Q-measures on the dual unit ball of a JB*-triple

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$C(\Omega)$
continuous complex functions on a compact Hausdorff space (Ω, τ)

$C(\Omega)^*$ - complex regular Borel measures

WHEN IS A SET THE SUPPORT OF A MEASURE?

Jordan *-triple

complex vector space A together with a triple product, i.e., a mapping

$$\{\ldots\} : A \times A \times A \to A$$
$$(a, b, c) \mapsto \{a \ b \ c\}$$

- symmetric and linear in the outer variables
- conjugate linear in the middle variable
- $D(a, b) : A \to A$ linear operator

$$D(a, b)c = \{a \ b \ c\}$$

$$[D(a, b), D(c, d)] = D(\{a \ b \ c\}, d) - D(c, \{b \ a \ d\})$$

JB*-triple is a Jordan *-triple A such that:

- A is a Banach space
- triple product is continuous
- $D(a, a)$ hermitian, with non-negative spectrum and

$$\|D(a, a)\| = \|a\|^2$$

JBW*-triple if A is the dual of a Banach space

Examples

- $C(\Omega)$ - continuous complex functions on a compact Hausdorff space (Ω, τ),
$$\{fgh\} = f\bar{g}h$$
- $B(H, K)$ H, K Hilbert spaces

$$\{a \ \ b \ \ c\} = \frac{1}{2}(ab^*c + cb^*a) \tag{1}$$

- C*-algebra is a JB*-triple   triple product (1)
- W*-algebra is JBW*-triple,   triple product (1)
- The bidual of a JB*-triple is a JBW*-triple

- $u \in A^{**}$ $\qquad u$ tripotent if $\{u\,u\,u\} = u$

- Peirce decomposition

$$A^{**} = A_0^{**}(u) \oplus A_1^{**}(u) \oplus A_2^{**}(u)$$

$$A_j^{**}(u) = \{a \in A^{**} : D(u,u)a = \tfrac{1}{2} j a\} \qquad \text{for} \qquad j = 0, 1, 2$$

- Peirce projections

$$P_2(u) = Q(u,u)^2 \qquad Q(u,u)a = \{u\,a\,u\}$$

$$P_1(u) = 2(D(u,u) - Q(u,u)^2) \qquad P_0(u) = \mathrm{I} - 2D(u,u) + Q(u,u)^2$$

Example

- $B(H)$ $\quad \{\mathbf{u\,u\,u}\} = \mathbf{u} \Leftrightarrow \mathbf{uu^*u} = \mathbf{u} \quad \rightarrow u$ partial isometry

- $A = \mathrm{M}_3(\mathbb{C})$

$$u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \left[\begin{array}{c|cc} A_2(u) & * & * \\ \hline * & \multicolumn{2}{c}{A_0(u)} \\ * & & \end{array}\right] \qquad * \leftrightarrow A_1(u)$$

- $u \in A^{**}$ $\qquad$ u tripotent if $\{u\,u\,u\} = u$

- u, v tripotents, $\qquad$ $\mathcal{U}(A^{**})$ set of tripotents

$$u \leq v \qquad \text{if} \qquad v - u \in A_0^{**}(u)$$

"$\leq$" partial order relation

- $\underbrace{\mathcal{U}(A^{**}) \cup \{\omega\}}_{\tilde{\mathcal{U}}(A^{**})}$ together with $\leq$ is a complete lattice

(ω greatest element)

- $u \in \mathcal{U}(A^{**})$ compact-G_δ relative to A if

$$\exists a \in A, \|a\| = 1 \qquad u = u(a),$$

where $\qquad u(a) = w^* - \lim a^{2n+1}$

- $u \in \tilde{\mathcal{U}}(A^{**})$ compact relative to A if $u = 0$
  or
  there exists decreasing net $(u_j)_{j\in\Lambda}$ in $\mathcal{U}(A^{**})$ of compact-G_δ tripotents relative to A such that

$$u = \wedge\{u_j : j \in \Lambda\}$$

- $\mathcal{U}_c(A)$ compact tripotents relative to A

$$\tilde{\mathcal{U}}_c(A^{**}) = \mathcal{U}_c(A) \cup \{\omega\}$$

V complex Banach space $\qquad V^*$ dual of V

V_1 unit ball of V $\qquad V_1^*$ unit ball of V^*

$\mathcal{F}_n(V_1)$ norm-closed faces of V_1 $\qquad \mathcal{F}_{w^*}(V_1^*)$ weak*-closed faces of V_1^*

$$C \subseteq V, \quad C \text{ convex}$$
$$E \subseteq C, \quad E \text{ convex}$$
$$E \text{ face of } C \quad \textit{if} \quad \text{for } z_1, z_2 \in C$$
$$\exists 0 < t < 1 \qquad tz_1 + (1-t)z_2 \in E \;\Rightarrow\; z_1, z_2 \in E$$

