Bloch functions of a bounded symmetric domain

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Joint work with

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4 Sep. 2014

at Queen Mary College, Lodon,

1980 Timoney [35]

Let $\Omega \subset \mathbb{C}^n$ be bounded homogeneous domain, $f : \Omega \to \mathbb{C}$ be holomorphic.

The following conditions on f are all equivalent:

(1) The function f is a Bloch function.

(2) The radii of the schlicht discs in the range of f are bounded above.

(3) As a function from the metric space (Ω, ρ_Ω) to the metric space $(\Omega, \|\cdot\|_e)$, the function f is uniformly continuous, where ρ_Ω is the Poincaré distance.

(4) The family $\{(f \circ \phi)(z) - f(\phi(z_0)); \phi \in \mathrm{Aut}(\Omega)\}$ is a normal family for some $z_0 \in \Omega$.

(5) $\sup\{\|D(f\circ\phi)(z_0)\|; \phi\in \mathrm{Aut}(\Omega)\} < \infty$.

(6) The family $\{f\circ g; g : U \to \Omega$, hol $\}$ is a family of Bloch functions with uniformly bounded Bloch norm.

(7) The family $\{(f\circ g)(z) - f(g(0)); g : U \to \Omega$, hol $\}$ is a normal family.

(8) $\sup\{Q_f^h(z) : z\in\Omega\} < \infty$,

where $Q_f^h(z) := \sup\left\{\frac{|Df(z)x|}{H_z(x,\bar{x})^{1/2}} : x\in\mathbb{C}^n\setminus\{0\}\right\}$,

H_z is the Bergman metric at z,

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f maps a domain in $\mathbb{U}$ biholomorphically onto a disc $D = D(f(z_0), r(f)) \subset f(\mathbb{U})$; center $f(z_0)$, radius $r(f)$,

(called a *schlicht disc*)

i.e., $\exists \varphi \in H(D)$ s.t. $f(\varphi(z)) = z$ for $z \in D$.

$r(f)$ is greater than some positive absolute constant.

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$r(f)$ is greater than some positive absolute constant.

The 'best possible' constant B for all such functions,
$\mathrm{B} := \inf\{r(f) : f$ is holomorphic on $\mathbb{U}$ and $f'(0) = 1\}$,
is called the **Bloch constant**.

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schlicht disc radius $r(f) \leq (1 - |z_0|^2)|f'(z_0)| \leq 4r(f)$

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The classical Bloch space $\mathcal{B}$

the space of holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ satisfying

$$\|f\|_{Bloch} := \sup_{z\in\mathbb{U}} (1 - |z|^2)|f'(z)| < \infty$$

endowed with the norm

$$\|f\|_{\mathcal{B}} := |f(0)| + \|f\|_{Bloch} < \infty$$

so that $\mathcal{B} = (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ becomes a Banach space.

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1986 Holland and Walsh

Bloch functions on the open unit disc $\mathbb{U}$ in $\mathbb{C}$

$\Leftrightarrow$

$$\sup_{z,w\in\mathbb{U},z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|}<\infty$$

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The concept of a Bloch function on $\mathbb{U}$ has been extended to various complex domains in higher dimensions.

Especially,

1980 Timoney [35]
for $\mathbb{C}$-valued Bloch functions $f : \Omega \to \mathbb{C}$ on a finite dim bounded homogeneous domain $\Omega \subset \mathbb{C}^n$,

(8) $\sup\{Q_f^h(z) : z \in \Omega\} < \infty,$

where $$Q_f^h(z) := \sup\left\{\frac{|Df(z)x|}{H_z(x,\bar{x})^{1/2}} : x \in \mathbb{C}^n \setminus \{0\}\right\},$$

H_z is the Bergman metric at z,

1992 X.Y.Liu [28]

for hol $f : \mathbb{B}^n \longrightarrow \mathbb{C}^n$ **on the Euclidean ball** $\mathbb{B}^n$ **of** $\mathbb{C}^n$**,**

the family $\{f \circ \varphi - (f \circ \varphi)(\mathbf{0}) : \varphi \in \mathrm{Aut}(\mathbb{B}^n)\}$ **is normal.**

