An H^p scale for Complete Pick Spaces

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Most Hilbert spaces come in a scale of Banach spaces.

$$\ell^1 \subsetneq \ell^p \subsetneq \ell^2 \subsetneq \ell^q \subsetneq \ell^\infty \qquad 1 < p < 2 < q < \infty$$

$$L^1 \supsetneq L^p \supsetneq L^2[0,1] \supsetneq L^q \supsetneq L^\infty$$

Hardy space $H^2 = \{f \in O(\mathbb{D}) : \sup_{0<r<1} \int |f(re^{i\theta})|^2 d\theta < \infty\}$

$$H^1 \supsetneq H^p \supsetneq H^2 \supsetneq H^q \supsetneq H^\infty$$

$$N^+ \supsetneq H^1 \supsetneq H^p \supsetneq H^2 \supsetneq H^q \supsetneq BMOA \supsetneq H^\infty$$

Smirnov class $N^+ = \{f/g : f, g \in H^\infty, g \text{ outer}\}$

$BMOA = (H^1)^*$

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¿ What about other Hilbert function spaces, *eg*

Dirichlet space $\mathcal{D} = \{f \in O(\mathbb{D}) : \int_{\mathbb{D}} |f'|^2 dxdy < \infty\}$

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Model is Hardy space, so start with H^1

(i) $H^1 = H^2 \cdot H^2$

(ii) $\mathrm{Mult}(H^1) = H^\infty = \mathrm{Mult}(H^2)$

(iii) Multiplier invariant subspaces are uH^1; $uH^1 \cap H^\infty$ dense in uH^1

(iv) $(H^1)^* = \{b : \int (fg)\overline{b} \text{ bounded by} \|f\|_{H^2}\|g\|_{H^2}\}$

(v) Complex Interpolation $[H^1, BMOA]_s = H^p, \quad p = \frac{1}{1-s}$

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Let $\mathcal{F}$ be Hilbert function space on X, kernel $k_x(y)$.

$$\mathcal{F} \odot \mathcal{F} = \{\sum f_n g_n : \sum \|f_n\| \|g_n\| < \infty\}$$

Thm. (Pisier, 96) $[\mathcal{F} \odot \mathcal{F}, (\mathcal{F} \odot \mathcal{F})^*]_{\frac{1}{2}} = \mathcal{F}$.

Define $\mathcal{H}^p := [\mathcal{F} \odot \mathcal{F}, (\mathcal{F} \odot \mathcal{F})^*]_{\frac{p-1}{p}}$.

In full generality, can't say much.

$$(\mathcal{H}^p)^* = \mathcal{H}^{p'}, \qquad p' = \frac{p}{p-1}$$

$$\|\delta_x\|_{(\mathcal{H}^p)^*} \leq k_x(x)^{1/p}, \quad 1 \leq p \leq 2$$

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Complete Pick spaces

$\mathcal{F}$ has Pick property if

$$\begin{array}{lcll} \forall\ T : k_{z_i} & \mapsto & w_i k_{z_i} & 1 \leq i \leq N \\ \exists\ \tilde{T} : k_z & \mapsto & \psi(z) k_z & \forall z \in X \end{array}$$

with $\|\tilde{T}\| = \|M^*_{\bar{\psi}}\| = \|T\|$

Complete Pick: analogous property for matrix-valued multipliers.

Examples: H^2 (Sarason, 1966); Drury-Arveson; Dirichlet; $W^1_2[0,1]$

Thm: (Agler-M): $\mathcal{F}$ is complete Pick if and only if $\exists\ b : X \to \mathbb{B}_d$ s.t.

$$k(x,y) = \frac{1}{1 - \langle b(x), b(y) \rangle_{\mathbb{C}^d}}$$

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Complete Pick spaces with Column-Row property

Complete Pick $k(x,y) = \frac{1}{1-\langle b(x),b(y)\rangle_{\mathbb{C}^d}}$ for some $b : X \to \mathbb{B}_d$

CR If $\begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \end{pmatrix} : \mathcal{F} \to \mathcal{F} \otimes \ell^2$ is bounded, so is $(\phi_1, \phi_2, \dots) : \mathcal{F} \otimes \ell^2 \to \mathcal{F}$

Radial derivative $\mathrm{R} = \sum_{j=1}^{d} z_j \frac{\partial}{\partial z_j}$; $\quad R^s[\sum_{n=0}^{\infty} \sum_{|\alpha|=n} a_\alpha z^\alpha] = \sum_{n=0}^{\infty} n^s \sum_{|\alpha|=n} a_\alpha z^\alpha$

Let ω be a radial weight on $\mathbb{B}_d$, with $\frac{\omega(r)}{(1-r)^m}$ increasing

Besov space $B^s_\omega = \{f \in O(\mathbb{B}_d) : \int_{\mathbb{B}_d} |R^s f|^2 \omega < \infty\}$

Theorem (AHMR): This is complete Pick with CR property if $s \geq \frac{m+d}{2}$

Examples: $\omega = 1, m = 0, s = \frac{d}{2}$: Drury-Arveson space H^2_d

$\omega = 1, m = 0, d = 1, s = 1$: Dirichlet space $\mathcal{D}$

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(ii) Corollary: Zero sets same in whole $\mathcal{H}^p$ scale

(iii) $(\mathcal{F} \odot \mathcal{F})^* = \{b : |\langle \phi f, b\rangle| \leq C\|\phi\|_{\mathcal{F}}\|f\|_{\mathcal{F}}, \qquad \phi \in \mathrm{Mult}(\mathcal{F}), f \in \mathcal{F}\}$

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$$\text{(v)} \qquad \|\delta_x\|_{(\mathcal{H}^p)^*} \approx k_x(x)^{1/p}, \quad 1 \leq p \leq 2$$

Questions:1. Does complete Pick $\Rightarrow$ CR property?
For complete Pick, always have analogue of Carleson's characterization of interpolating sequences, don't always have corona theorem.
2. Does complete Pick & CR $\Rightarrow$ corona theorem?

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