Banach Lattices in Infinite Dimensional Complex Analysis

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1. Holomorphy — some history

- Hilbert (1909): Holomorphic functions on $\mathbb{C}^{\mathbb{N}}$ defined locally by **monomial expansions**:

$$f(z) = \sum_{\alpha} c_{\alpha}(z-a)^{\alpha}$$

$\alpha = (\alpha_j)$: a multi-index.
The monomial $z^{\alpha} = \prod_j z_j^{\alpha_j}$ has degree $|\alpha| = \sum_j \alpha_j$

- Fréchet, Gâteaux, Michael, Taylor,...: local representation by power series of homogeneous polynomials:

$$f(z) = \sum_{n} P_n(z-a)$$

$P_n(z) = \sum_{|\alpha|=n} z^{\alpha}$: (bounded) n-homogeneous polynomials.

Equivalent to Fréchet differentiability.

- Grothendieck, Nachbin, Gupta (1950's and 60's): Duality in terms of **nuclear functions**/tensor products:

$$\mathcal{P}(^n E') = \Big(\mathcal{P}_N(^n E)\Big)' \quad \text{(subject to AP)}$$

- Boland, Dineen (1970's): Holomorphic functions on **nuclear locally convex spaces**. For suitable nuclear spaces with basis, the monomials are a basis for the space of holomorphic functions.
- Matos, Nachbin (1980's and 90's): Monomial expansions for holomorphic functions on a Banach space with unconditional basis.
- Defant, Díaz, García, Kalton, Maestre (2001 and 2005) proved Dineen's conjecture: if E is a Banach space and $n \geqslant 2$, then $\mathcal{P}(^n E)$ has an unconditional basis if and only if E is finite dimensional.

2. Grothendieck and Fremlin

Grothendieck's projective tensor product linearizes bounded bilinear forms:

$$\mathcal{B}(E,F) \cong \left(E\widehat{\otimes}_\pi F\right)'$$

RR extended this to polynomials:

$$\mathcal{P}(^n E) \cong \big(\widehat{\bigotimes}_{n,\pi,s} E\big)'$$

A bilinear form B on a product $E \times F$ of (real) Banach lattices is **positive** if $B(x, y) \geqslant 0$ for all positive x, y and **regular** if it the difference of two positive bilinear forms.

Fremlin constructed a tensor product $\mathcal{B}_r(E, F)$ that linearizes regular bilinear forms:

$$\mathcal{B}_r(E, F) \cong \left(E \widehat{\otimes}_{|\pi|} F\right)'$$

John Loane (2007) extended this to polynomials:

$$\mathcal{P}_r(^n E) \cong \Big(\widehat{\bigotimes}_{n,|\pi|,s} E\Big)'$$

3. The Matos-Nachbin approach

E: a Banach space with unconditional Schauder basis (e_j).

Every $P \in \mathcal{P}(^n E)$ has a monomial expansion:

$$P(z) = A(z, \dots, z) = \sum_{|\alpha|=n} c_\alpha z^\alpha$$

But this expansion is only conditionally convergent in general.

$\mathcal{P}_v(^n E)$: the subspace of polynomials for which the monomial expansion is unconditionally convergent at every point.

If $P \in \mathcal{P}_\nu({}^n E)$, then

$$\tilde{P}(z) := \sum_{|\alpha|=n} |c_\alpha| z^\alpha$$

also belongs to $\mathcal{P}_\nu({}^n E)$. A norm is defined on $\mathcal{P}_\nu({}^n E)$ by

$$\nu(P) := \|\tilde{P}\| = \sup\Big\{ \Big| \sum |c_\alpha| z^\alpha \Big| : \|z\| \leqslant 1 \Big\}$$

$\mathcal{P}_\nu({}^n E)$ is a Banach space with this norm.

$$\mathcal{P}_N({}^n E) \subset \mathcal{P}_\nu({}^n E) \subset \mathcal{P}({}^n E)$$

Grecu–RR (2004):

$$\mathcal{P}_\nu({}^n E) = \mathcal{P}_r({}^n E)$$

with equality of norms.

. 4. Fremlin's Theorem

Fremlin (1972): Let K be a compact, Hausdorff space. Every positive bilinear form B on $C(K) \times C(K)$ is integral:

$$B(x, y) = \int_{K \times K} x(s)y(t)\, d\mu(s, t)$$

where μ is a regular Borel measure on $K \times K$.

1. Connects nuclearity and regularity.
2. $C(K)$ spaces are building blocks for Banach lattices.

5. Orthogonally Additivity

A function $f\colon E \to \mathbb{R}$ is **orthogonally additive** if

$$f(x+y) = f(x) + f(y)$$

whenever x, y are disjoint.

Sundaresan (1991): Let $1 \leqslant p < \infty$.

(a) If $1 \leqslant n \leqslant p$, then an n-homogeneous polynomial P on $L_p[0,1]$ is orthogonally additive if and only if there is a (unique) $\xi \in L_r[0,1]$, where $r = p/(p-n)$ ($r = \infty$ if $n = p$), such that

$$P(x) = \int x^n \xi \, d\mu$$

where μ is Lebesgue measure.

(b) If $n > p$, then there are no non-zero orthogonally additive n-homogeneous polynomials on $L_p[0,1]$.

Bu–Buskes (2012):

The vector space of orthogonally additive n-homogeneous polynomials on E can be identified with

(a) the dual Banach space $\left[\left(\bigotimes_{n,s,\pi} E\right)/J\right]'$,

where J is the closed subspace generated by $\{x_1 \otimes \cdots \otimes x_n : x_i \perp x_j$ for some $i, j\}$

or

(b) the dual Banach lattice $\left[\left(\bigotimes_{n,s,|\pi|} E\right)/I\right]'$,

where I is the closed ideal generated by $\{x_1 \otimes \cdots \otimes x_n : x_i \perp x_j$ for some $i, j\}$

Geometry of the space of orthogonally additive polynomials

(a) Carando–Lassalle–Zalduendo (2006):
The extreme points of the closed unit ball of the space $\big(\mathcal{P}_o({}^n C(K)), \|\cdot\|_r\big)$ are the n-homogeneous polynomials $\pm\delta_t^n$, where $t \in K$ and

$$\delta_t^n(x) = x(t)^n\,.$$

(b) Boyd–RR–Snigireva (2018):

(i) n odd: the extreme points of the closed unit ball of $\big(\mathcal{P}_o({}^n C(K)), \|\cdot\|_\infty\big)$ are the polynomials $P = \pm\delta_t^n$.

(ii) n even: the extreme points of the closed unit ball of $\big(\mathcal{P}_o({}^n C(K)), \|\cdot\|_\infty\big)$ are of two types:

- $P = \pm\delta_t^n$ for some $t \in K$;
- $P = \delta_s^n - \delta_t^n$, where s, t are distinct points in K, and

$$(\delta_s^n - \delta_t^n)(x) = x(s)^n - x(t)^n\,.$$