff there exists a defining function of $b\Omega$, $r : U \supset b\Omega \to \mathbb{C}$, such that there exist constants $L, M > 0$ satisfying that for all $p \in U$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2$,
2. $|\nabla r(p)| \geq M$.

Strong pseudoconvexity

A domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly (Levi) pseudoconvex when moreover we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} > 0, \; \forall t \neq 0 \in T_p^{\mathbb{C}}(bU) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}, \quad (2)$$

for r a defining function of $b\Omega$ and all $p \in b\Omega$.

Proposition (*HFIRSCV* of Range, 1986)

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is strongly pseudoconvex iff there exists a defining function of $b\Omega$, $r : U \supset b\Omega \to \mathbb{C}$, such that there exist constants $L, M > 0$ satisfying that for all $p \in U$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2$,
2. $|\nabla r(p)| \geq M$.

Strong pseudoconvexity

A domain of holomorphy $\Omega \subset \mathbb{C}^n$ with C^2 boundary is called strictly (Levi) pseudoconvex when moreover we have

$$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} > 0, \; \forall t \neq 0 \in T_p^{\mathbb{C}}(bU) := \{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial r}{\partial z_j}(p)\xi_j = 0\}, \quad (2)$$

for r a defining function of $b\Omega$ and all $p \in b\Omega$.

Proposition (*HFIRSCV* of Range, 1986)

A domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary is strongly pseudoconvex iff there exists a defining function of $b\Omega$, $r : U \supset b\Omega \to \mathbb{C}$, such that there exist constants $L, M > 0$ satisfying that for all $p \in U$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2$,
2. $|\nabla r(p)| \geq M$.

Plurisubharmonicity

The mentioned condition on the complex Hessian of a C^2 function $f : U \subset \mathbb{C}^n \to \mathbb{R}$ given by

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 \text{ for all } p \in U,\ t \in \mathbb{C}^n,$$

is referred to as f being a C^2 strongly plurisubharmonic function, since it is more than plurisubharmonic.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is called plurisubharmonic if f is upper semicontinuous and for each $a \in U$ and $b \in \mathbb{C}^n$ such that $a + \overline{\mathbb{D}} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

Remark

If $f \in C^2(U)$, f is plurisubharmonic iff $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$.

Plurisubharmonicity

The mentioned condition on the complex Hessian of a C^2 function $f : U \subset \mathbb{C}^n \to \mathbb{R}$ given by

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 \text{ for all } p \in U,\ t \in \mathbb{C}^n,$$

is referred to as f being a C^2 strongly plurisubharmonic function, since it is more than plurisubharmonic.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is called plurisubharmonic if f is upper semicontinuous and for each $a \in U$ and $b \in \mathbb{C}^n$ such that $a + \overline{\mathbb{D}} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

Remark

If $f \in C^2(U)$, f is plurisubharmonic iff $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$.

Plurisubharmonicity

The mentioned condition on the complex Hessian of a C^2 function $f : U \subset \mathbb{C}^n \to \mathbb{R}$ given by

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 \text{ for all } p \in U,\ t \in \mathbb{C}^n,$$

is referred to as f being a C^2 strongly plurisubharmonic function, since it is more than plurisubharmonic.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is called plurisubharmonic if f is upper semicontinuous and for each $a \in U$ and $b \in \mathbb{C}^n$ such that $a + \overline{\mathbb{D}} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

Remark

If $f \in C^2(U)$, f is plurisubharmonic iff $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$.

Plurisubharmonicity

The mentioned condition on the complex Hessian of a C^2 function $f : U \subset \mathbb{C}^n \to \mathbb{R}$ given by

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L \|t\|^2 \text{ for all } p \in U,\ t \in \mathbb{C}^n,$$

is referred to as f being a C^2 strongly plurisubharmonic function, since it is more than plurisubharmonic.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is called plurisubharmonic if f is upper semicontinuous and for each $a \in U$ and $b \in \mathbb{C}^n$ such that $a + \overline{\mathbb{D}} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

Remark

If $f \in C^2(U)$, f is plurisubharmonic iff $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$.

Plurisubharmonicity

The mentioned condition on the complex Hessian of a C^2 function $f : U \subset \mathbb{C}^n \to \mathbb{R}$ given by

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L \|t\|^2 \text{ for all } p \in U,\ t \in \mathbb{C}^n,$$

is referred to as f being a C^2 strongly plurisubharmonic function, since it is more than plurisubharmonic.