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2014 Blasco, Galindo and Miralles [3]

for hol $f : \mathbb{B}_H \to \mathbb{C}$ **on the open unit ball** $\mathbb{B}_H$ **of a Hilbert space** H**,**

$$\|f\|_{Bloch} := \sup_{z \in \mathbb{B}_H} (1 - \|z\|^2)\|Df(z)\| < \infty$$

Preliminaries

Def. (homogeneous)

A domain G is said to be *homogeneous*
if for $\forall x$, $\forall y \in G$, $\exists g \in \mathrm{Aut}(G)$ s.t. $g(x) = y$

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W.Kaup [26] showed that the bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples which are complex Banach spaces equipped with a Jordan triple structure.

A complex Banach space X is a JB*-triple
if, and only if, the open unit ball of X is homogeneous.

About JB*-triple

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A JB*-triple is the cpx Banach space X equipped with triple product $\{\cdot,\cdot,\cdot\} : X^3 \to X$: for $a, b, x, y, z \in X$,

(i) conjugate linear in the middle, linear and symmetric in the others

(ii) $\{a, b, \{x, y, z\}\}$

$= \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$

(iii) $a\Box a : x \in X \mapsto \{a, a, x\} \in X$ **: hermitian with nonneg spectr**

(iv) $\|\{a, a, a\}\| = \|a\|^3$

Kobayashi metric κ for $\mathbb{B}_X$

$$\kappa(z,x) := \inf\{\eta > 0 : \exists\phi \in H(\mathbb{U},\mathbb{B}_X),\ \phi(0) = z, D\phi(0)\eta = x\}$$

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$$\kappa(z,x) := \inf\{\eta > 0 : \exists\phi \in H(\mathbb{U},\mathbb{B}_X),\ \phi(0) = z, D\phi(0)\eta = x\}$$

Then $\kappa(0,x) = \|x\|$ for all $x \in X$.

In particular, $\kappa(z,\alpha x) = |\alpha|\kappa(z,x)$ for all $\alpha \in \mathbb{C}$ gives

$$\kappa(z,x) \leq \frac{\|x\|}{1-\|z\|^2} \quad (z \in \mathbb{B}_X, x \in X). \tag{1}$$

Lemma 1

Let $\mathbf{dim}\ \boldsymbol{X} < \infty$,

$\boldsymbol{H}_{\boldsymbol{z}}(\boldsymbol{x}, \overline{\boldsymbol{x}})$ *denote the Bergman metric on* $\mathbb{B}_{\boldsymbol{X}}$.

Then $\exists \boldsymbol{C} > \mathbf{0}$, *s.t.*

$$\boldsymbol{\kappa}(\boldsymbol{z}, \boldsymbol{x}) \leq \boldsymbol{H}_{\boldsymbol{z}}(\boldsymbol{x}, \overline{\boldsymbol{x}})^{\mathbf{1/2}} \leq \boldsymbol{C}\boldsymbol{\kappa}(\boldsymbol{z}, \boldsymbol{x}), \quad \boldsymbol{z} \in \mathbb{B}_{\boldsymbol{X}}, \boldsymbol{x} \in \boldsymbol{X}.$$

Lemma 1

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Then $\exists \boldsymbol{C} > \mathbf{0}$, *s.t.*

$$\boldsymbol{\kappa(z, x) \leq H_z(x, \overline{x})^{1/2} \leq C\kappa(z, x), \quad z \in \mathbb{B}_X, x \in X.}$$

Remark. <u>**In infinite dimensional bounded domains**</u>**,** <u>**the Bergman metric is not available.**</u>

We use the Kobayashi metric instead to circumvent this difficulty.

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The Kobayashi distance ρ on $\mathbb{B}_X$, which is the integral form of the infinitesimal Kobayashi metric κ and generalizes the Poincaré distance $\rho_{\mathbb{U}}$ on $\mathbb{U}$, can be described by a Möbius transformation:

$$\rho(a, b) = \tanh^{-1} \|g_{-a}(b)\| \text{ for } a, b \in \mathbb{B}_X.$$

In particular, $\rho(a, 0) = \tanh^{-1} \|a\|$.

