Geometry of vector arrays and operator ampliations

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General motivation...

- Earlier work on using Hilbert space geometry in probability (Christopher Small and D.L. McLeish 1990s) and time series analysis (own work with Masaya Matsuura and Yasunori Okabe 2000s).
- Technical aspects inspired by some articles of Richard Timoney (e.g. with Rupert Levene from 2012) and problems in pluricomplex analysis involving Bergman functions (own work with Marta Kosek and Azza Alghamdi).
- Block matrix calculations important in computer science.
- General idea: use an approach to tedious block matrix algebra reminiscent of Hilbert frames theory and just as easy and geometric. It should accommodate several matrix dimensions simultaneously.
- C^*-module/algebra generalizations of frames (Michael Frank and David Larson 2002) are not covering the cases considered here.

Notation and preliminaries

Arrays, Gram products, ampliations ...

- $(H, \langle\cdot,\cdot\rangle)$ – a complex separable Hilbert space (similarly $H_1, H_2, \ldots$ etc.).
- $H^{m\times n}$ – $(m\times n)$-arrays of elements of H, $H^m \equiv H^{m\times 1}$.
- We have the natural matrix like product:

$$\mathbb{C}^{p\times m}\times H^{m\times n} \ni (A,\mathbf{x}) \longmapsto A\mathbf{x} \in H^{p\times n}.$$

- ***Vectorization*** of arrays: if $\mathbf{x} = [x_{ij}] \in H^{m\times n}$, then $\mathrm{vec}([x_{ij}]) := [x_{11},\ldots,x_{m1},x_{12},\ldots,x_{m2},\ldots,x_{1n},\ldots,x_{mn}]^T$.
- ***The Gram product:***

$$\langle\langle \mathbf{x},\mathbf{y}\rangle\rangle := [\langle x_i, y_j\rangle] \in \mathbb{C}^{m\times n}$$

for $\mathbf{x} = [x_1,\ldots,x_m]^T \in H^m$, $\mathbf{y} = [y_1,\ldots,y_m]^T \in H^n$;
if $\mathbf{x}\in H^{k\times\ell}$ and $\mathbf{y}\in H^{m\times n}$, we define the block matrix

$$\langle\langle \mathbf{x},\mathbf{y}\rangle\rangle := \langle\langle \mathrm{vec}(\mathbf{x}),\mathrm{vec}(\mathbf{y})\rangle\rangle = [\langle\langle \mathbf{x}_{:\,i},\mathbf{y}_{:\,j}\rangle\rangle]_{1\le i\le \ell, 1\le j\le n} \in \left(\mathbb{C}^{k\times m}\right)^{\ell\times n}.$$

NOTE: The (p,q)-entry is in the $(\lceil p/k\rceil, \lceil q/m\rceil)$-block at the position $(p-\lfloor p/k\rfloor k, q-\lfloor q/m\rfloor m)$.

- If $\mathbf{x} = [x_1, \ldots, x_m]^T \in H^m$, $\mathbf{y} = [y_1, \ldots, y_m]^T \in H^n$, then
$$\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \langle\langle \mathbf{y}, \mathbf{x} \rangle\rangle^*$$
and for compatible size matrices A, B we have
$$A\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle B = \langle\langle A\mathbf{x}, B^*\mathbf{y} \rangle\rangle.$$
In particular, we get a natural quaternary product $[A, \mathbf{x}, \mathbf{y}, B]$.
- ***Ampliation:*** if $T \in \mathcal{L}(H_1, H_2)$ and $m, n \in \mathbb{N}$, then we define the ***ampliation*** of T:
$$T^{(m\times n)} : H_1^{m\times n} \ni [x_{ij}] \longmapsto [T(x_{ij})] \in H_2^{m\times n}.$$
In particular, we have ampliations of matrix multiplication from the left: if $A \in \mathbb{C}^{p\times k}$ and $B = [B_{ij}] \in \left(\mathbb{C}^{k\times m}\right)^{\ell\times n}$, then
$$A^{(\ell\times n)}B = \left[AB_{ij}\right] \in (\mathbb{C}^{p\times m})^{\ell\times n}.$$
- If $\mathbf{x} = [x_{ij}] \in H^{m\times n}$, then $\mathbf{Span}(\mathbf{x})$ denotes the closed linear span of all x_{ij} (similar notation for a list or a set of $\mathbf{x}$'s). Then $\mathbf{y} \in \mathbf{Span}(\mathbf{x})^\ell \iff \exists C_1, \ldots, C_n \in \mathbb{C}^{\ell\times m} \ : \ \mathbf{y} = \sum_{j=1}^n C_j x_{:j}$.(NOTE: Uniqueness of $C_j \Leftrightarrow$ linear independence of all x_{ij}.)

