Hessian Manifolds and Their Submanifolds

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1. Hessian manifolds

Statistical manifolds, introduced in 1985 by Amari, have been studied in terms of information geometry. Since the geometry of such manifolds includes the notion of dual connections, also called conjugate connections in affine geometry, it is closely related to affine differential geometry. Moreover, a statistical structure being a generalization of a Hessian one, it connects Hessian geometry.

A *statistical manifold* is a Riemannian manifold $(\tilde{M}, g)$ of dimension $n+k$, endowed with a pair of torsion-free affine connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$ satisfying

$$Zg(X,Y) = g(\tilde{\nabla}_Z X, Y) + g(X, \tilde{\nabla}^*_Z Y),$$

for any X, Y and $Z \in \Gamma(T\tilde{M})$.

$$g(\tilde{R}^*(X,Y)Z, W) = -g(Z, \tilde{R}(X,Y)W).$$

A statistical manifold is said to be of constant curvature $c \in \mathbb{R}$ if

$$R(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y], \forall X, Y, Z \in \Gamma(T\tilde{M}).$$

A statistical manifold of constant curvature 0 is called a Hessian manifold.

Sectional curvature on statistical manifolds

Let $(\tilde{M}, g)$ be a statistical manifold of dimension $n + k$ endowed with dual connections $\tilde{\nabla}$ and $\tilde{\nabla}^*$. Unfortunately, the $(0, 4)$-tensor field $g(R(X, Y)Z, W)$ is not skew-symmetric with respect to Z and W. Then we cannot define a sectional curvature on $\tilde{M}$ by the standard definition.

We define a skew-symmetric $(0, 4)$-tensor field on $\tilde{M}$ by

$$T(X, Y, Z, W) = \frac{1}{2}[g(R(X, Y)Z, W) + g(R^*(X, Y)Z, W)]$$

for all $X, Y, Z, W \in \Gamma(T\tilde{M})$.

Then we are able to define a sectional curvature on $\tilde{M}$ by the formula

$$K(X \wedge Y) = \frac{T(X,Y,X,Y)}{g(X,X)g(Y,Y) - g^2(X,Y)},$$

for any linearly independent tangent vectors X, Y at $p \in \tilde{M}$.

B. Opozda [Ann. Global Anal. Geom., 2015],
B. Opozda [Linear Algebra Appl., 2016].

Hessian curvature

A statistical structure $(\tilde{\nabla}, g)$ of constant curvature 0 is known as a *Hessian structure.* It follows that $(\tilde{\nabla}^*, g)$ is also Hessian.
On a Hessian manifold $(\tilde{M}^m, \tilde{\nabla})$, let $\gamma = \tilde{\nabla} - \tilde{\nabla}^0$. The tensor field $\tilde{Q}$ of type (1,3) defined by

$$\tilde{Q}(X, Y) = [\gamma_X, \gamma_Y], \quad X, Y \in \Gamma(T\tilde{M}^m)$$

is called the *Hessian curvature tensor* for $\tilde{\nabla}$ (see Shima, Opozda), which satisfies

$$\tilde{R}(X, Y) + \tilde{R}^*(X, Y) = 2\tilde{R}^0(X, Y) + 2\tilde{Q}(X, Y).$$

By using the Hessian curvature tensor $\tilde{Q}$, Hessian sectional curvatures can be defined on a Hessian manifold. In fact, let $p \in \tilde{M}^m$ and π a plane section in $T_p\tilde{M}^m$. Take an orthonormal basis $\{X, Y\}$ of π. Then the Hessian sectional curvature is defined by

$$\tilde{K}(\pi) = g(\tilde{Q}(X, Y)Y, X),$$

which is independent of the choice of an orthonormal basis.
A Hessian manifold has constant Hessian sectional curvature c if and only if (see Shima)

$$\tilde{Q}(X, Y, Z, W) = \frac{c}{2}\{g(X, Y)g(Z, W) + g(X, W)g(Y, Z)\},$$

for all vector fields on $\tilde{M}^m$.

It is known (Shima) that a Hessian manifold of constant Hessian sectional curvature c is a Riemannian space form of constant sectional curvature $-c/4$.

