On some subsets of the space of regular compact sets

Marta Kosek

Institute of Mathematics
Faculty of Mathematics and Computer Science
Jagiellonian University

JAGIELLONIAN UNIVERSITY
IN KRAKOW

Dublin, 8^{th} May 2019

The Green function

Let $D \subset \mathbb{C}$ be an unbounded domain. We consider its Green function with the pole at the infinity $g(\cdot, \infty) : D \longrightarrow \mathbb{R}$, i.e. the function such that

- $g(\cdot, \infty)$ is harmonic and positive in D;
- $g(z, \infty)$ tends to 0 as $z \to \partial D$;
- $g(z, \infty) - \log |z|$ tends to a finite number γ as $z \to \infty$.

If such $g(\cdot, \infty)$ exists, it is unique.

Consider a compact set $E \subset \mathbb{C}$. Let D_∞ be the unbounded component of $\mathbb{C} \setminus E$ and $\widehat{E} := \mathbb{C} \setminus D_\infty$
If $E = \widehat{E}$ we say that E is polynomially convex.

If $g(\cdot, \infty)$ as above exists for D_∞, we say that E is regular. We put

$$g_E(z) = \begin{cases} g(z, \infty), & \text{if } z \in D_\infty \\ 0, & \text{if } z \in \widehat{E} \end{cases}$$

and call this continuous function the Green function of E. Note that by definition $g_E = g_{\widehat{E}}$.

The pluricomplex case

Let E be a compact subset of $\mathbb{C}^N$. Its pluricomplex Green function is defined via

$$g_E(z) = \sup\{u(z) : u \in \mathcal{L},\ u|_E \leq 0\},$$

where $\mathcal{L}$ is the family of all plurisubharmonic functions on $\mathbb{C}^N$ such that $\sup\{u(z) - \log^+ |z| : z \in \mathbb{C}^N\} < \infty$.

E is (pluri)regular (or L-regular) if g_E is continuous.

$$\widehat{E} := \{z \in \mathbb{C}^N | \quad \forall p \in \mathbb{C}[z_1, ..., z_N] : |p(z)| \leq ||p||_E\}.$$

E is polynomially convex if $E = \widehat{E}$.
Note that

$$g_E = \log \Phi_E, \qquad \Phi_E(z) = \sup\{|p(z)|^{1/\deg p} \mid \quad p \in \mathbb{C}[z_1, ..., z_N] \setminus \mathbb{C},\ ||p||_E \leq 1\}.$$

Φ_E – Siciak extremal function

Space $\mathcal{R}$ of compact regular polynomially convex sets

$\mathcal{R}_\star$ is the family of all compact (pluri)regular sets in $\mathbb{C}^N$.
$\mathcal{R} = \{E \in \mathcal{R}_\star : E = \widehat{E}\}$.

$\mathcal{R} \subset \mathcal{R}_\star \subset \kappa(\mathbb{C}^N)$ – the family of all nonempty compact subsets of $\mathbb{C}^N$.

In $\kappa(\mathbb{C}^N)$ consider the Hausdorff metric

$$\chi(E, F) \quad = \quad \max\{||\mathrm{dist}(\cdot, E)||_F, ||\mathrm{dist}(\cdot, F)||_E\}.$$

In $\mathcal{R}_\star$ define

$$\Gamma(E, F) \quad = \quad \max\{||g_E||_F, ||g_F||_E\}.$$

Γ is a pseudodistance (semimetric) on $\mathcal{R}_\star$ and a metric on $\mathcal{R}$.

E.g. via Siciak and Lundin formulas

$$\Gamma([-1,1]^N, \Delta_1^N) = \Gamma(\mathbb{B}_1, \mathbb{B}_1 \cap \mathbb{R}^N) = \log(1+\sqrt{2}).$$

Dilations and augmentations

Let E be compact and nonempty, $\varepsilon > 0$.

ε-dilation of E: $E^\varepsilon := \{z : \mathrm{dist}(z, E) \leq \varepsilon\}$.

$E \in \kappa(\mathbb{C}^N) \implies E^\varepsilon \in \mathcal{R}_\star$.

$$\bigcap_{\varepsilon>0} E^\varepsilon = E \quad \text{and} \quad \bigcap_{\varepsilon>0} \widehat{E^\varepsilon} = \widehat{E} \quad \text{and} \quad \chi(E^\varepsilon, E) = \varepsilon.$$

ε-augmentation of E: $E_\varepsilon := \{z : g_E(z) \leq \varepsilon\}$.

Theorem (M. Mazurek, 1981)

$g_{E_\varepsilon} = \max(0, g_E - \varepsilon)$.

$E \in \mathcal{R}_\star \implies E_\varepsilon \in \mathcal{R}$.

$$\bigcap_{\varepsilon>0} E_\varepsilon = \widehat{E} \quad \text{and} \quad \Gamma(E_\varepsilon, E) = \varepsilon.$$

Theorem (M. Klimek, 1995)

$(\mathcal{R}, \Gamma)$ *is a complete metric space, but it is not compact.*

Corollary (M. Klimek, 1995)

If $E \in \mathcal{R}$, *then* $\{\{z : g_E < \varepsilon\}\}_{\varepsilon > 0}$ *is the basis of neighbourhoods of* E.

