Biholomorphic functions on Banach Space

Mary Lilian Lourenço
University of São Paulo - Brazil
Joint work with H. Carrion and P. Galindo

AGA 2019 - Conference dedicated to the memory of Richard Timoney.

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- *He gave me the opportunity to attend the library and seminars in the TCD*
- *He hepled me to rent an apartament in the Trinity House,*
- *He invited me to a special and formal dinner in the TCD.*
- *He and his wife invited me to special dinner in their house.*

AGA 2019 - Dedicated to the memory of Richard Timoney.

Gmail — Mary Lilian Lourenco <lourencomarylilian@gmail.com>

(Reference Number 18048), Proc Edinb Math

2 mensagens

Mary Lilian Lourenco <mllouren@ime.usp.br> — 24 de outubro de 2018 16:19

Para: "Richard M.Timoney" <richardt@maths.tcd.ie>

Dear Richard

Thank you for the answer and accept it our article (Reference Number 18048)
Good news indeed!

A latex version and a pdf version are attached.
Any other request, we are at your disposal
Thank you again

All the best

Lilian

Dear Mary Lilian,

I'm happy to report that your paper has been accepted, the longer of the two versions you sent me.

What I need now are (a) the LaTeX source code and (b) the pdf [which I already have]. I've to check that the source compiles ok and then send the lot to the journal office. Once I do that, the paper will be dealt with from my point of view, although it may take a while for the copy editor to deal with your paper. Maybe the first hint that they are getting to your paper will be a copyright form.

Yours,

2 anexos

Bihilomo-revisado-amply1final.tex
47K

Bihilomo-revisado-amply1final.pdf

Notation

- *Let E be a complex Banach space.*
- *$\mathcal{L}(E)$ space of all bounded linear operator.*
- *Let $T \in \mathcal{L}(E)$, $\sigma(T)$ denotes its spectrum of T.*
- *Let V be an open set of E, $\mathcal{H}(V, E)$ the space of all holomorphic mappings from V into E.*
- *If f belongs to $\mathcal{H}(V, E)$, df_p denotes the derivative of f at the point $p \in V$*
- $\Delta(0,1) = \{z \in \mathbb{C} : |z| < 1\}$ *and* $S(0,1) = \{z \in \mathbb{C} : |z| = 1\}$.

- *If V is a domain in E, we say f is biholomorphic on V, if*
  1. *$f : V \longrightarrow V$ is bijective and holomorphic mapping*
  2. *f^{-1} is holomorphic mapping.*

For finite dimensional spaces $(1) \Longrightarrow (2)$

Lemma (Classical Schwarz Lemma)

Let $f : D \longrightarrow D$ be a holomorphic mapping such that $f(0) = 0$, then

$$|f(z)| \leq |z| \text{ and } |f'(0)| \leq 1.$$

Moreover, if $|f'(0)| = 1$, then f is a rotation.

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2. *L. Harris , A continuous form of Schwarz's Lemma in normed linear spaces, Pacific J. Math. 38 (1971), 635-639.*

Theorem

Let X and Y be complex normed linear spaces. Let $h : B_X \to \overline{B_Y}$ be a holomorphic functions with $h(0) = 0$. Put $L = dh(0)$ and let $\mathcal{U}$ be the set of all linear isometries of X onto Y. Suppose $\mathcal{U}$ is nonempty. Then

$$||h(x) - L(x)|| \leq \frac{8||x||^2}{(1 - ||x||)^2} d(L, \mathcal{U}), \ \forall x \in B_X,$$

where $d(L, \mathcal{U})$ denote the distance of L from $\mathcal{U}$ in the operator norm.

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Theorem (Carathéodory-Cartan-Kaup-Wu)

If a holomorphic mapping $f : \Omega \longrightarrow \Omega$ of a bounded domain $\Omega \subset \mathbb{C}^n$, satisfies $f(p) = p$ for some $p \in \Omega$ and $|det\{df_p\}| = 1$, then f is a biholomorphic mapping.