- $S \subseteq V_1$

$$\underbrace{S' = \{x \in V_1^* \colon x(a) = 1,\ \forall a \in S\}}_{\text{weak*-closed face of } V_1^*}$$

- $W \subseteq V_1^*$

$$W_\prime = \underbrace{\{a \in V_1 \colon x(a) = 1,\ \forall x \in W\}}_{\text{norm-closed face of } V_1}$$

(Edwards, Fernández-Polo, Hoskin, Peralta, Rüttimann)

Theorem

1. $\tilde{\mathcal{U}}_c(A^{**})$ *together with inherited partial ordering is a complete atomic lattice*
2. *The map*
$$\tilde{\mathcal{U}}(A^{**}) \to \mathcal{F}_n(A_1^*)$$
$$u \mapsto \{u\},$$
is an order isomorphism.
3. *The map*
$$\tilde{\mathcal{U}}_c(A^{**}) \to \mathcal{F}_{w^*}(A_1^*)$$
$$u \mapsto \{u\},$$
is an order isomorphism.

A JB*-triple, $x \in A^*$, $\|x\| = 1$

- $e(x)$ support tripotent of x the unique tripotent such that

$$\underbrace{\{e(x)\}_{\prime}}_{\uparrow}$$

the smallest norm-closed face of A_1^* containing x

- $e_c(x)$ compact support tripotent of x the unique tripotent compact relative to A such that

$$\underbrace{\{e_c(x)\}_{\prime}}_{\uparrow}$$

the smallest weak*-closed face of A_1^* containing x

Lemma

Let A be a JB^*-triple. Then, for each element w in $\tilde{\mathcal{U}}(A^{**})$, there exists uniquely a smallest element w^{c} in $\tilde{\mathcal{U}}_c(A)$ such that

$$w \leq w^{\mathrm{c}}$$

given by

$$w^{\mathrm{c}} = \wedge_{\{v \in \tilde{\mathcal{U}}_c(A) : w \leq v\}} v,$$

for which

$$(\{w^{\mathrm{c}}\}_{\prime})_{\prime} = (\{w\}_{\prime})_{\prime}.$$

- w in $\tilde{\mathcal{U}}(A^{**})$

  w^{c} of $\tilde{\mathcal{U}}_c(A)$ Q-closure of w

- v of $\tilde{\mathcal{U}}_c(A)$

  w in $\tilde{\mathcal{U}}(A^{**})$ is said to be Q-dense in v if

$$w^{\mathrm{c}} = v$$

Lemma

Let A be a JB^*-triple. Then, for each element x in A^* of norm one,

$$e(x)^{\mathrm{c}} = e_c(x).$$

x and y of A^* are said to be L-orthogonal if

$$\|x \pm y\| = \|x\| + \|y\|$$

Lemma

Let A be a JB*-triple, $v \in \mathcal{U}_c(A)$, $w \in \mathcal{U}(A^{**})$ and $w \leq v$

$\{z_j : j \in \Lambda_1\}$ maximal L-orthogonal family of elements of $\{w\}$,

$\{z_j : j \in \Lambda_2\}$ maximal L-orthogonal family of elements of $\{v - w\}$,

$\Lambda = \Lambda_1 \cup \Lambda_2$

Then

$$w = \vee_{j \in \Lambda_1} e(z_j), \quad v - w = \vee_{j \in \Lambda_2} e(z_j), \quad v = \vee_{j \in \Lambda} e(z_j).$$

Moreover, if w is Q-dense in v, then

$$v = (\vee_{j \in \Lambda_1} e(z_j))^{c} = (\vee_{j \in \Lambda_1} e_c(z_j))^{c}.$$

u and v are said to be orthogonal if v lies in $A_0^{**}(u)$

$u \in \tilde{\mathcal{U}}(A^{**})$ is said to be *σ-finite* if it does not majorize an uncountable orthogonal subset of $\tilde{\mathcal{U}}(A^{**})$.

Theorem

A be a JB^*-triple, $v \in \tilde{\mathcal{U}}_c(A)$
Then, there exists an element x, of norm one, in the unit ball A_1^* in A^* such that

$$v = e_c(x)$$

if and only if there exists a σ-finite element w of $\tilde{\mathcal{U}}(A^{**})$ that is Q-dense in v.