A function $f : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is called plurisubharmonic if f is upper semicontinuous and for each $a \in U$ and $b \in \mathbb{C}^n$ such that $a + \overline{\mathbb{D}} \cdot b \subset U$,

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + e^{i\theta} b) d\theta.$$

Remark

If $f \in C^2(U)$, f is plurisubharmonic iff $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq 0$.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \text{ in } \mathscr{D}(U) \text{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Plurisubharmonicity

This has been generalized in the sense of distribution: Given $f \in L^1(U, loc)$, f is called plurisubharmonic in distribution if the distribution it induces is plurisubharmonic, while a distribution $T \in \mathscr{D}'(U)$ is called plurisubharmonic when

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq 0, \text{ for all } \phi \geq 0 \text{ in } \mathscr{D}(U) \text{ and } t \in \mathbb{C}^n.$$

Proposition (*Notions of Convexity* of Hörmander, 1994)

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is plurisubharmonic, then $f \in L^1(U, loc)$ and f is plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is plurisubharmonic then there exists $f \in L^1(U, loc)$ plurisubharmonic such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ plurisubharmonic such that $f = g$ λ-a.e.

Strict plurisubharmonicity

Let us say that a distribution $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic when there exists $\psi \in C^\infty(U)$ positive such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq \left(\int_U \psi \cdot \phi \, d\lambda\right) \|t\|^2, \text{ for all } \phi \geq 0 \text{ in } \mathscr{D}(U) \text{ and } t \in \mathbb{C}^n,$$

and in case ψ is a constant $M > 0$, call T strongly plurisubharmonic.

Given an upper semicontinuous function $g : U \subset X \to [-\infty, \infty)$, let us call it strictly plurisubharmonic on average when we can find a positive $\varphi \in C^\infty(U)$ such that for all $a \in U$ and $b \in X$ of small norm,

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta, \quad (3)$$

and in case φ is a constant $M > 0$, call g strongly plurisubharmonic on average.

Strict plurisubharmonicity

Let us say that a distribution $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic when there exists $\psi \in C^\infty(U)$ positive such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq \left(\int_U \psi \cdot \phi \, d\lambda\right) \|t\|^2, \text{ for all } \phi \geq 0 \text{ in } \mathscr{D}(U) \text{ and } t \in \mathbb{C}^n,$$

and in case ψ is a constant $M > 0$, call T strongly plurisubharmonic.

Given an upper semicontinuous function $g : U \subset X \to [-\infty, \infty)$, let us call it strictly plurisubharmonic on average when we can find a positive $\varphi \in C^\infty(U)$ such that for all $a \in U$ and $b \in X$ of small norm,

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta, \tag{3}$$

and in case φ is a constant $M > 0$, call g strongly plurisubharmonic on average.

Strict plurisubharmonicity

Let us say that a distribution $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic when there exists $\psi \in C^\infty(U)$ positive such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq \left(\int_U \psi \cdot \phi \, d\lambda\right) \|t\|^2, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n,$$

and in case ψ is a constant $M > 0$, call T strongly plurisubharmonic.

Given an upper semicontinuous function $g : U \subset X \to [-\infty, \infty)$, let us call it strictly plurisubharmonic on average when we can find a positive $\varphi \in C^\infty(U)$ such that for all $a \in U$ and $b \in X$ of small norm,

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta, \tag{3}$$

and in case φ is a constant $M > 0$, call g strongly plurisubharmonic on average.

Strict plurisubharmonicity

Let us say that a distribution $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic when there exists $\psi \in C^\infty(U)$ positive such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq \left(\int_U \psi \cdot \phi \, d\lambda\right) \|t\|^2, \text{ for all } \phi \geq 0 \text{ in } \mathscr{D}(U) \text{ and } t \in \mathbb{C}^n,$$

and in case ψ is a constant $M > 0$, call T strongly plurisubharmonic.

Given an upper semicontinuous function $g : U \subset X \to [-\infty, \infty)$, let us call it strictly plurisubharmonic on average when we can find a positive $\varphi \in C^\infty(U)$ such that for all $a \in U$ and $b \in X$ of small norm,

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta, \tag{3}$$

and in case φ is a constant $M > 0$, call g strongly plurisubharmonic on average.