J.A. Cima, I. Graham, K. T. Kim and S. Krantz, The Carathéodory-Cartan-Kaup-Wu Theorem on an infinite dimensional Hilbert Space. Nagoya Math. J. Vol.185 (2007), *17-30.*

Theorem

Let H be a separable Hilbert space and $\Omega \subset H$, be a bounded convex domain. Fix a point $p \in \Omega$. Let $f : \Omega \longrightarrow \Omega$ be a holomorphic mapping such that

- $f(p) = p$;
- *the differential df_p is triangularizable;*
- $\sigma(df_p) \subset S(0, 1)$.

Then f is biholomorphic

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Theorem

Let E be the dual of a Banach space and $G \subset E$ a bounded domain with the separation property E such that its weak closure coincides with its norm closure. Let $f : G \to G$ be a holomorphic mapping such that*

1. $f(p) = p$.
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Then f is a biholomorphic mapping.

Corollary

If f *is a holomorphic sef-map of the unit ball* B_H*of a separable Hilber space* H*, with* $f(0) = 0$ *and*

$$| < df_0(e_i), e_i > | = 1$$

for all vector e_i *in an orthornormal basis, then* f *is biholomorphic.*

Definition

Let $T \in \mathcal{L}(E)$ *be a bounded linear operator.* T *is* **triangularizable** *if there is a total, i.e., with dense span, linearly independent sequence* $\{e_1, e_2, ...e_n, ...\}$ *of* E *such that for all* $x \in span\{e_1, e_2, ..., e_n\}$

$$T(x) \in span\{e_1, e_2, ..., e_n\}$$

for every $n \in \mathbb{N}$*. In this case,* $T(e_k) = \sum_{j=1}^{k} \beta_j^k e_j$*, for all* $k = 1, 2, \cdots, n$*.*

So, the matrix of T *when restricted the subspace generated by* $\{e_1, e_2, ..., e_k\}$ *is an upper-triangular matrix, whose main diagonal is given by* $\beta_1^1, \beta_2^2, \cdots \beta_k^k$*. We shall refer to the sequence* (β_k^k) *as the diagonal entries.*

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Definition

We say that an open set $A \subset E$ has the separation property if for every $u \in \overline{A} \setminus A$, there is an analytic function h in a neighborhood of $\overline{A}$ such that $h(u) = 1$ and $|h(x)| < 1$ for all $x \in A$.

Examples

- *Any convex domain Ω has the separation property.* *Indeed, if $u \notin \Omega$, by the Hahn-Banach separation theorem, then there are $\alpha \in \mathbb{R}$ and $\varphi \in E^*$ such that $\Re e \varphi(w) < \alpha \leq \Re e \varphi(u)$ for all $w \in \Omega$. Thus $h := e^{\varphi - \varphi(u)}$ has absolute value less than 1 on Ω and $h(u) = 1$.*

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- *If A is a relatively compact strictly pseudoconvex open set in $\mathbb{C}^n$ with a $\mathcal{C}^2$ boundary, then it has the separation property*

Proposition

Let U be a domain and $A \subset E$ be an open set with the separation property, and let $f \in \mathcal{H}(U, \overline{A})$. Then either $f(U) \subset \overline{A} \backslash A$ or $f(U) \subset A$.

- *H. Carrión, P. Galindo and M.L. Lourenço, Biholomorphic Mappings on Banach Spaces, Proc of Edinburgh Math. Soc.*

Definition

An operator is said to be **power bounded** *if*

$$\sup_n \|T^n\| < \infty.$$

where $T^n = T \circ T \circ \cdots \circ T$, is the n-times self composition of T.

If E is a Banach space, this is equivalent to

$$\sup_n \|T^n(x)\| < \infty$$

for each $x \in E$ according to the Uniform Boundedness Principle.