It follows that if $(E_n)_{n=1}^{\infty} \subset \mathcal{R}$ is decreasing and $E = \bigcap_{n=1}^{\infty} E_n \in \mathcal{R}$, then $E_n \longrightarrow E$ with respect to both metrics (χ and Γ).

$\left(\mathbb{B}_{1/n}\right)_{n=1}^{\infty}$ is convergent in $(\kappa(\mathbb{C}^N), \chi)$ to $\{0\} \notin \mathcal{R}$.

Theorem (M. Klimek, 2001)

The space $(\mathcal{R}, \Gamma)$ *is separable.*

Theorem (A. Alghamdi, M. Klimek, 2017)

The space $(\mathcal{R}, \Gamma)$ *is not proper, i.e. closed balls do not have to be compact.*

Γ versus polynomiality

Let $\mathcal{P}_d^\star$ be the family of all polynomials $P : \mathbb{C} \longrightarrow \mathbb{C}$ of degree $d \geq 2$ (or all regular polynomial mappings $P : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ of degree $d \geq 2$, where regularity means that $H_d^{-1}(0) = \{0\}$).

Theorem (A. Alghamdi, M. Klimek, M.K., 2018)

(a) If $P \in \mathcal{P}_d^\star$, then $A_P : \mathcal{R} \ni E \longmapsto P^{-1}(E) \in \mathcal{R}$ is a contractive similitude with the contraction ratio $1/d$.
(b) The mapping

$$\mathcal{P}_d^\star \times \mathcal{R} \ni (P, K) \longmapsto P^{-1}(K) \in \mathcal{R}$$

is continuous with respect to the product topology on $\mathcal{P}_d^\star \times \mathcal{R}$.

The fixed point of A_P is the filled-in Julia set of P, i.e.

$$J[P] = \{z : (P^n(z))_{n=1}^\infty \text{ is bounded}\} = \lim_{n\to\infty} P^{-n}(E), \quad E \in \mathcal{R}.$$

It has nice properties: in particular its Green function is Hölder continuous.

Hölder Continuity Property

If $E \in \mathcal{R}$ and g_E is Hölder continuous, we say that E has Hölder Continuity Property and write $E \in \mathcal{R}_{\mathrm{HCP}}$.
Note that

$$E \in \mathcal{R}_{\mathrm{HCP}} \iff \exists U - \text{ bounded neighbourhood of } E$$
$$\exists A, \alpha > 0 \, \forall z \in U : \; g_E(z) \leq A(\mathrm{dist}(z, E))^{\alpha}.$$

Hölder Continuity Property is important in applications, since it implies Markov inequality. We say namely that E is a Markov set if

there exist $M, m > 0$ such that for each polynomial $p : \mathbb{C}^N \longrightarrow \mathbb{C}$ and each multiindex $\alpha \in \mathbb{N}^N$

$$\|D^{\alpha} p\|_E \leq M(\deg p)^{m|\alpha|} \|p\|_E.$$

$\mathcal{R}_{\mathrm{HCP}}$ is not closed

Example (W. Pleśniak, 1990)

Let E be a Cantor set constructed as follows. Given a sequence $\{a_n\}_{n=1}^{\infty}$ of positive numbers such that $a_{n+1} < a_n/2, a_1 < 1/2$, we put $E_1 = I_1^1 \cup I_1^2 = [0, a_1] \cup [1 - a_1, 1]$. If $E_n = \bigcup_{j=1}^{2^n} I_j^n$ is constructed, with each I_j^n being a closed interval of length a_n, then E_{n+1} is obtained by deleting an open concentric subinterval of length $a_n - 2a_{n+1}$ from each I_j^n. We now put $E = \bigcap_{n=1}^{\infty} E_n$.

We have

$$\sum_{n=1}^{\infty} 2^{-n} \log \frac{1}{a_n} < \infty \quad \iff \quad E \in \mathcal{R}.$$

$$\limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{a_n} = \infty \quad \implies \quad E \notin \mathcal{R}_{\mathrm{HCP}}.$$

e.g. $a_n = 2^{-n^2-1}$

$(E_n)_{n=1}^{\infty} \subset \mathcal{R}_{\mathrm{HCP}}$ and is convergent with respect to both metrics to E (if $\sum_{n=1}^{\infty} 2^{-n} \log(1/a_n) < \infty$).

$\mathcal{R}_{\mathrm{HCP}}$ is not closed

Example (M. Klimek, 2001)

Let $(\lambda_n)_{n=1}^{\infty} \subset (4, \infty)$ *be such that*

$$\sum_{n=1}^{\infty} 2^{-n} \log \lambda_n < \infty \qquad \textit{and} \qquad \limsup_{n\to\infty} (\lambda_1 \lambda_2 \ldots \lambda_n)^{1/n} = \infty.$$

Let $f_{\lambda_n} : \mathbb{C} \ni z \longmapsto \lambda_n z(1-z) \in \mathbb{C}$ *and* $E_n = (f_{\lambda_n} \circ \ldots \circ f_{\lambda_1})^{-1}([0,1])$.