Definition 2 *Let* $\mathbb{B}_X$ *be a bounded symmetric domain realized as the open unit ball of a JB*-triple* X. *A function* $f \in H(\mathbb{B}_X, \mathbb{C})$ *is called a Bloch function if*

$$\sup\{Q_f(z) : z \in \mathbb{B}_X\} < \infty,$$

where

$$Q_f(z) := \sup\left\{\frac{|Df(z)x|}{\kappa(z,x)} : x \in X \setminus \{0\}\right\}.$$

In finite dimensions, this definition coincides with the one in Timoney [35, Definition 3.3], where Q_f^h was defined in terms of the Bergman metric.

For each $f \in H(\mathbb{B}_X, \mathbb{C})$**, we define the** *Bloch semi-norm* **of** f **by**

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} := \sup\left\{\|D(f \circ g)(0)\| : g \in \mathrm{Aut}(\mathbb{B}_X)\right\}.$$

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For finite dimensional domains $\mathbb{B}_X$, the Bloch semi-norm $\|\cdot\|_{\mathcal{B}(\mathbb{B}_X),s}$ defined above is equivalent to the Bloch 'norm' defined in Timoney [35, Definition 4.1], by Lemma 1 and the following lemma.

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Lemma 3 *Given $f \in H(\mathbb{B}_X, \mathbb{C})$, we have*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} = \sup\{Q_f(z) : z \in \mathbb{B}_X\}.$$

We equip $\mathcal{B}(\mathbb{B}_X)$ with a norm,

called the *Bloch norm*, **defined by**

$$\|f\|_{\mathcal{B}(\mathbb{B}_X)} := |f(0)| + \|f\|_{\mathcal{B}(\mathbb{B}_X),s} \quad (f \in \mathcal{B}(\mathbb{B}_X))$$

and call $\mathcal{B}(\mathbb{B}_X)$ the *Bloch space* **on $\mathbb{B}_X$, which is a Banach space.**

Definition 4 *Let* $f : \mathbb{B}_X \to \mathbb{C}$ *be a holomorphic function. A disc* $\Delta = \{w \in \mathbb{C} : |w - w_0| < r\}$, *where* $w_0 \in \mathbb{C}$ *and* $r > 0$, *is called a schlicht disc in the range of* f

$\Leftrightarrow$

there exists a holomorphic map $h : \mathbb{U} \to \mathbb{B}_X$ *so that* $f \circ h$ *maps* $\mathbb{U}$ *biholomorphically onto* Δ.

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For each $z_0 \in \mathbb{B}_X$**, we define a family** $F_f(z_0)$ **of functions on** $\mathbb{B}_X$ **by**

$$F_f(z_0) = \{f \circ g - (f \circ g)(z_0) : g \in \mathrm{Aut}(\mathbb{B}_X)\}.$$

We recall that a family $\mathcal{F} \subset H(\mathbb{U}, \mathbb{C})$ is said to be normal if every sequence in $\mathcal{F}$ admits a subsequence which converges uniformly on compact subsets of $\mathbb{U}$.

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A classical result states that $\mathcal{F}$ is normal if and only if it is uniformly bounded on compact sets in $\mathbb{U}$ (cf. Ahlfors [1, p. 216]).

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The next our theorem give some characterisations, which is a generalization of Timoney [35, Theorem 3.4] to infinite dimensional bounded symmetric domains.

Chu, Hamada, Honda, Kohr (J. Fuct. Anal. (2017))

Theorem 5 Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple X and let $f \in H(\mathbb{B}_X, \mathbb{C})$. The following conditions are equivalent:

(1) f is a Bloch function.

(2) The radii of the schlicht discs in the range of f are bounded above.

(3) f is uniformly continuous as a function from the metric space $(\mathbb{B}_X, \rho)$ to the metric space $(\mathbb{C}, |\cdot|_e)$.

(4) The family $F_f(z_0)$ is bounded on $\mathbb{B}(0, r)$ for $0 < r < 1$ and $z_0 \in \mathbb{B}_X$.

(5) $\|f\|_{\mathcal{B}(\mathbb{B}_X),s} < \infty$.

