Boundary Value Problems

Anthony G. O'Farrell, Maynooth

In memoriam RMT

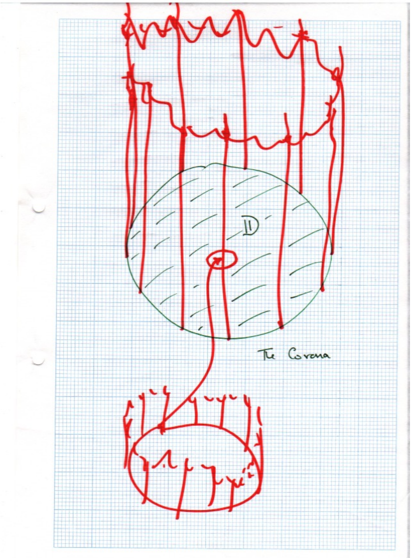
TCD, 8 May 2019


65
66

Theorem

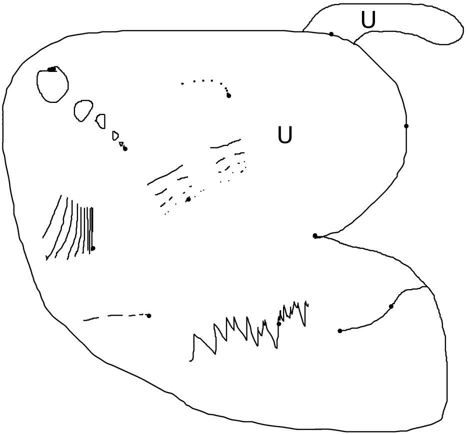
Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, and let $\check{f}$ be the induced self-map of the maximal ideal space M of $H^\infty(\mathbb{D})$. Then either f has a fixed point in M, or there is an analytic disk $D \subset M$ on which $\check{f}$ acts as a (hyperbolic) Möbius map.

(Dineen, Feinstein, O'F and Timoney. PRIA 94A (1994) 77-84.)

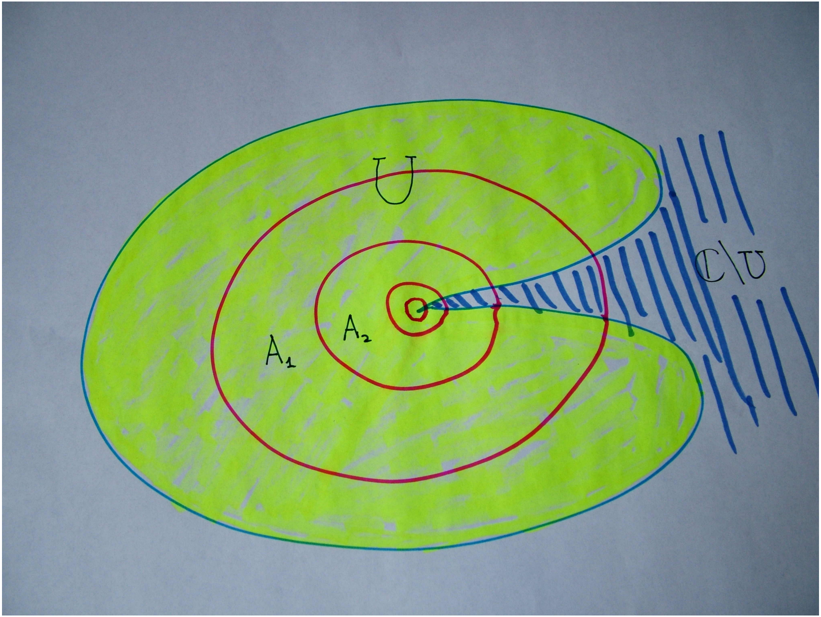

D
The Corona

An open set $U \subset \mathbb{C}$ and some boundary points:

B+W version:



Wiener series: $\sum_{n=1}^{\infty} 2^n c(A_n \setminus U)$



problem $\leftrightarrow$ capacity

Wiener (1924): Regular points for the Dirichlet problem on $\mathbb{R}^d$: $\leftrightarrow$ Newtonian capacity.
Mel'nikov (1966): $\leftrightarrow$ Peak points for $R(X)$: analytic capacity.
Peak points for $A(U)$: $\leftrightarrow$ continuous analytic capacity.
Gamelin and Garnett (1970): Bounded holomorphic functions on U: $\leftrightarrow$ analytic capacity.
Hedberg (1969): $R^p(X)$: $\leftrightarrow$ a condenser capacity.

Kinds of boundary value

- Concrete: limit taken in some set.
- Abstract: continuous linear functional on some Banach space $B \subset \mathcal{H}ol(U)$:

Abstract version should be sensible on some dense subset of B, and norm-continuous there, so that it has a unique extension to B.
Concept: continuous point evaluation.