2. Statistical submanifolds

Let M be an n-dimensional submanifold of $\tilde{M}$. Then, for any $X, Y \in \Gamma(TM)$, the corresponding Gauss formulae are:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y),$$

where h and $h^*$are symmetric and bilinear, called the imbedding curvature tensor of M in $\tilde{M}$ for $\tilde{\nabla}$ and the imbedding curvature tensor of M in $\tilde{M}$ for $\tilde{\nabla}^*$, respectively.
It is known that (∇, g) and (∇^*, g) are dual statistical structures on M.

Since h and $h^*$are bilinear, we have the linear transformations A_ξ and A^*_ξ defined by

$$g(A_\xi X, Y) = g(h(X, Y), \xi),$$

$$g(A^*_\xi X, Y) = g(h^*(X, Y), \xi),$$

for any $\xi \in \Gamma(TM^\perp)$ and $X, Y \in \Gamma(TM)$.
The Weingarten formulae are

$$\tilde{\nabla}_X \xi = -A^*_\xi X + \nabla^\perp_X \xi,$$

$$\tilde{\nabla}^*_X \xi = -A_\xi X + \nabla^{*\perp}_X \xi,$$

for any $\xi \in \Gamma\left(TM^\perp\right)$ and $X \in \Gamma\left(TM\right)$. The connections $\nabla^\perp$ and $\nabla^{*\perp}$ are Riemannian dual connections with respect to induced metric on $\Gamma(T^\perp M)$.

The corresponding Gauss, Codazzi and Ricci equations are given by the following.
Let $\tilde{\nabla}$ be a dual connection on $\tilde{M}$ and ∇ the induced connection on M. Let $\tilde{R}$ and R be the Riemannian curvature tensors of $\tilde{\nabla}$ and ∇, respectively. Then,

$$g\left(\tilde{R}(X,Y)Z,W\right)=g\left(R(X,Y)Z,W\right)+g\left(h(X,Z),h^{*}(Y,W)\right)-$$
$$-g\left(h^{*}(X,W),h(Y,Z)\right),$$
$$\left(\tilde{R}(X,Y)Z\right)^{\perp}=\nabla_X^{\perp}h(Y,Z)-h(\nabla_XY,Z)-h(Y,\nabla_XZ)-$$
$$-\left\{\nabla_Y^{\perp}h(Y,Z)-h(\nabla_YX,Z)-h(X,\nabla_YZ)\right\},$$
$$g\left(R^{\perp}(X,Y)\xi,\eta\right)=g\left(\tilde{R}(X,Y)\xi,\eta\right)+g\left(\left[A_{\xi}^{*},A_{\eta}\right]X,Y\right),$$

where $R^{\perp}$ is Riemannian curvature tensor on $T^{\perp}M$, $\xi,\eta\in\Gamma\left(T^{\perp}M\right)$ and $\left[A_{\xi}^{*},A_{\eta}\right]=A_{\xi}^{*}A_{\eta}-A_{\eta}A_{\xi}^{*}$.

Let $p \in M$ and $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_{n+k}\}$ orthonormal bases of T_pM and $T_p^{\perp}M$, respectively.
The mean curvature vector fields are given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right) e_{n+\alpha}, \ \ h_{ij}^{\alpha} = \tilde{g}\left(h(e_i, e_j), e_{n+\alpha}\right),$$

$$H^* = \frac{1}{n}\sum_{i=1}^{n} h^*(e_i, e_i) = \frac{1}{n}\sum_{\alpha=1}^{k}\left(\sum_{i=1}^{n} h_{ii}^{*\alpha}\right) e_{n+\alpha},$$

$$h_{ij}^{*\alpha} = \tilde{g}\left(h^*(e_i, e_j), e_{n+\alpha}\right).$$

Proposition 1 (M.E. Aydin, A. Mihai, I. Mihai [Filomat, 2015]). *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$ of constant curvature $c \in \mathbb{R}$. Assume that the imbedding curvature tensors h and h^* satisfy*

$$h(X,Y) = g(X,Y)H \text{ and } h^*(X,Y) = g(X,Y)H^*,$$

for any $X, Y \in \Gamma(TM)$. Then M is also a statistical manifold of constant curvature $c + g(H, H^*)$ whenever $g(H, H^*)$ is constant.