Strict plurisubharmonicity

Let us say that a distribution $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic when there exists $\psi \in C^\infty(U)$ positive such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 T}{\partial z_j \partial \overline{z_k}}(\phi) t_j \overline{t_k} \geq \left(\int_U \psi \cdot \phi \, d\lambda\right) \|t\|^2, \text{ for all } \phi \geq 0 \textit{ in } \mathscr{D}(U) \textit{ and } t \in \mathbb{C}^n,$$

and in case ψ is a constant $M > 0$, call T strongly plurisubharmonic.

Given an upper semicontinuous function $g : U \subset X \to [-\infty, \infty)$, let us call it strictly plurisubharmonic on average when we can find a positive $\varphi \in C^\infty(U)$ such that for all $a \in U$ and $b \in X$ of small norm,

$$\varphi(a)\|b\|^2 + g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta} b) d\theta, \tag{3}$$

and in case φ is a constant $M > 0$, call g strongly plurisubharmonic on average.

Strict plurisubharmonicity

Proposition

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is strictly plurisubharmonic on average, then f is strictly plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic then there exists $f \in L^1(U, loc)$ strictly plurisubharmonic on average such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is strictly plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ strictly plurisubharmonic on average such that $f = g$ λ-a.e.

Remark

In finite dimension, strict plurisubharmonicity of a C^2 function $F : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is known to being equivalent to having, for each $a \in U$, a nonempty ball $B(a, r) \subset U$ and a constant $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic on $B(a, r)$, but this is no longer true in infinite dimension.

Strict plurisubharmonicity

Proposition

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is strictly plurisubharmonic on average, then f is strictly plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic then there exists $f \in L^1(U, loc)$ strictly plurisubharmonic on average such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is strictly plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ strictly plurisubharmonic on average such that $f = g$ λ-a.e.

Remark

In finite dimension, strict plurisubharmonicity of a C^2 function $F : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is known to being equivalent to having, for each $a \in U$, a nonempty ball $B(a, r) \subset U$ and a constant $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic on $B(a, r)$, but this is no longer true in infinite dimension.

Strict plurisubharmonicity

Proposition

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is strictly plurisubharmonic on average, then f is strictly plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic then there exists $f \in L^1(U, loc)$ strictly plurisubharmonic on average such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is strictly plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ strictly plurisubharmonic on average such that $f = g$ λ-a.e.

Remark

In finite dimension, strict plurisubharmonicity of a C^2 function $F : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is known to being equivalent to having, for each $a \in U$, a nonempty ball $B(a, r) \subset U$ and a constant $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic on $B(a, r)$, but this is no longer true in infinite dimension.

Strict plurisubharmonicity

Proposition

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is strictly plurisubharmonic on average, then f is strictly plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic then there exists $f \in L^1(U, loc)$ strictly plurisubharmonic on average such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is strictly plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ strictly plurisubharmonic on average such that $f = g$ λ-a.e.

Remark

In finite dimension, strict plurisubharmonicity of a C^2 function $F : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is known to being equivalent to having, for each $a \in U$, a nonempty ball $B(a,r) \subset U$ and a constant $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic on $B(a,r)$, but this is no longer true in infinite dimension.

Strict plurisubharmonicity

Proposition

Suppose that U is a connected domain in $\mathbb{C}^n$. If $f \neq -\infty$ is strictly plurisubharmonic on average, then f is strictly plurisubharmonic in distribution. Conversely, if $T \in \mathscr{D}'(U)$ is strictly plurisubharmonic then there exists $f \in L^1(U, loc)$ strictly plurisubharmonic on average such that f induces the distribution T. As a corollary, if $f \in L^1(U, loc)$ is strictly plurisubharmonic in distribution then there exists $g \in L^1(U, loc)$ strictly plurisubharmonic on average such that $f = g$ λ-a.e.

Remark

In finite dimension, strict plurisubharmonicity of a C^2 function $F : U \subset \mathbb{C}^n \to [-\infty, \infty)$ is known to being equivalent to having, for each $a \in U$, a nonempty ball $B(a,r) \subset U$ and a constant $\varepsilon > 0$ such that $F - \varepsilon\|\cdot\|^2$ is plurisubharmonic on $B(a,r)$, but this is no longer true in infinite dimension.