(6) The family $\{f \circ h : h \in H(\mathbb{U}, \mathbb{B}_X)\}$ consists of Bloch functions on $\mathbb{U}$ with uniformly bounded Bloch semi-norm.

(7) The family $\{f \circ h - (f \circ h)(0) : h \in H(\mathbb{U}, \mathbb{B}_X)\}$ is normal.

(8) $\sup\{Q_f(z) : z \in \mathbb{B}_X\} < \infty$, where

$$Q_f(z) := \sup\left\{\frac{|Df(z)x|}{\kappa(z,x)} : x \in X \setminus \{0\}\right\}.$$

Corollary 6 *(C-H-H-K (CpxAnal.Oper.Theory (2019)))* *Let f be a Bloch function on $\mathbb{B}_X$. Then we have*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

Corollary 6 *(C-H-H-K (CpxAnal.Oper.Theory (2019)))* *Let* f *be a Bloch function on* $\mathbb{B}_X$*. Then we have*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\rho(z, w)}.$$

Using Corollary 6, we obtain the following generalization of Blasco, Galindo and Miralles [2, Theorem 3.3] to infinite dimensional bounded symmetric domains.

Theorem 7 *Let* $\{f_n\}$ *be a sequence of Bloch functions on a bounded symmetric domain* $\mathbb{B}_X$ *converging uniformly on compact subsets of* $\mathbb{B}_X$ *to some* $f \in H(\mathbb{B}_X, \mathbb{C})$.

Theorem 7 *Let* $\{f_n\}$ *be a sequence of Bloch functions on a bounded symmetric domain* $\mathbb{B}_X$ *converging uniformly on compact subsets of* $\mathbb{B}_X$ *to some* $f \in H(\mathbb{B}_X, \mathbb{C})$.

If the sequence $\{\|f_n\|_{\mathcal{B}(\mathbb{B}_X),s}\}$ *is bounded,*

then f *is a Bloch function and*

$$\|f\|_{\mathcal{B}(\mathbb{B}_X),s} \leq \liminf_{n\to\infty} \|f_n\|_{\mathcal{B}(\mathbb{B}_X),s}.$$

In other words, the Bloch seminorm $\|\cdot\|_{\mathcal{B}(\mathbb{B}_X),s}$ *on* $\mathcal{B}(\mathbb{B}_X)$ *is lower semi-continuous in the compact open topology.*

Holland-Walsh's characterization

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1986 Holland and Walsh

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$$\sup_{z,w\in\mathbb{U},z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|}<\infty$$

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$$\sup_{z,w\in\mathbb{U},z\neq w}(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}\frac{|f(z)-f(w)|}{|z-w|}<\infty$$

Let $\mathbb{B}$ be the open unit ball of a complex Banach space. For $f\in H(\mathbb{B},\mathbb{C})$, we define

$$S(f):=\sup_{z,w\in\mathbb{B},z\neq w}(1-\|z\|^2)^{1/2}(1-\|w\|^2)^{1/2}\frac{|f(z)-f(w)|}{\|z-w\|}.$$

2005 Ren and Tu

f **is a Bloch function on the Euclidean balls in** $\mathbb{C}^n$

$$\Longleftrightarrow \qquad S(f) < \infty$$

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$$\iff \qquad \boldsymbol{S(f) < \infty}$$

C-H-H-K (Cpx Anal.Oper.Theory (2019))

Theorem 8 *Let $\mathbb{B}_{\boldsymbol{H}}$ be a Hilbert ball and $\boldsymbol{f} \in \boldsymbol{H}(\mathbb{B}_{\boldsymbol{H}}, \mathbb{C})$. Then $\boldsymbol{f}$ is a Bloch function* $\iff$ $\boldsymbol{S(f) < \infty}$.

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C-H-H-K (Cpx Anal.Oper.Theory (2019))

Theorem 8 *Let* $\mathbb{B}_H$ *be a Hilbert ball and* $f \in H(\mathbb{B}_H, \mathbb{C})$. *Then* f *is a Bloch function* $\iff \quad S(f) < \infty$.