Curvature invariants on statistical submanifolds

Curvature invariants are the main Riemannian invariants and the most natural ones. Curvature invariants also play key roles in physics. For instance, the magnitude of a force required to move an object at constant speed, according to Newton's laws, is a constant multiple of the curvature of the trajectory. The motion of a body in a gravitational field is determined, according to Einstein's general theory of relativity, by the curvatures of spacetime. All sort of shapes, from soap bubbles to red cells are determined by various curvatures.

Classically, among the curvature invariants, the most studied were sectional, scalar and Ricci curvatures.

M.E. Aydin, A. Mihai, I. Mihai [Filomat, 2015] studied classical curvature invariants on statistical submanifolds in statistical manifolds of constant curvature.

The *scalar curvature* of a statistical submanifold is given by

$$\tau = \sum_{1 \leq i < j \leq n} g(R(e_i, e_j)e_j, e_i) = \tau^*.$$

Proposition 2. *Let $\tilde{M}(c)$ be an $(n+k)$-dimensional statistical manifold of constant curvature $c \in \mathbb{R}$ and M an n-dimensional statistical submanifold of $\tilde{M}(c)$. We have*

$$2\tau \geq n(n-1)c + n^2 g(H, H^*) - \|h\| \|h^*\|.$$

The *Ricci tensor* S (of type $(0,2)$) of M is defined by

$$S(Y,Z) = \text{trace}\,\{X \mapsto R(X,Y)Z\}.$$

Theorem 3. *Let M be an n-dimensional statistical submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$. For each $X \in T_p(M)$ unit, we have*

$$Ric(X) \geq 2Ric^0(X) - \frac{n^2}{8}g(H,H) - \frac{n^2}{8}g(H^*,H^*) + (n-1)c-$$

$$-2(n-1)\max \tilde{K}^0(X \wedge \cdot).$$

Particular Case: M is a minimal submanifold. Because $H^0 = 0$, we have $H + H^* = 0$. Then the previous inequality implies:

Corollary 4. *Let M be a minimal n-dimensional statistical submanifold of an $(n+k)$-dimensional statistical manifold $\tilde{M}(c)$. For each $X \in T_p(M)$ unit, we have*

$$Ric(X) \geq 2Ric^0(X) + \frac{n^2}{4} g(H, H^*) + (n-1)c - 2(n-1) \max \tilde{K}^0(X \wedge \cdot).$$

M.E. Aydin, I. Mihai [Math. Ineq. Appl., 2019] proved a Wintgen inequality for statistical surfaces.
For a statistical surface M^2, we can define a Gauss curvature by

$$G = K(e_1 \wedge e_2),$$

for any orthonormal frame $\{e_1, e_2\}$ on M^2.

We state a version of **Euler inequality** for surfaces in 3-dimensional statistical manifolds of constant curvature.

Proposition 5. *Let M^2 be surface in a 3-dimensional statistical manifold of constant curvature c. Then its Gauss curvature satisfies*:

$$G \leq 2||H|| \cdot ||H^*|| - c.$$

A normal curvature of a statistical surface M^2 in an orientable 4-dimensional statistical manifold $\tilde{M}^4$ can be defined as follows. Let $\{e_1, e_2, e_3, e_4\}$ be a positive oriented orthonormal frame on $\tilde{M}^4$ such that e_1, e_2 are tangent to M^2.

$$G^{\perp} = \frac{1}{2}\left[g\left(R^{\perp}(e_1, e_2)\, e_3, e_4\right) + g\left(R^{*\perp}(e_1, e_2)\, e_3, e_4\right)\right].$$

Theorem 6. (**Wintgen inequality for statistical surfaces**) *Let M^2 be a statistical surface in a 4-dimensional statistical manifold $\left(\tilde{M}^4, c\right)$ of constant curvature c. Then*

$$G + \left|G^{\perp}\right| + 2G^0 \leq \frac{1}{2}\left(\|H\|^2 + \|H^*\|^2\right) - c + 2\tilde{K}^0\left(e_1 \wedge e_2\right).$$

In particular, for $c = 0$ we derive the following.