Strict plurisubharmonicity

Example (O.-C. and Santillán Zerón, 2018)

Consider the space of complex sequences $(z_n) \subset \mathbb{C}^{\mathbb{N}}$ such that

$$\|(z_n)\| = \sqrt{\sum_{k=1}^{\infty} |z_k|^2/k^2} < \infty,$$

which is a Hilbert space, call it X.

Consider the function on X, $F(z) = \sum_{k=1}^{\infty} |z_k|^2/k^3$, which has positive definite complex Hessian

$$D'D''F(z)(w,w) = \sum_{k=1}^{\infty} |w_k|^2/k^3.$$

Strict plurisubharmonicity

Example (O.-C. and Santillán Zerón, 2018)

Consider the space of complex sequences $(z_n) \subset \mathbb{C}^{\mathbb{N}}$ such that

$$\|(z_n)\| = \sqrt{\sum_{k=1}^{\infty} |z_k|^2/k^2} < \infty,$$

which is a Hilbert space, call it X.

Consider the function on X, $F(z) = \sum_{k=1}^{\infty} |z_k|^2/k^3$, which has positive definite complex Hessian

$$D'D''F(z)(w,w) = \sum_{k=1}^{\infty} |w_k|^2/k^3.$$

Strict plurisubharmonicity

Example (O.-C. and Santillán Zerón, 2018)

Consider the space of complex sequences $(z_n) \subset \mathbb{C}^{\mathbb{N}}$ such that

$$\|(z_n)\| = \sqrt{\sum_{k=1}^{\infty} |z_k|^2/k^2} < \infty,$$

which is a Hilbert space, call it X.

Consider the function on X, $F(z) = \sum_{k=1}^{\infty} |z_k|^2/k^3$, which has positive definite complex Hessian

$$D'D''F(z)(w,w) = \sum_{k=1}^{\infty} |w_k|^2/k^3.$$

Strict plurisubharmonicity

Example (O.-C. and Santillán Zerón, 2018)

However, this function $F(z) = \sum_{k=1}^{\infty} |z_k|^2/k^3$ does not admit $\varepsilon > 0$ such that, for z near to z_0,

$$F(z) - \varepsilon\|z\|^2 = \sum_{k=1}^{\infty} |z_k|^2/k^3 - \varepsilon \sum_{k=1}^{\infty} |z_k|^2/k^2 = \sum_{k=1}^{\infty} |z_k|^2(1/k^3 - \varepsilon/k^2)$$

is plurisubharmonic, since its complex Hessian at z_0,

$$D'D''(F - \varepsilon\|\cdot\|^2)(z_0)(w,w) = \sum_{k=1}^{\infty} |w_k|^2(1/k^3 - \varepsilon/k^2),$$

eventually has eigenvalues which are negative.

Strict plurisubharmonicity

Example (O.-C. and Santillán Zerón, 2018)

However, this function $F(z) = \sum_{k=1}^{\infty} |z_k|^2/k^3$ does not admit $\varepsilon > 0$ such that, for z near to z_0,

$$F(z) - \varepsilon \|z\|^2 = \sum_{k=1}^{\infty} |z_k|^2/k^3 - \varepsilon \sum_{k=1}^{\infty} |z_k|^2/k^2 = \sum_{k=1}^{\infty} |z_k|^2(1/k^3 - \varepsilon/k^2)$$

is plurisubharmonic, since its complex Hessian at z_0,

$$D'D''(F - \varepsilon\|\cdot\|^2)(z_0)(w,w) = \sum_{k=1}^{\infty} |w_k|^2(1/k^3 - \varepsilon/k^2),$$

eventually has eigenvalues which are negative.

Strict plurisubharmonicity

Proposition

If U is an open domain in a Banach space X, a function $f \in C^2(U,\mathbb{R})$ is strictly plurisubharmonic iff it is strictly plurisubharmonic on average.

Remark

A function $F: U \subset X \to [-\infty, \infty)$ that locally admits an $\varepsilon > 0$ such that $F - \varepsilon\|\cdot\|^2$ is plurisubharmonic may not be strictly plurisubharmonic on average, since the squared norm is plurisubharmonic but may not be strictly plurisubhamonic on average. This is the case in $X = \ell_\infty$.

