The Pedersen Rigidity Problem

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How do we get the action back?

Given an action α of a locally compact group G on a C^*-algebra A, our first reaction is to form the crossed product C^*-algebra $A \rtimes_\alpha G$.

Question

How do we recover the action from the crossed-product?

Short answer

We can't.

Example

If G acts on $C_0(G)$ by translation, then $C_0(G) \rtimes G \simeq \mathcal{K}(L^2(G))$.
But also
$(C_0(G) \otimes \mathcal{K}(L^2(G))) \rtimes G \simeq \mathcal{K}(L^2(G)) \otimes \mathcal{K}(L^2(G)) \simeq \mathcal{K}(L^2(G))$,
where G acts trivially on $\mathcal{K}$.

Now, $C_0(G) \otimes \mathcal{K}$ is not isomorphic to $C_0(G)$, but they are at least Morita equivalent.

Even worse

Example (KOQ)

Examples arising from number theory (and which began with Cuntz) give nonisomorphic commutative C^*-algebras carrying actions of a discrete abelian group G with isomorphic crossed products. Note that by commutativity, A and B are not Morita equivalent.

So, there's no hope of recovering (A, α) just from $A \rtimes G$, even up to Morita equivalence.

The dual action

Standing assumption: G is abelian.

Then there is a dual action $\widehat{\alpha}$ of the dual group $\widehat{G}$ on $A \rtimes_\alpha G$.

We assume that we know the group G, the crossed product $A \rtimes G$, and the dual action $\widehat{\alpha}$.

We want to know what other information we need to recover the C^*-algebra A and the action $\alpha : G \to \operatorname{Aut} A$, at least in some sense.

Can be generalized

A lot of this can be done with nonabelian groups, but that's a

Takesaki-Takai duality

Theorem (Takesaki-Takai)

$A \rtimes_{\alpha} G \rtimes_{\widehat{\alpha}} \widehat{G} \simeq A \otimes \mathcal{K}(L^2(G))$.

Moreover, $\widehat{\widehat{\alpha}}$ *corresponds to* $\alpha \otimes \operatorname{Ad}\rho$.

Here ρ is the (right, but who cares since G is abelian) regular representation of G.

This recovers (A, G, α) from the dual action $(A \rtimes_{\alpha} G, \widehat{\alpha})$ up to Morita equivalence.

To recover more, we need to keep track of more information about the the crossed product.

More data

The crossed product $A \rtimes_\alpha G$ is generated by a universal covariant representation

$$(i_A, i_G) : (A, G) \to M(A \rtimes G)$$

The extra data we consider involves i_A, i_G.

Landstad duality

Landstad asks for one more piece of information: i_G.

Landstad constructs (what we now call) the *generalized fixed-point algebra* $\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G)$, which is a C^*-subalgebra of $M(A \rtimes_\alpha G)$.

Theorem (Landstad)

The original action (A, α) is isomorphic to $(\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G), \mathrm{Ad}\, i_G)$.

Indeed, $i_A : A \to M(A \rtimes_\alpha G)$ gives the isomorphism.

What's the underlying structure?

Definition

An *equivariant action* of $\widehat{G}$ is a triple (C, γ, v), where (C, γ) is an action of $\widehat{G}$ and $v : G \to M(C)$ is a (fine print: strictly continuous) unitary homomorphism such that $\gamma_\chi(v_s) = \chi(s)v_s$ for $\chi \in \widehat{G}$ and $s \in G$.

Thus $(A \rtimes_\alpha G, \widehat{\alpha}, i_G)$ is an equivariant action.

Landstad duality says that the generalized fixed-point algebra $A := \mathrm{Fix}(C, \gamma, v)$ is a C^*-subalgebra of $M(C)$, $\mathrm{Ad}\, v$ gives an action α of G on A, and $(A \rtimes_\alpha G, \widehat{\alpha}, i_G) \simeq (C, \gamma, v)$.

Explore the middle

Question

How much does the generalized fixed-point algebra $\mathrm{Fix}(C, \gamma, v)$ *depend upon* v*?*

In other words, could we recover v *from the action* (C, γ) *if we also knew the generalized fixed-point algebra* $\mathrm{Fix}(C, \gamma, v)$*?*

Pedersen's theorem

Definition

A *cocycle* for an action (A, G, α) is a (fine print: strictly continuous) unitary map $u : G \to M(A)$ such that $u_{st} = u_s \alpha_s(u_t)$. Then $\operatorname{Ad} u \circ \alpha$ is another action on A, said to be *exterior equivalent* to α.

Theorem (Pedersen)

Two actions α and β of G on A are exterior equivalent if and only if there is an isomorphism $\theta : (A \rtimes_\alpha G, \widehat{\alpha}) \xrightarrow{\simeq} (A \rtimes_\beta G, \widehat{\beta})$ such that $\theta \circ i_A^\alpha = i_A^\beta$.

Escape from A

Definition

Actions (A, α) and (B, β) of G are *outer conjugate* if there is an action γ exterior equivalent to β such that $(A, \alpha) \simeq (B, \gamma)$.

Theorem (Pedersen (+ KOQ))

Two actions (A, α) and (B, β) of G are outer conjugate if and only if there is an isomorphism $(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$ taking $\mathrm{Fix}(A \rtimes_\alpha G, \widehat{\alpha}, i_G^\alpha)$ *to* $\mathrm{Fix}(B \rtimes_\beta G, \widehat{\beta}, i_G^\beta)$.

So, Pedersen says we can recover the action (A, α) of G up to outer conjugacy if we know the dual action (C, γ) of $\widehat{G}$ and the generalized fixed-point algebra $\mathrm{Fix}(C, \gamma, v)$, but perhaps not the homomorphism $v : G \to M(C)$ itself.

So the dual action really isn't enough?

If we knew anything about outer conjugacy, we should be able to find examples of the following: actions (A, α) and (B, β) of G such that

1. $(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$, but
2. α and β are not outer conjugate.

Equivalently, by Pedersen's theorem we want there to exist an isomorphism between the dual actions $\widehat{\alpha}$ and $\widehat{\beta}$, but not one that preserves the generalized fixed-point algebras.

But it's too hard...

Somehow irritating, we are frustrated by our inability to find any examples of this phenomenon.

No-go theorems

We know that there are no examples in any of the following cases:

1. G is discrete
2. A and B are stable
3. A and B are commutative
4. α or β is inner
5. G is compact, and α and β are faithful and ergodic
6. A and B are continuous trace and α and β are *locally unitary*

Question

Can No-Go Theorem 6 be extended to pointwise unitary actions?

Pedersen Rigidity Problem

Question

If (A, α) and (B, β) are actions of G such that $(A \rtimes_\alpha G, \widehat{\alpha}) \simeq (B \rtimes_\beta G, \widehat{\beta})$, are α and β outer conjugate?

We call this the “Pedersen Rigidity Problem”, because an affirmative answer would mean that Pedersen’s condition, namely that the isomorphism preserves the generalized fixed-point algebras, is superfluous. The No-Go Theorems are evidence hinting at an affirmative answer.