Hilbert frames
(Duffin, Schaeffer 1950s; recent - Pete Casazza and others)

Let J be a linearly ordered finite or countable set. An ordered collection $F = (x_j)_{j \in J} \subset H$ is a *frame* if $\exists A, B > 0$ (***frame bounds***) s.t.

$$A\|x\|^2 \leq \sum_{j \in J} |\langle x, x_j \rangle|^2 \leq B\|x\|^2, \qquad x \in H.$$

Any frame for H is linearly dense in H. The ***analysis operator***:

$$T_F : H \ni x \mapsto \big(\langle x, x_j \rangle\big)_{j \in J} \in \ell^2(J)$$

is injective and its adjoint is the surjective ***synthesis operator***:

$$T_F^* : \ell^2(J) \ni (\lambda_j)_{j \in J} \mapsto \sum_{j \in J} \lambda_j x_j \in H.$$

The *frame operator* is defined as $S_F = T_F^* T_F$ and it is an isomorphism. Moreover $\|T_F\| \leq \sqrt{B}$, $\|T_F^*\| \leq \sqrt{B}$ and $\|S_F\| \leq B$. Frames satisfy the ***resolution of identity***:

$$x = \sum_{j \in J} \underbrace{\langle x, S_F^{-1}(x_j) \rangle}_{\text{canonical coeffs.}} x_j, \qquad x \in H.$$

Such a representation may be very redundant!!!

Block frames

Extension of earlier joint work with M. Matsuura and Y. Okabe (2009)

Let $\mathbf{x}_j = [x_{1\,j}, \ldots, x_{d\,j}]^T \in H^d$, $j \in J$ and $M = \mathbf{Span}(\mathbf{x}_j\ :\ j \in J)$. We say that $F = \{\mathbf{x}_j : j \in J\}$ is a ***block frame*** for M^d, if the family $\underline{F} = \{x_{ij} : i = 1, \ldots, d,\ j \in J\}$ is a frame for M. We call $\underline{F}$ is the ***underlying frame*** for F.

NOTE: ***a block frame is not necessarily a frame for M^d !!!***

Define the ***block frame operator*** for F as $S_F := \left(S_{\underline{F}}\right)^{(d\times 1)}$. Then

$$S_F(\mathbf{z}) = \sum_{j\in J} \langle\langle \mathbf{z}, \mathbf{x}_j \rangle\rangle \mathbf{x}_j, \qquad \mathbf{z} \in M^d.$$

The operator is bijective with $S_F^{-1} = \left(S_{\underline{F}}^{-1}\right)^{(d\times 1)}$, ***block homogeneous*** (i.e. $S_F(C\mathbf{z}) = CS_F(\mathbf{z})$ for $\mathbf{z} \in M^d,\ C \in \mathbb{R}^{d\times d}$) and ***block self-adjoint*** (i.e. $\langle\langle S_F(\mathbf{x}), \mathbf{y} \rangle\rangle = \langle\langle \mathbf{x}, S_F(\mathbf{y}) \rangle\rangle$) and hence adjoint (because the inner product in H^d is the trace of the Gram product).