Concept: continuous point derivation of order k.
Hallström (1969): $\exists$ on $R(X)$:

$$\sum_{n=1}^{\infty} 2^{(k+1)n}\gamma(A_n(b)\setminus X) < +\infty.$$

Hedberg (1972): $\exists$ on $R^p(X)$:

$$\sum_{n=1}^{\infty} 2^{(k+1)qn}\Gamma_q(A_n(b)\setminus X) < +\infty.$$

Lipschitz class holomorphic function spaces

Fix $0 < \alpha < 1$, $U \in \mathbb{C}$ open and consider

$$A^{\alpha}(U) := \{f \in \text{lip}(\alpha) : f \text{ is holo on } U\}.$$

Lipschitz class holomorphic function spaces

Fix $0 < \alpha < 1$, $U \in \mathbb{C}$ open and consider

$$A^{\alpha}(U) := \{f \in \mathrm{lip}(\alpha) : f \text{ is holo on } U\}.$$

Lemma

$\mathcal{A} := \{f \in A^{\alpha}(U) : f \text{ is holo near } a\}$ *is dense in A in Lipα norm.*

Lipschitz class holomorphic function spaces

Fix $0 < \alpha < 1$, $U \in \mathbb{C}$ open and consider

$$A^\alpha(U) := \{f \in \mathrm{lip}(\alpha) : f \text{ is holo on } U\}.$$

Lemma

$\mathcal{A} := \{f \in A^\alpha(U) : f$ is holo near $a\}$ *is dense in A in Lip*α *norm.*

Dolzhenko: The boundary point b is removable for $A^\alpha(U)$ if and only if $M_*^{1+\alpha}(B \sim U) = 0$ for some ball B about b.

Lipschitz class holomorphic function spaces

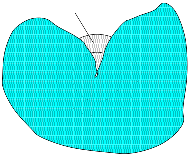
Fix $0 < \alpha < 1$, $U \in \mathbb{C}$ open and consider

$$A^\alpha(U) := \{f \in \mathrm{lip}(\alpha) : f \text{ is holo on } U\}.$$

Lemma

$\mathcal{A} := \{f \in A^\alpha(U) : f$ is holo near $a\}$ *is dense in* A *in Lip*α *norm.*

Dolzhenko: The boundary point b is removable for $A^\alpha(U)$ if and only if $M_*^{1+\alpha}(B \sim U) = 0$ for some ball B about b.

Lord-OF:  $A^\alpha(U)$ admits a continuous point derivation at b if and only if

$$\sum_{n=1}^{\infty} 4^n M_*^{1+\alpha}(A_n(b) \sim U) < +\infty.$$

Concrete results go back to work of Wang and O'F from the 1970's, in the case of bounded functions. More recently, I showed that when cpd's exist on Lipα holomorphic functions, then they may be evaluated using limits:

Q: Suppose $A^\alpha(U)$ admits a continuous point derivation at b, and let ∂ be the normalised derivation there. Is there a set $E \subset U$ such that

$$(*) \lim_{a \to b, a \in E} \frac{f(a) - f(b)}{a - b} = \partial f, \ \forall f \in A?$$

Q: Suppose $A^\alpha(U)$ admits a continuous point derivation at b, and let ∂ be the normalised derivation there. Is there a set $E \subset U$ such that

$$(*) \lim_{a \to b, a \in E} \frac{f(a) - f(b)}{a - b} = \partial f, \ \forall f \in A?$$

Theorem

(1) (2014) If U contains an open triangle with vertex at b, then (*) holds when E is the angle bisector at b.
(2) (2016) In general, (*) holds for some E having full area density at b.

Proofs involve duality. (2) needed some interesting new technical facts about iterated potentials.

New proof of (1), by Stephen Deterding (2018): 

This raises the question: *what do M^β and M_*^β have to do with the boundary behaviour of analytic functions when* $0 < \beta < 1$?. What is the significance of the condition

$$\sum_{n=1}^{\infty} 2^n M^\beta(A_n \setminus U) < +\infty,$$

when $0 < \beta < 1$, where U is a bounded open subset of $\mathbb{C}$ and $b \in \partial U$?

The answer involves the so-called '*negative Lipschitz spaces*'.

Theorem

Let $0 < \beta < 1$ *and* $s = \beta - 1$. *Let* $U \subset \mathbb{C}$ *be a bounded open set, and* $b \in \partial U$. *Then* $A^s(U)$ *admits a continuous point evaluation at* b *if and only if*

$$\sum_{n=1}^{\infty} 2^n M_*^{\beta}(A_n \setminus U) < +\infty,$$

Theorem

Let $0 < \beta < 1$ *and* $s = \beta - 1$. *Let* $U \subset \mathbb{C}$ *be a bounded open set, and* $b \in \partial U$. *Then* $B^s(U)$ *admits a weak-star continuous point evaluation at* b *if and only if*

$$\sum_{n=1}^{\infty} 2^n M^{\beta}(A_n \setminus U) < +\infty,$$

Examples

There are similar theorems about bounded point derivations, and the theorems have versions that are about ordinary harmonic functions.
We might explain some details of this story, and the conceptual framework around it.