Corollary 7. *Let M^2 be a statistical surface of a Hessian 4-dimensional statistical manifold $\tilde{M}^4$.* Then:

$$G + \left|G^{\perp}\right| + 2G^0 \leq \frac{1}{2}\left(\|H\|^2 + \|H^*\|^2\right).$$

Generalized Wintgen inequality (M.E. Aydin, A. Mihai, I. Mihai [Bull. Math. Sci., 2017]).

Notations:

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} K(e_i \wedge e_j),$$

where τ is the scalar curvature, and the normalized scalar normal curvature by

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1 \le i < j \le n} \sum_{1 \le \alpha < \beta \le m} (R^{\perp}(e_i, e_j, \xi_\alpha, \xi_\beta))^2}.$$

Theorem 8. (**Generalized Wintgen inequality for statistical submanifolds**)
Let M^n be a submanifold in a statistical manifold $\left(\tilde{M}^{n+m}, c\right)$ of constant curvature c. Then

$$\rho^{\perp} + 3\rho \leq \frac{15}{2}\|H\|^2 + \frac{15}{2}\|H^*\|^2 + 12g\left(H, H^*\right) - 3c + 30\left(\tilde{\rho}^0 - \rho^0\right).$$

The equality case was characterized in terms of the shape operator.

3. Chen inequalities on statistical submanifolds in Hessian manifolds of constant Hessian curvature

A. Mihai, I. Mihai [Mathematics **6**, 2018]

Theorem 9. *Let M^n be a statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. Then the scalar curvature satisfies*:

$$2\tau \geq -\frac{n^2}{4}\|H - H^*\|^2 + n(n-1)\frac{c}{4}.$$

Moreover, the equality holds at any pont $p \in M^n$ if and only if $h = h^*$. In this case, the scalar curvature is constant, $2\tau = n(n-1)\frac{c}{4}$.

Theorem 10. *Let M^n be a statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. Then the Ricci curvature of a unit vector $X \in T_pM^n$ satisfies*:

$$Ric(X) \geq (n-1)\frac{c}{2} - \frac{n^2}{8}\|H\|^2 - \frac{n^2}{8}\|H^*\|^2 + Ric^0(X).$$

Moreover, the equality case holds if and only if

$$\begin{cases} 2h(X,X) = nH(p),\ h(X,Y) = 0,\ \forall Y \in T_pM^n \text{ orthogonal to } X, \\ 2h^*(X,X) = nH^*(p),\ h^*(X,Y) = 0,\ \forall Y \in T_pM^n \text{ orthogonal to } X. \end{cases}$$

A Chen first inequality

Theorem 11 (B.Y. Chen, A. Mihai, I. Mihai [submitted]).
Let M^n be a statistical submanifold of a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature. Then

$$\tau - K(\pi) \geq \tau_0 - K_0(\pi) + (n-2)(n+1)\frac{c}{4} - \frac{n^2(n-2)}{4(n-1)}(\|H\|^2 + \|H^*\|^2).$$

The proof is based on Gauss equation and on the algebraic lemma below.

Lemma. *Let $n \geq 3$ be an integer and let $a_1, ..., a_n$ be n real numbers. Then one has:*

$$\sum_{1 \leq i < j \leq n} a_i a_j - a_1 a_2 \leq \frac{n-2}{2(n-1)} \left(\sum_{i=1}^{n} a_i \right)^2 .$$

Moreover, the equality holds if and only if $a_1 + a_2 = a_3 = ... = a_n$.

An immediate consequence of Theorem 11 is the following.

Corollary. *Let M^n be a statistical submanifold in a Hessian manifold $\tilde{M}^m(c)$ of constant Hessian curvature c. If there exist a point $p \in M^n$ and a plane section π at p such that*

$$(\tau - K(\pi)) - (\tau_0 - K_0(\pi)) < (n-2)(n+1)\frac{c}{4},$$

then M^n is non-minimal in $\tilde{M}^m(c)$, i.e., either $H \neq 0$ or $H^* \neq 0$.

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