NOTE: adjoint $\not\Rightarrow$ block self adjoint

The block versions of the ***analysis and synthesis operators***:

$$T_F : M^d \ni \mathbf{z} \mapsto \left(\langle\langle \mathbf{z}, \mathbf{x}_j \rangle\rangle\right)_{j\in J} \in \ell^2(J)^{d\times d}.$$

and

$$T_F^* : \ell^2(J)^{d\times d} \ni \left(C_j\right)_{j\in J} \mapsto \sum_{j\in J} C_j \mathbf{x}_j \in M^d.$$

It can be shown that T_F is injective, T_F^* is surjective, $S_F = T_F^* T_F$ and T_F^* is the adjoint of T_F: indeed, for $C = \left(C_j\right)_{j\in J} \in \ell^2(J)^{d\times d}$:

$$\begin{aligned} \langle T_F(\mathbf{z}), C \rangle &= \sum_{j\in J} \text{trace}\left[\langle\langle \mathbf{z}, \mathbf{x}_j \rangle\rangle C_j^*\right] = \sum_{j\in J} \text{trace}\langle\langle \mathbf{z}, C_j \mathbf{x}_j \rangle\rangle \\ &= \text{trace}\langle\langle \mathbf{z}, \sum_{j\in J} C_j \mathbf{x}_j \rangle\rangle = \langle \mathbf{z}, T_F^*(C) \rangle. \end{aligned}$$

If in addition F, G are block frames for M, N respectively, then $F \cup G$ is a block fame for $M \oplus N$ and

$$S_{F\cup G}(\mathbf{x}+\mathbf{y}) = S_F(\mathbf{x}) + S_G(\mathbf{y}) \qquad \mathbf{x} \in M^d, \ \mathbf{x} \in N^d. \qquad (1)$$

If M is a closed subspace of H, then by $\mathbf{P}_M$ we denote the orthogonal projection of H onto M. We have

$$\mathbf{P}_{M^d} = (\mathbf{P}_M)^{(d\times 1)} : H^d \longrightarrow H^d.$$

The ***block projection error*** is the matrix:

$$\mathbf{Err}_{M^d}(\mathbf{x}) = \langle\langle \mathbf{P}_{(M^\perp)^d}(\mathbf{x}), \mathbf{P}_{(M^\perp)^d}(\mathbf{x}) \rangle\rangle = \langle\langle \mathbf{P}_{(M^\perp)^d}(\mathbf{x}), \mathbf{x} \rangle\rangle.$$

The next three theorems can be seen as a toolbox of formulas most useful in manipulation of block frames. The first one describes the two most basic geometric properties of block projections and block frame operators:

Theorem 1

Let $F = (\mathbf{x}_j)_{j \in J}$ be a block frame for M^d and let $d' \in \mathbb{N}$.

1. For all $\mathbf{y} \in H^{d'}$:

$$\mathbf{P}_{M^{d'}}(\mathbf{y}) = \sum_{j \in J} \langle\langle \mathbf{y}, S_F^{-1}(\mathbf{x}_j) \rangle\rangle \mathbf{x}_j$$

and $\mathbf{P}_{M^{d'}}$ is block self-adjoint.

2. Given $\mathbf{y} \in H^{d'}$, define the affine subspace

$$N(\mathbf{y}) = \left\{ Y = (Y_j)_{j \in J} \in \ell^2(J)^{d' \times d} \;:\; \mathbf{P}_{M^{d'}}(\mathbf{y}) = \sum_{j \in J} Y_j \mathbf{x}_j \right\}.$$

Then the orthogonal projection of the origin in $\ell^2(J)^{d' \times d}$ onto $N(\mathbf{y})$ has the canonical block frame coefficients of $\mathbf{y}$ as components

$$Y = (Y_j)_{j \in J} = \left(\langle\langle \mathbf{y}, S_F^{-1}(\mathbf{x}_j) \rangle\rangle \right)_{j \in J}$$

and furnishes the minimum norm solution to the equation

$$\mathbf{P}_{M^{d'}}(\mathbf{y}) = \sum_{j \in J} Y_j \mathbf{x}_j.$$

The next theorem describes in terms of block frame coefficients how the projection $\mathbf{P}_{M^d}$ and the block projection error $\mathbf{Err}_{M^d}$ change, when an additional block vector is appended to M.