Hilbert space case 2014 Blasco, Galindo and Miralles [3]

f **: a Bloch function on** $\mathbb{B}_H \iff$

$$\|f\|_{\mathcal{B}(\mathbb{B}_H),s} := \sup_{z \in \mathbb{B}_H} (1 - \|z\|^2)\|Df(z)\| < \infty$$

Lemma 9 *Let $\mathbb{B}$ be the unit ball of a cpx Banach space $\boldsymbol{X}$ and let $\boldsymbol{f} \in \boldsymbol{H}(\mathbb{B}, \mathbb{C})$. Then*

$$\boldsymbol{S}(\boldsymbol{f}) < \infty \iff \sup_{\boldsymbol{z} \in \mathbb{B}} (1 - \|\boldsymbol{z}\|^2) \|\boldsymbol{D} \boldsymbol{f}(\boldsymbol{z})\| < \infty.$$

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Bounded Symmetric Domain $\mathbb{B}_X$ Case

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Bounded Symmetric Domain $\mathbb{B}_X$ Case

$\boldsymbol{f}$ **: Bloch ft on** $\mathbb{B}_X \Rightarrow \boldsymbol{\sup_{z \in \mathbb{B}_X}(1 - \|z\|^2)\|Df(z)\| < \infty}$**.**

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$\boldsymbol{f}$ **: Bloch ft on** $\mathbb{B}_X \Rightarrow \boldsymbol{\sup_{z \in \mathbb{B}_X} (1 - \|z\|^2) \|Df(z)\| < \infty}$.

The following result follows from this and Lemma 9.

Proposition 10 *Let* $\boldsymbol{f}$ *be a Bloch function on a bounded symmetric domain* $\mathbb{B}_X$. $\Longrightarrow \boldsymbol{S(f) < \infty}$.

Lemma 9 *Let* $\mathbb{B}$ *be the unit ball of a cpx Banach space* X *and let* $f \in H(\mathbb{B}, \mathbb{C})$. *Then*

$$S(f) < \infty \Longleftrightarrow \sup_{z \in \mathbb{B}} (1 - \|z\|^2)\|Df(z)\| < \infty.$$

Bounded Symmetric Domain $\mathbb{B}_X$ Case

f **: Bloch ft on** $\mathbb{B}_X \Rightarrow \sup_{z \in \mathbb{B}_X}(1 - \|z\|^2)\|Df(z)\| < \infty$**.**

The following result follows from this and Lemma 9.

Proposition 10 *Let* f *be a Bloch function on a bounded symmetric domain* $\mathbb{B}_X$. $\Longrightarrow S(f) < \infty$.

Question. Is the converse of the previous result O.K. ?

Question. **that is,** $S(f) < \infty \implies$ **?**

f **is a Bloch function on a bdd symmetric domain** $\mathbb{B}_X$**.**

Question. that is, $S(f) < \infty \Longrightarrow$?

f is a Bloch function on a bdd symmetric domain $\mathbb{B}_X$.

We recall that the Bergman operator $B(z,z)^{-1/2}$ for the bidisc $\mathbb{D} = \mathbb{U} \times \mathbb{U} \subset \mathbb{C}^2$ is given by

$$B(z,z)^{-1/2}(x) = \left(\frac{x_1}{1-|z_1|^2}, \frac{x_2}{1-|z_2|^2}\right)$$

for $z = (z_1, z_2) \in \mathbb{U} \times \mathbb{U}$ and $x = (x_1, x_2) \in \mathbb{C}^2$.

Example 11 *(counter-example)*

Let $\mathbb{D} = \mathbb{U} \times \mathbb{U}$ *be the bidisc and let* $f : \mathbb{D} \to \mathbb{C}$ *be defined by*

$$f(z_1, z_2) = (1 - z_2) \log \frac{1}{1 - z_1} \qquad (z_1, z_2) \in \mathbb{D}.$$

Then we have $S(f) < \infty$, *but* f *is not a Bloch function.*

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Then we have $S(f) < \infty$, *but* f *is not a Bloch function.*

This counter-example for the bidisc suggests that the criterion of Bloch functions in Theorem 8 for Hilbert balls is atypical for bounded symmetric domains.

Thank you for your attention !

We pray

Prof. Richard Timoney rest in peace.

References

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Thank you for your attention !