- $v \in \mathcal{U}_{\rm c}(A) \qquad v \neq 0$

$$\mathcal{C}_v = \{q \in \tilde{\mathcal{U}}_c(A) : 0 \leq q \leq v\},$$

- $w \in \tilde{\mathcal{U}}(A^{**}) \qquad$ Q-dense in v

  $\{z_j : j \in \Lambda\} \qquad$ maximal L-orthogonal family contained in $\{w\}$,

$$w = \vee_{j \in \Lambda} e(z_j)$$

For each element j in Λ, let

$$\mathcal{D}^{(j)} = \{q \in \mathcal{C}_v \setminus \{v\} : q \wedge e(z_j) \neq 0\}$$

Lemma

Let A be a JB^*-triple. Then, $\mathcal{D}^{(j)}$ is an anti-order ideal in $\mathcal{C}_v$, closed under finite orthogonal sums, such that, for all $q \in C_v \setminus \{0\}$,

$$q = (\vee_{j \in \Lambda} q^{(j)})^{\mathrm{c}},$$

where, for j in Λ,

$$q^{(j)} = q \wedge e_c(z_j).$$

Lemma

For each finite subset S *of* $\mathcal{D}^{(j)}$

$$S = \{q_1, q_2, \ldots, q_n\}$$

the covering number $c(S)$ *of* S*, defined by*

$$c(S) = \inf_{y \in \{v\},} \Sigma_{k=1}^{n} n^{-1} y(q_k),$$

is less than one.

Theorem

A be a JB*-triple, $v \in \mathcal{U}_{\mathrm{c}}(A)$. Then, there exists an element x in A^* of norm one such that v is the compact support tripotent $e_c(x)$ of x if and only if the family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$ is countable.

(Ω, τ) - compact Hausdorff space
$C^{\tau}(\Omega)$ - continuous complex functions on (Ω, τ)
$C^{\tau}(\Omega)^{**} \sim C^{\psi}(\hat{\Omega})$

u non-zero tripotent in $C^{\psi}(\hat{\Omega})$
there exists a unique ψ-clopen subset E_u of Ω given by

$$E_u = \{t \in \hat{\Omega} : |u(t)| = 1\}$$

such that

$$u = \chi_{E_u} u$$

in which case

$$E_u = \hat{\Omega} \setminus \{t \in \hat{\Omega} : u(t) = 0\}.$$

Theorem

Let (Ω, τ) *be a compact Hausdorff space and let* $(\hat{\Omega}, \psi)$ *be its hyper-Stonean envelope. For each non-zero tripotent* u *in* $C^\psi(\hat{\Omega})$*, let*

$$u = \chi_{E_u} u$$

and, for u *equal to zero, let* E_u *be the empty set. Then, the mapping*

$$u \mapsto E_u \cap \Omega$$

is an order isomorphism from the complete atomic lattice $\tilde{\mathcal{U}}_c(C^\tau(\Omega))$ *of tripotents* u *in* $C^\psi(\hat{\Omega})$ *compact relative to* $C^\tau(\Omega)$ *onto the complete atomic lattice of* τ*-compact subsets of* Ω*.*

Theorem

Let x be a regular Borel probability measure on the compact Hausdorff space (Ω, τ), with compact support tripotent $e_c(x)$, and let

$$\mathrm{supp}_{(\Omega,\tau)} x = \Omega \setminus (\cup\{U \in \tau : x(\chi_U) = 0\}),$$

be the support of x. Then,

$$\mathrm{supp}_{(\Omega,\tau)} x = E_{e_c(x)} \cap \Omega.$$

v tripotent in $\tilde{\mathcal{U}}_c(C^\tau(\Omega))$
$\mathcal{C}_v$ be the set of ψ-clopen subsets of $\hat{\Omega}$ the intersections with $E_v \cap \Omega$ of which are τ-closed, and, for each element j in Λ, let

$$\mathcal{D}^{(j)} = \{E_q \in \mathcal{C}_v \setminus \{E_v\} : E_q \cap E_{e(z_j)} \cap \Omega \neq \emptyset\}.$$

Corollary

Let (Ω, τ) be a compact Hausdorff space with hyper-Stonean envelope $(\hat{\Omega}, \psi)$ and let E be a ψ-clopen subset of $\hat{\Omega}$ such that $E \cap \Omega$ is τ-closed. Then, the following conditions are equivalent.

(i) *There exists a regular Borel probability measure x on (Ω, τ) with support $E \cap \Omega$.*

(ii) *The family $\{\mathcal{D}^{(j)} : j \in \Lambda\}$, consisting of sets of ψ-clopen subsets of $\hat{\Omega}$ the intersections of which with Ω are τ-closed, is countable.*