We will use the ampersand symbol & to denote ***concatenation*** of linearly ordered sets. So if F and G are such sets, then $F\&G$ is also linearly ordered with any element of F regarded as preceding any element of G. If $G = \{g\}$ is a singleton, we will write $F\&g$ rather that $F\&G$.

For any matrix $A \in \mathbb{C}^{m\times n}$, let $A^\dagger$ denote the ***Moore-Penrose pseudoinverse*** of A. If A is treated as an element of $\mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$, then

$$A^\dagger = \left(A\Big|_{\mathrm{Ker}(A)^\perp} : \underbrace{\mathrm{Ker}(A)^\perp}_{=\mathcal{R}(A^*)} \longrightarrow \mathcal{R}(A) \right)^{-1} \circ \mathbf{P}_{\mathcal{R}(A)},$$

where $\mathcal{R}(A)$ denotes the range of A.

Theorem 2

Let $F = (\mathbf{x}_j)_{j \in J}$ be a block frame for M^d and let $\mathbf{y} \in H^d$. Let $M_\mathbf{y} = M + \mathbf{Span}(\mathbf{y})$, $F_\mathbf{y} = F \& \mathbf{y}$ and let $d' \in \mathbb{N}$. Then

$$\mathbf{P}_{M_\mathbf{y}^{d'}}(\mathbf{z}) = \mathbf{P}_{M^{d'}}(\mathbf{z}) + \langle\langle \mathbf{z}, S_{F_\mathbf{y}}^{-1}(\mathbf{y}) \rangle\rangle \mathbf{P}_{(M^\perp)^d}(\mathbf{y}), \qquad \mathbf{z} \in H^{d'},$$

If $\mathbf{z} \in H^{d'}$, where $F_\mathbf{z} = F \& \mathbf{z}$, $M_\mathbf{z} = M + \mathbf{Span}(\mathbf{z})$, then

$$\mathbf{Err}_{M_\mathbf{y}^{d'}}(\mathbf{z}) = \left(\mathbf{I}_{d'} - \langle\langle \mathbf{z}, S_{F_\mathbf{y}}^{-1}(\mathbf{y}) \rangle\rangle \langle\langle \mathbf{y}, S_{F_\mathbf{z}}^{-1}(\mathbf{z}) \rangle\rangle \right) \mathbf{Err}_{M^{d'}}(\mathbf{z}),$$

where $\mathbf{I}_{d'}$ denotes here the $d' \times d'$ identity matrix. Moreover

$$\langle\langle \mathbf{z}, S_{F_\mathbf{y}}^{-1}(\mathbf{x}_j) \rangle\rangle = \langle\langle \mathbf{z}, S_F^{-1}(\mathbf{x}_j) \rangle\rangle - \langle\langle \mathbf{z}, S_{F_\mathbf{y}}^{-1}(\mathbf{y}) \rangle\rangle \langle\langle \mathbf{y}, S_F^{-1}(\mathbf{x}_j) \rangle\rangle,$$

$$\langle\langle \mathbf{z}, S_{F_\mathbf{y}}^{-1}(\mathbf{y}) \rangle\rangle \mathbf{Err}_{M^d}(\mathbf{y}) = \mathbf{Err}_{M^{d'}}(\mathbf{z}) \langle\langle S_{F_\mathbf{z}}^{-1}(\mathbf{z}), \mathbf{y} \rangle\rangle,$$

for all $j \in J$. In particular, if $\mathbf{y} \perp M^d$, then

$$\mathbf{P}_{(M_\mathbf{y})^{d'}}(\mathbf{z}) = \mathbf{P}_{M^{d'}}(\mathbf{z}) + \langle\langle \mathbf{z}, \mathbf{y} \rangle\rangle \langle\langle \mathbf{y}, \mathbf{y} \rangle\rangle^\dagger \mathbf{y}, \qquad \mathbf{z} \in H^{d'}.$$