Remark

Strictly plurisubharmonic functions on average without two degrees of differentiability exist in infinite-dimensional ambient spaces, such as the norm of ℓ_1: $\|(z_n)\|_1 = \sum_{n=1}^{\infty} |z_n|$.

Strict plurisubharmonicity

Proposition

If U is an open domain in a Banach space X, a function $f \in C^2(U, \mathbb{R})$ is strictly plurisubharmonic iff it is strictly plurisubharmonic on average.

Remark

A function $F : U \subset X \to [-\infty, \infty)$ that locally admits an $\varepsilon > 0$ such that $F - \varepsilon \| \cdot \|^2$ is plurisubharmonic may not be strictly plurisubharmonic on average, since the squared norm is plurisubharmonic but may not be strictly plurisubhamonic on average. This is the case in $X = \ell_\infty$.

Remark

Strictly plurisubharmonic functions on average without two degrees of differentiability exist in infinite-dimensional ambient spaces, such as the norm of ℓ_1: $\|(z_n)\|_1 = \sum_{n=1}^{\infty} |z_n|$.

Strict plurisubharmonicity

Proposition

If U is an open domain in a Banach space X, a function $f \in C^2(U, \mathbb{R})$ is strictly plurisubharmonic iff it is strictly plurisubharmonic on average.

Remark

A function $F : U \subset X \to [-\infty, \infty)$ that locally admits an $\varepsilon > 0$ such that $F - \varepsilon \| \cdot \|^2$ is plurisubharmonic may not be strictly plurisubharmonic on average, since the squared norm is plurisubharmonic but may not be strictly plurisubhamonic on average. This is the case in $X = \ell_\infty$.

Remark

Strictly plurisubharmonic functions on average without two degrees of differentiability exist in infinite-dimensional ambient spaces, such as the norm of ℓ_1: $\|(z_n)\|_1 = \sum_{n=1}^{\infty} |z_n|$.

Strict plurisubharmonicity

Proposition

If U is an open domain in a Banach space X, a function $f \in C^2(U, \mathbb{R})$ is strictly plurisubharmonic iff it is strictly plurisubharmonic on average.

Remark

A function $F : U \subset X \to [-\infty, \infty)$ that locally admits an $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic may not be strictly plurisubharmonic on average, since the squared norm is plurisubharmonic but may not be strictly plurisubhamonic on average. This is the case in $X = \ell_\infty$.

Remark

Strictly plurisubharmonic functions on average without two degrees of differentiability exist in infinite-dimensional ambient spaces, such as the norm of ℓ_1: $\|(z_n)\|_1 = \sum_{n=1}^{\infty} |z_n|$.

Strict plurisubharmonicity

Proposition

If U is an open domain in a Banach space X, a function $f \in C^2(U, \mathbb{R})$ is strictly plurisubharmonic iff it is strictly plurisubharmonic on average.

Remark

A function $F : U \subset X \to [-\infty, \infty)$ that locally admits an $\varepsilon > 0$ such that $F - \varepsilon \|\cdot\|^2$ is plurisubharmonic may not be strictly plurisubharmonic on average, since the squared norm is plurisubharmonic but may not be strictly plurisubhamonic on average. This is the case in $X = \ell_\infty$.

Remark

Strictly plurisubharmonic functions on average without two degrees of differentiability exist in infinite-dimensional ambient spaces, such as the norm of ℓ_1: $\|(z_n)\|_1 = \sum_{n=1}^{\infty} |z_n|$.

Strict plurisubharmonicity

Example

Spaces whose norm is strictly plurisubharmonic on average include the 2-uniformly PL-convex spaces, where a Banach space X is 2-uniformly PL-convex if for some $0 < q < \infty$ there exists $\lambda > 0$ such that,

$$(\frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta}b\|^q d\theta)^{1/q} \geq (\|a\|^2 + \lambda\|b\|^2)^{1/2}$$

for all a and b in X.

Davis, Garling and Tomczak-Jaegermann proved in 1984 that for $p \in [1,2]$, $L_p(\Sigma, \Omega, \mu)$ is 2-uniformly PL-convex.

