Universal Dirichlet series

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Joint work with

R.M. Aron, F. Bayart, P. Gauthier and V. Nestoridis

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THEOREM. W. Luh 1970, C.K. Chui and M. N. Parnes 1971

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Given K a compact subset of $\mathbb{C}$ and defined $A(K)$ as the algebra of all functions $g : K \to \mathbb{C}$ that are continuous on K and holomorphic on its interior. It holds that

$$A(K) = \overline{\mathcal{P}(\mathbb{C})}^{\|.\|_K}$$

if and only if $\mathbb{C} \setminus K$ is a connected set.

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We endow $\mathcal{D}_a(\mathbb{C}_+)$ with the Fréchet topology induced by the semi-norms $\|\cdot\|_\sigma, \sigma > 0$.

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Let $K \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \leqslant 0\}$ be compact with connected complement, $f \in \mathcal{D}_a(\mathbb{C}_+)$, $g \in A(K)$, $\sigma > 0$ and $\varepsilon > 0$. Then there exists a Dirichlet polynomial $h = \sum_{n=1}^{N} a_n n^{-s}$ such that $\|h - g\|_{C(K)} < \varepsilon$ and $\|h - f\|_\sigma < \varepsilon$.

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Lemma

Let X be a Banach space and let $(x_n)_{n\geqslant 1}$ be a sequence in X. Assume that $\sum_{n=1}^{+\infty} |\langle x^*, x_n\rangle| = +\infty$ for every nonzero continuous linear functional $x^* \in X^*$. Then, for every $N \in \mathbb{N}$, the set $\left\{\sum_{n=N}^{M} a_n x_n;\ M \geqslant N, |a_n| \leqslant 1\right\}$ is dense in X.

KEY ELEMENT

Let $K \subset \{z \in \mathbb{C} \mid \operatorname{Re} z \leqslant 0\}$ be compact with connected complement. The key point is to prove that for every $\delta > 0$, the set

$$\left\{ \sum_{n=N}^{M} \frac{a_n}{n^{s+1+\delta}};\ M \geqslant N, |a_n| \leqslant 1 \right\}$$

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COROLLARY.

Let $K \subset \overline{\mathbb{C}_-}$ be compact with connected complement, for every $f \in D_a(\mathbb{C}_+)$, g entire, $\sigma, \varepsilon > 0$, and $N \in \mathbb{N}$, there exists a Dirichlet polynomial h such that

$$\sup_{0 \leqslant l \leqslant N} \|h^{(l)} - g^{(l)}\|_{C(K)} < \varepsilon \text{ and } \|h - f\|_\sigma < \varepsilon.$$

THEOREM

There exists a Dirichlet series $D = \sum_{n=1}^{\infty} a_n n^{-s}$, absolutely convergent in $\mathbb{C}_+$, with partial sums $S_N = \sum_{n=1}^{N} a_n n^{-s}$, such that, for every entire function g there exists a sequence (N_k), so that for every ℓ in $\{0,1,2,\ldots\}$ the derivatives $S_{N_k}^{(\ell)}$ converge to $g^{(\ell)}$ uniformly on each compact subset of $\{z \in \mathbb{C} \mid \operatorname{Re} z \leqslant 0\}$.

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Moreover, the set of such Dirichlet series is dense and a G_δ in the space $\mathcal{D}_a(\mathbb{C}_+)$ and it contains a dense vector subspace of it, apart from 0.

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Hence given $\varepsilon = 1/j$, there exists a polynomial $P_j : \mathbb{C} \to \mathbb{C}$ such that

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By induction there exists $(N_j) \subset \mathbb{N}$ an strictly increasing sequence such that

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for all j.

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