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Theorem

Let E be a Banach space. Let $T \in \mathcal{L}(E)$ be a power bounded closed range operator. Assume that:

- *There is a subsequence $\left(T^{n(k)}\right) \subset (T^n)$ which converges pointwise to an operator $S \in \mathcal{L}(E)$.*
- *If either S is onto and T is one-to-one, or S is invertible.*

Then T is an invertible operator and $\sigma(T) \subset S(0,1)$.
Further, if $m(k) = n(k+1) - n(k)$, then $\left(T^{m(k)}\right)$ converges pointwise to the identity mapping.

In particular, if E is a Hilbert space, then T is similar to a unitary operator.

If E is a Hilbert space, there is an invertible self-adjoint operator Q such that QTQ^{-1} is unitary.

B. Sz. Nagy, On uniformly bounded linear transformations in Hilbert space. Acta Univ. Szeged. Sect. Sci. Math. 11, (1947). 152-157

Theorem

For each power bounded linear operator T in a Hilbert space there is a self-adjoint operator Q such that $Q^{-1}TQ$ is a unitary operator.

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Example

The backward shift operator

$$B\big((x_n)\big) := (x_2, x_3, \dots)$$

acting on ℓ_2 *is power bounded since* $\|B\| \leq 1$ *and the sequence of iterates* $B^m\big((x_n)\big) = (x_{m+1}, x_{m+2}, \dots)$ *converges pointwise to a non surjective operator, the null one. So in Theorem above some assumptions about* S *are necessary.*

Example

Let $(\lambda_n)_n \subset S(0,1)$. *Consider the Hilbert space* $E = \ell_2(\mathbb{R})$. *Put* $\mathbb{R} \equiv I \times \mathtt{N}$ *for an uncountable set* I. *For every element in the canonical basis* $\{e_{i,n} : I \times \mathbb{N}\}$ *define*

$$T(e_{i,n}) = \lambda_n e_{i,n}.$$

$T \in \mathcal{L}(E)$, *and* $\|T\| = 1$, *and hence* $\|T^m\| \leq 1$.
Since $(\lambda_n)_n \subset S(0,1)$, *by the Cantor's diagonalization process there exists a subsequence of positive integer numbers* $(m(k))_k$ *such that*

$$\lim_k \lambda_n^{m(k)} = 1$$

for all n. *It turns out that* T^{m_k} *converges pointwise to* $Id_{|_E}$ *since for all pairs* (i,n), $\lim_k T^{m_k}(e_{i,n}) = \lim_k (\lambda_n)^{m_k} e_{i,n} = e_{i,n}$.
Notice that T *is not triangularizable.*

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Notice that T *is not triangularizable.*

Theorem

Let E be a dual Banach space and let $U \subset E$ a bounded domain with the separation property such that its weak closure coincides with its norm closure. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that*

- $f(p) = p$.
- df_p *is a one-to-one closed range operator*
- *there is a subnet* $\left(df_p^{n(k)}\right)$ *which converges pointwise to an onto operator.*

Then f is a biholomorphic mapping.

Remark

Observe that Theorem does not provide a necessary condition for f to be biholomorphic: Consider $E = \ell_2(\mathbb{Z})$ *and* f *the shift operator in* E,

$$f((x_n)) = (x_{n+1}).$$

- f *is an automorphism of the unit ball,*
- $f = df_0$, *and* $f^m(e_n) = e_{n+m}$,

so the sequence $(f^k(e_n))_k$ *does not have a Cauchy subsequence and, therefore,* $(f^k)_k$ *can not have a pointwise convergent subsequence. In addition,* f *is known to be not a triangularizable operator.*

The idea of the proof:

- *Let $E = X^*$.Let $G \subset E$ be a bounded domain and $f : G \to G$ be a holomorphic mapping such that there is $p \in G$ with $f(p) = p$. For each $m \in \mathbb{N}$ we have*

$$\|(df^m(p))\| \leq \frac{\sup_{x \in G} \|f(x)\|}{d}$$

where $d = dist(p, \partial G)$ and $f^m = f \circ f \circ \cdots \circ f$, is the m-times self composition of f.