Strict plurisubharmonicity

Example

Spaces whose norm is strictly plurisubharmonic on average include the 2-uniformly PL-convex spaces, where a Banach space X is 2-uniformly PL-convex if for some $0 < q < \infty$ there exists $\lambda > 0$ such that,

$$(\frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta} b\|^q d\theta)^{1/q} \geq (\|a\|^2 + \lambda \|b\|^2)^{1/2}$$

for all a and b in X.

Davis, Garling and Tomczak-Jaegermann proved in 1984 that for $p \in [1,2]$, $L_p(\Sigma, \Omega, \mu)$ is 2-uniformly PL-convex.

Strict plurisubharmonicity

Example

Spaces whose norm is strictly plurisubharmonic on average include the 2-uniformly PL-convex spaces, where a Banach space X is 2-uniformly PL-convex if for some $0 < q < \infty$ there exists $\lambda > 0$ such that,

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \|a + e^{i\theta} b\|^q d\theta\right)^{1/q} \geq \left(\|a\|^2 + \lambda \|b\|^2\right)^{1/2}$$

for all a and b in X.

Davis, Garling and Tomczak-Jaegermann proved in 1984 that for $p \in [1,2]$, $L_p(\Sigma, \Omega, \mu)$ is 2-uniformly PL-convex.

Strong pseudoconvexity

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|}\|b\|^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L\|b\|^2$$

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|}||b||^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L||b||^2$$

Strong pseudoconvexity

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^n \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|}||b||^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L||b||^2$$

Strong pseudoconvexity

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|} ||b||^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L||b||^2$$

Strong pseudoconvexity

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|}||b||^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L||b||^2$$

Strong pseudoconvexity

Theorem

Let U be an open domain in $\mathbb{C}^n$ with C^2 boundary. Then U is strongly pseudoconvex if and only if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^2(V)$ a defining function of bU such that,

- $\sum_{j,k=1}^n \frac{\partial^2(-\log|\rho|)}{\partial z_j \partial \overline{z_k}}(a) b_j \overline{b_k} \geq \frac{L}{|\rho(a)|}\|b\|^2$ *for all* $a \in U \cap V$ *and* $b \in \mathbb{C}^n$.
- $|\nabla \rho(a)| \geq M$ *for all* $a \in U \cap V$.

If $U \subset \mathbb{C}^n$ is an open domain with C^1 boundary, let us call U strongly pseudoconvex if there exist a positive constant L, a neighborhood V of bU and $\rho \in C^1(V)$ a defining function of bU such that for $a \in U \cap V$ and $b \in \mathbb{C}^n$ of small norm,

$$\int_0^{2\pi} -\log|\rho|(a + e^{i\theta} b) d\theta \geq -\log|\rho|(a) + L\|b\|^2$$

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0,$
2. $|\nabla r_n(p)| \geq M > 0.$

The ball of ℓ_1^n is strictly pseudoconvex.

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Strong pseudoconvexity

Proposition

A domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary is strongly pseudoconvex when there exists a C^1 defining function of $b\Omega$ which is strongly plurisubharmonic on average.

For $p \in (1,2]$, the space $L_p(\Sigma, \Omega, \mu)$ has strongly pseudoconvex unit ball.

Let us call a bounded domain in $\mathbb{C}^n$ strongly pseudoconvex when it is exhausted by strongly pseudoconvex domains Ω_n with C^1 boundary having defining functions r_n of $b\Omega_n$, i.e. $r_n : U \supset b\Omega \to \mathbb{R}$ for which there exist constants $L, M > 0$ satisfying that for all $p \in U \cap \Omega_n$ and $t \in \mathbb{C}^n$,

1. $\sum_{j,k=1}^{n} \frac{\partial^2(-\log|r_n|)}{\partial z_j \partial \overline{z_k}}(p) t_j \overline{t_k} \geq L\|t\|^2 > 0$,
2. $|\nabla r_n(p)| \geq M > 0$.

The ball of ℓ_1^n is strictly pseudoconvex.