Then $(E_n)_{n=1}^{\infty}$ *is convergent* $(\mathcal{R}, \Gamma)$ *to a set* $E \in \mathcal{R} \setminus \mathcal{R}_{\mathrm{HCP}}$, *while* $E_n \in \mathcal{R}_{\mathrm{HCP}}$, $n \in \mathbb{N}$.

e.g. $\lambda_n = 2^{n+2}$.

Compare with the conditions from the previous example:

$$\sum_{n=1}^{\infty} 2^{-n} \log \frac{1}{a_n} < \infty \qquad \text{and} \qquad \limsup_{n\to\infty} \frac{1}{n} \log \frac{1}{a_n} = \infty.$$

e.g. $a_n = 2^{-n^2-1}$.

Łojasiewicz-Siciak inequality

Let $E \in \mathcal{R}$.
Recall:

$$E \in \mathcal{R}_{\mathrm{HCP}} \iff \exists U - \text{ bounded neighbourhood of } E \\ \exists A, \alpha > 0\ \forall z \in U: \ g_E(z) \le A(\mathrm{dist}(z,E))^{\alpha}.$$

Belghiti, Gendre introduced:

$$E \in \mathcal{R}_{\mathrm{ŁS}} \iff \exists U - \text{ bounded neighbourhood of } E \\ \exists B, \beta > 0\ \forall z \in U: \quad g_E(z) \ge B(\mathrm{dist}(z,E))^{\beta}.$$

(Łojasiewicz-Siciak inequality)

examples: finite unions of pairwise disjoint balls or polydiscs, pluriregular subsets of $\mathbb{R}^N$, some polynomially convex holomorphic polyhedra, some planar Cantor type sets...

$\mathcal{R}_{\text{ŁS}}$ is not closed

Example (J. Siciak, 2011)

$$K := \Delta(-2,2) \cup \Delta(2,2) \in \mathcal{R} \setminus \mathcal{R}_{\text{ŁS}}.$$

Example (R. Pierzchała, 2014)

$$K_n := \Delta\left(-2, \sqrt{4+n^{-2}}\right) \cup \Delta\left(2, \sqrt{4+n^{-2}}\right) \in \mathcal{R}_{\text{ŁS}}.$$

$K_n \longrightarrow K$ with respect to both metrics.

The same example shows that $\mathcal{R}_{\text{ŁS}} \cap \mathcal{R}_{\text{HCP}}$ is not closed.

Refinements of metric Γ

Assume $\mathcal{A} \subset \mathcal{R}_\star$ and we have a pseudodistance d on $\mathcal{A}$.

If either d or Γ is a metric on $\mathcal{A}$, then $\Gamma + d$ is a metric on $\mathcal{A}$.

e.g. $\Gamma + \chi$ is a metric on $\mathcal{R}_\star$.

e.g. $(E, F) \longmapsto \Gamma(E, F) + \chi(\partial E, \partial F)$ is a metric on $\mathcal{R}$. It might be appropriate for investigation of the classical Julia sets in the complex plane (which are boundaries of the filled-in Julia sets).

Hölder continuity revisited

Let $f : \Omega \longrightarrow (W, \|\cdot\|)$ be a mapping from a non-empty open set $\Omega \subset \mathbb{C}^N$ to a normed space. Fix $\alpha \in (0,1]$.
Recall that if

$$\mathrm{HC}_\alpha(f) := \sup\left\{ \frac{\|f(x) - f(y)\|}{|x-y|^\alpha} : x, y \in \Omega, x \neq y \right\},$$

then $H_\alpha(\Omega, W) := \{f \in W^\Omega : \mathrm{HC}_\alpha(f) < \infty\}$ is a vector space and HC_α is a seminorm.

Define $\mathcal{R}_\alpha := \{E \in \mathcal{R} : g_E \in H_\alpha(\mathbb{C}^N, \mathbb{R})\}$

and $\Gamma_\alpha(E,F) := \Gamma(E,F) + \mathrm{HC}_\alpha(g_E - g_F), \quad E, F \in \mathcal{R}_\alpha, \quad \alpha \in (0,1].$

Theorem (M. Klimek, 2018)

$\forall \alpha \in (0,1] : \quad (\mathcal{R}_\alpha, \Gamma_\alpha)$ *is a complete metric space.*

Note that $\mathcal{R}_{\mathrm{HCP}} = \bigcup_{\alpha \in (0,1]} \mathcal{R}_\alpha$.

Thank you for your attention!

Maciej Klimek, Marta Kosek, *On the metric space of pluriregular sets*, Dolomites Research Notes on Approximation, 11 (2018), 51-61.