There is a constant $C > 0$ such that $\left\|df_p^n\right\| < C$ for all n. By the power bounded Theorem we have that df_p is invertible, which implies that df_p^{-1} does exist, and also that there is a subsequence $\left(df_p^{n(k)}\right)$ pointwise convergent to the identity.

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There is a constant $C > 0$ *such that* $\left\|df_p^n\right\| < C$ *for all* n*. By the power bounded Theorem we have that* df_p *is invertible, which implies that* df_p^{-1} *does exist, and also that there is a subsequence* $\left(df_p^{n(k)}\right)$ *pointwise convergent to the identity.*

Let $\mathcal{A} = \{f^{n(k)}, k \in I\} \subset \mathcal{H}(U, U)$*. It is possible to show, there are a subnet* $\left(f^{n(k_i)}\right)_i$ *and* $g \in \mathcal{H}(U, E)$*, such that for every compact* $K \subset U$ *we have that*

$$\lim_i \left(\sup_K \left\{ | < f^{n(k_i)}(x) - g(x), \rho > | \right\} \right) = 0$$

for all $\rho \in X$*.*

Then $g(z) \in \overline{U}^{w^*} = \overline{U}$*, for all* $z \in U$ *and* $g(p) = p$*. Since* U *is an open set with separation property we have that* $g(U) \subset U$*. Moreover, for every compact* $K \subset E$ *we have that*

$$\lim_k \left(\sup_K \left\{ | < (df_p)^{n(k)}(x) - dg_p(x), \rho > | \right\} \right) = 0$$

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Theorem

(Cartan) Let $G \subset E$ be an open bounded set, $h \in \mathcal{H}(G, G)$ and $p \in G$, such that $h(p) = p$ and $dh_p = I_E$. Then $h = I_G$.

Using Cartan's Theorem $g = I_U$.

Again it is possible to get a subnet of the bounded subnet $\left(f^{n(k_i)-1}\right)_i$ and a holomorphic function $h \in \mathcal{H}(U, E)$ such that for every compact $K \subset G$ we have that

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Clearly, $h(p) = p$ and $h(z) \in \overline{U}^{w^*} = \overline{U}$ for all $z \in U$, we have that $h \in \mathcal{H}(U, U)$.

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Clearly, $h(p) = p$ and $h(z) \in \overline{U}^{w^*} = \overline{U}$ for all $z \in U$, we have that $h \in \mathcal{H}(U, U)$.

Moreover, for $z \in U$,

$$z = \lim f\left(f^{n(k_i)-1}\right)(z) = \lim f^{n(k_i)-1}(f(z)) = h(f(z)),$$

which shows that $h \circ f = I_U$. Finally, we show that $f \circ h = I_U$. Since $h \circ f = I_U$, we have

$$dh_p \circ df_p = I_E$$

and since df_p^{-1} exists, it follows that $dh_p = df_p^{-1}$.

Therefore $df_p \circ dh_p = I_E$, using Cartan's Theorem again, we obtain

$$f \circ h = I_U.$$

Corollary

Let E be a reflexive Banach space and let $U \subset E$ be a convex bounded domain. Let $f \in \mathcal{H}(U, U)$ and $p \in U$ such that

- $f(p) = p$,
- df_p *is a one-to-one operator*
- *There is a subsequence* $\left((df_p)^{m_k}\right)_k$ *that converges pointwise to an onto operator in* $\mathcal{L}(E)$.

Then f is biholomorphic.

In the Theorem about power bounded operator, the assumption of T having closed range may be replaced by the weak compactness of T. Since df_p is weakly compact, the result follows.

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Thank you!