Summary

Table: Plurisubharmonicity notions

Concept	Conditions	Equation	Examples
Plurisubharmonic $f : U \to [-\infty, \infty)$	upper semicont.	$f(a) \leq \frac{1}{2\pi}\int_0^{2\pi} f(a + e^{i\theta}b)d\theta$ $a \in U$, b arbitrary	Re(F), Im(F), $(\|F\|_p)^m$, $\log(\|F\|_p)$ for F holomorphic and $1 \leq p \leq \infty$
C^2 plurisubharmonic f on U	C^2 smooth	$\sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(a) b_j \bar{b}_k \geq 0$ $a \in U$, b arbitrary	Re(F), Im(F), $(\|F\|_p)^m$, $\log(\|F\|_p)$ for F holomorphic and $1 < p < \infty$
Strictly p.s.h. on av. $f : U \to [-\infty, \infty)$	f upper semicont.	$\varphi(a)\|b\|^2 + f(a) \leq \frac{1}{2\pi}\int_0^{2\pi} f(a + e^{i\theta})d\theta$ $\varphi > 0 \in C^\infty(U)$, $a \in U$, $\|b\|$ small	$\|\cdot\|_p$ for $1 \leq p \leq 2$, $\sum_{k=1}^\infty \|z_k\|^2/k^3$, $\sum_{k=1}^\infty \|z_k\|^2/k^2 < \infty$
C^2 strictly p.s.h. f on U	C^2 smooth	$\sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(a) b_j \bar{b}_k \geq \varphi(a)\|b\|^2$ $\varphi > 0 \in C^\infty(U)$, $a \in U$, b any	$\|\cdot\|_p$ for $p = 2$ outside $\{0\}$, $\sum_{k=1}^\infty \|z_k\|^2/k^3$, $\sum_{k=1}^\infty \|z_k\|^2/k^2 < \infty$
Strongly p.s.h. on av.	upper semicont.	$L\|b\|^2 + f(a) \leq \frac{1}{2\pi}\int_0^{2\pi} f(a + e^{i\theta})d\theta$	$\|\cdot\|_p$ for $1 \leq p \leq 2$
C^2 strongly p.s.h.	C^2 smooth	$\sum_{j,k=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_k}(a) b_j \bar{b}_k \geq L\|b\|^2$	$\|\cdot\|_p$ for $p = 2$ outside $\{0\}$

Summary

Table: Pseudoconvexity in $\mathbb{C}^n$

Concept	Conditions	Equation	Examples
Ψ-convex domain	inc. union of C^2 Ψ-convex domains	$\sum_{j,k=1}^{n} \frac{\partial^2 r_n}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k \geq 0$ $\zeta \in T_p^{\mathbb{C}}(b\Omega_n), p \in b\Omega_n$	convex domains & domains of convergence
C^2 Ψ-convex domain	C^2 smooth boundary def. by r	$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k \geq 0$ $\zeta \in T_p^{\mathbb{C}}(b\Omega)$, $p \in b\Omega$	$B_{\ell_p^n}$, $1 < p < \infty$; $\{z \in \mathbb{C}^2 : \lvert z_1 z_2\rvert < 1\}$; $\{z \in \mathbb{C} : \text{Im} z < (\text{Re} z)^2\}$
Strongly Ψ-convex	inc. union of C^2 str. Ψ-convex w. same bounds$^+$	$\sum_{j,k=1}^{n} \frac{\partial^2 r_n}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k > L\lVert\zeta\rVert^2$ & $\lvert\nabla r_n(p)\rvert \geq M$; $p \in U \supset b\Omega$	$B_{\ell_p^n}$, $1 \leq p \leq 2$
C^2 strictly Ψ-convex	C^2 smooth boundary def. by r	$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k > 0$ $\zeta \in T_p^{\mathbb{C}}(b\Omega)$, $p \in b\Omega$	$B_{\ell_p}^n$, $p = 2$
C^2 strongly Ψ-convex	C^2 smooth boundary def. by r	$\sum_{j,k=1}^{n} \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k}(p)\zeta_j\bar{\zeta}_k > L\lVert\zeta\rVert^2$ & $\lvert\nabla r(p)\rvert \geq M$; $p \in U \supset b\Omega$	$B_{\ell_p^n}$, $p = 2$

Bibliography

- M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer Verlag, Berlin (1986).
- W. J. Davis, D. J. H. Garling, N. Tomczak-Jaegermann, *The Complex Convexity of Quasi-normed Linear Spaces*, Journal of Functional Analysis 55 (1984), pp. 110-150.
- L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, 3rd edition, North Holland, Amsterdam, 1990.
- L. Hörmander, *Notions of Convexity*, Birkhaüser Basel, Boston, 1994.

THANK YOU!