If $F = \{\mathbf{x}_j : j \in J\}$ is a ***block frame*** for M^d and $\#J = n$, then F can be identified with $\mathbf{x} = [\mathbf{x}_1, \dots, \mathbf{x}_m] \in H^{d \times n}$.
The corresponding ***block Gram matrix*** is the block matrix of size $(nd \times nd)$

$$R_F = \langle\langle \mathbf{x}, \mathbf{x} \rangle\rangle$$

whose (i, j)-th block of size $(d \times d)$ is

$$R_F(i, j) = \langle\langle \mathbf{x}_i, \mathbf{x}_j \rangle\rangle$$

Clearly $R_F = R_{\underline{F}}$, and thus every positive semi-definite Hermitian matrix of size $(nd \times nd)$ is the block Gram matrix for some $\mathbf{x}_1, \dots, \mathbf{x}_n$.

The next theorem is an alternative block frame interpretation of the ***blueprint algorithm*** (M. Matsuura 2005):

Theorem 3

If $F = \{\mathbf{x}_j : 1 \le j \le n\}$ is a block frame for $M^d \subset H^d$, $\mathbf{x}_{n+1} \in H^d$ and all entries of $R = R_{F\&\mathbf{x}_{n+1}}$ are known, then one can calculate recursively the matrices

$$\lambda_R(j) = \langle\langle \mathbf{x}_{n+1}, S_F^{-1}(\mathbf{x}_j)\rangle\rangle \in \mathbb{R}^{d\times d}, \qquad j = 1, \dots, n.$$

Furthermore,

$$R(n+1,k) = \sum_{j=1}^{n} \lambda_R(j) R(j,k), \qquad k = 1, \dots, n,$$

$$R(n+1,n+1) = \sum_{j=1}^{n} \lambda_R(j) R(j,n+1) + \mathbf{Err}_{M^d}(\mathbf{x}_{n+1}).$$

Three observations are useful in practice:

1. The calculation of $\lambda_R(j)$ uses as the only input data the entries of the block Gram matrix R, with the exception of $\langle\langle \mathbf{x}_{n+1}, \mathbf{x}_{n+1} \rangle\rangle$.
2. If $n = 1$, then $\lambda_R(1) = \langle\langle \mathbf{x}_2, \mathbf{x}_1 \rangle\rangle \langle\langle \mathbf{x}_1, \mathbf{x}_1 \rangle\rangle^\dagger$ (by Theorem 2) and $\mathbf{Err}_{M^d}(\mathbf{x}_2)$ can be calculated from the definition. The induction step is a little more complex.
3. Let $F^{(i)} = \{\mathbf{x}_j^{(i)} : 1 \leq j \leq n\}$, where $i = 1, 2$, be finite block frames for $M_{(i)}^d \subset H^d$. Let $\mathbf{x}_{n+1}^{(i)} \in H^d$, for $i = 1, 2$. If

$$R_{F^{(1)} \& \mathbf{x}_{n+1}^{(1)}} = R_{F^{(2)} \& \mathbf{x}_{n+1}^{(2)}},$$

   then

$$\|\mathbf{P}_{M_{(1)}^d}(\mathbf{x}_{n+1}^{(1)})\| = \|\mathbf{P}_{M_{(2)}^d}(\mathbf{x}_{n+1}^{(2)})\|.$$

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