Symmetric concrete spaces (SCS)

The general framework concerns the relation between problems about a given SCS F, in combination with an elliptic operator L, and an appropriate associated capacity, the L-F-cap.
A Symmetric Concrete Space (SCS) on $\mathbb{R}^d$ is a complete locally-convex topological vector space F over the field $\mathbb{C}$, such that

- $\mathcal{D} \hookrightarrow F \hookrightarrow \mathcal{D}^*$;
- F is a topological $\mathcal{D}$-module under the usual product $\varphi \cdot f$ of a test function and a distribution;
- F is closed under complex conjugation;
- The affine group of $\mathbb{R}^d$ acts by composition on F, and each compact set of affine maps gives an equicontinuous family of composition operators.

T_s and C_s

The Poisson kernel:

$$P_t(z) := \frac{t}{\pi(t^2 + |z|^2)^{\frac{3}{2}}}, \ (t > 0, z \in \mathbb{C}).$$

The Poisson transform: $F(z,t) := (P_t * f)(z)$
Let $s < 0$ and $f \in \mathcal{E}^*$. Then $f \in T_s$ if

$$\|f\|_s := \sup\{t^s |F(z,t)| : z \in \mathbb{C}, t > 0\} < +\infty,$$

and $f \in C_s$ if, in addition,

$$\lim_{t \downarrow 0} t^s \sup\{|F(z,t)| : z \in \mathbb{C}\} = 0.$$

These extend $\mathrm{Lip}(s)_{\mathrm{cs}}$ and $\mathrm{lip}(s)_{\mathrm{cs}}$ to negative s.
Complete them to Banach spaces.

$A^s(U)$ and $B^s(U)$:

For an open set $U \subset \mathbb{C}$, and $s \in \mathbb{R}$, let

$$A^s(U) := \{f \in C_s : f \text{ is holomorphic on } U\},$$

and

$$B^s(U) := \{f \in T_s : f \text{ is holomorphic on } U\}.$$

Lemma

For each $s \in \mathbb{R}$, each open set $U \subset \mathbb{C}$ and each $b \in \mathbb{C}$, The set $\{f \in A^s(U) : f \text{ is holomorphic on some neighbourhood of } b\}$ is dense in $A^s(U)$.

A tool

$$N_k(\varphi) := \mathrm{diam}(\mathrm{spt}(\varphi))^k \cdot \sup |\nabla^k \varphi|.$$

$$N_k(\kappa \cdot \varphi) = \kappa \cdot N_k(\varphi).$$

$$N_0(\varphi) \leq N_1(\varphi) \leq N_2(\varphi) \leq \cdots ,$$

$$N_k(\varphi \cdot \psi) \leq 2^k N_k(\varphi) N_k(\psi)$$

$$\psi(x) := \varphi(r \cdot x) \implies N_k(\psi) = N_k(\varphi).$$

Estimating $\langle \varphi, f \rangle$

Beyond Cauchy-Schwartz-Hölder.

$$|\langle \varphi, f \rangle| \stackrel{?}{\leq} K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot (1\text{-}F\text{-cap})(\mathrm{spt}(\varphi \cdot f)).$$

Version:

$$|\langle L^t \varphi, f \rangle| \stackrel{?}{\leq} K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot (L\text{-}F\text{-cap})(\mathrm{spt}(\varphi \cdot f)).$$

cf. Vitushkin (d-bar), Mazaloff (Δ).
Simple example:

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_0(\varphi) \cdot \|f\|_{L^\infty} \cdot \mathrm{volume}(\mathrm{spt}(\varphi \cdot f)).$$

Scaling and covering

For $F = T_s$, with $-2 < s < 0$, scaling gives

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot r^{s+2},$$

whenever $\mathrm{spt}(\varphi \cdot f) \in \mathbb{B}(0, r)$. Then a covering argument gives

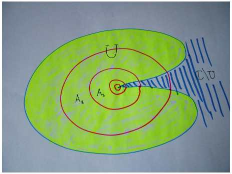
$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot M^{s+2}(\mathrm{spt}(\varphi \cdot f)).$$

and for $f \in C^s$,

$$|\langle \varphi, f \rangle| \leq K \cdot \mathbb{N}_k(\varphi) \cdot \|f\|_F \cdot M_*^{s+2}(\mathrm{spt}(\varphi \cdot f)).$$

If the series converges:

$$f(0) = -\sum_{n=1}^{N} \left\langle \frac{\varphi_n}{\pi z}, \frac{\partial f}{\partial \bar{z}} \right\rangle$$



where

- $\mathrm{spt}\varphi_n$ is contained in 3 annuli,
- $\sum_n \varphi_n = 1$ on $\mathbb{B}(0,1) \setminus \{0\}$, and
- $N_3(\varphi_n) \leq K$, hence $N_3(\frac{\varphi_n}{\pi z}) \leq K \cdot 2^n$.

$$|f(0)| \leq K \cdot \sum_{n=1}^{\infty} 2^n \cdot M_*^{\beta}(A_n \setminus U) \cdot \|f\|_s.$$

Let's remember the good days!