Bivariate interpolation and cubature on the even-odd product nodes

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Even-odd nodes

Given $h_0 > h_1 > \cdots > h_m$, $\tilde{h}_0 > \tilde{h}_1 > \cdots > \tilde{h}_m$

$\mathcal{N} = \{(h_n, \tilde{h}_q) : 0 \le n, q \le m\}$ (product nodes)

Lagrange polys degree $\le 2m$

$$P_{n,q}(s,t) = \prod_{j \neq n} \frac{s - h_j}{h_n - h_j} \prod_{j \neq q} \frac{t - \tilde{h}_j}{\tilde{h}_q - \tilde{h}_j}$$

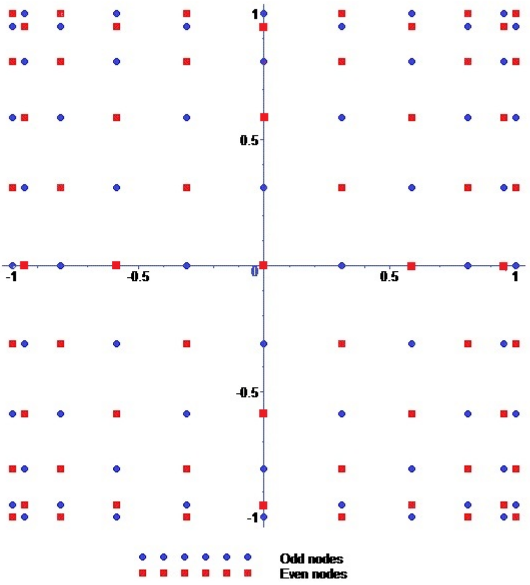
Choose subsets so that Lagrange polys have degree $\le m$.

$\mathcal{N}_0 = \{(h_n, \tilde{h}_q) \in \mathcal{N} : n, q \text{ same parity}\}$ even nodes

$\mathcal{N}_1 = \{(h_n, \tilde{h}_q) \in \mathcal{N} : n, q \text{ opposite parity}\}$ odd nodes

$\mathcal{N} = \mathcal{N}_0 \cup \mathcal{N}_1$, $\mathcal{N}_0 \cap \mathcal{N}_1 = \emptyset$, $0 \le n(\mathcal{N}_0) - n(\mathcal{N}_1) \le 1$

Chebyshev nodes for m = 10



Application

Chebyshev nodes – case $h_n = \tilde{h}_n = \cos(\frac{n\pi}{m}),\ 0 \leq n \leq m$

Morrow-Patterson 1978, Yuan Xu 1996, Bojanov-Petrova 1997 obtained Lagrange polys and cubature for the Chebyshev nodes

Markov's theorem for Banach spaces follows from the alternation of the directional derivative of Lagrange polynomials on rows of even and odd Chebyshev nodes (2010)

Objective is to extend theorems on Lagrange polys and cubature to arbitrary even and odd nodes.

Combined notation for even and odd nodes

$\mathcal{N}_k = \{(h_n, \tilde{h}_q) :\ (n, q) \in Q_k\}, \quad k = 0, 1$

$Q_k = \{(n, q) :\ 0 \leq n, q \leq m, \quad n - q = k \bmod 2\}$ index set

Orthogonal polynomials

Let μ be a measure on $\mathbb{R}$ with $\int_{\mathbb{R}} |x|^n \, d\mu < \infty$ for all $n = 0, 1, \ldots$.

$\mathcal{P}(\mathbb{R})$ - all real polynomials of single variable

Define $(p, q) = \int_{\mathbb{R}} pq \, d\mu$ for $p, q \in \mathcal{P}_m(\mathbb{R})$

Suppose $(p, p) = 0$ implies $p = 0$ for $p \in \mathcal{P}_m(\mathbb{R})$

Def. $\{p_n\}$ is a finite or infinite sequence of orthogonal polys if

$p_0 = 1$, p_n has degree n,

$(p_n, p_m) = 0$ when $n \neq m$

Can start with basis of polys (eg. x^n) and apply Gram-Schmidt

Ex Chebyshev polynomials $T_n(\cos\theta) = \cos n\theta$

$$d\mu = \frac{2\,dx}{\pi\sqrt{1-x^2}}\ dx \text{ on } (-1, 1)$$

$$T_0(x) = 1,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \text{ for } n \geq 1$$

3-term recurrence

Theorem If $\{p_n\}$ is a sequence of orthogonal polynomials then

$$p_0(x) = 1, \quad p_1(x) = a_0 x + b_0,$$

$$p_{n+1}(x) = (a_n x + b_n)p_n(x) - c_n p_{n-1}(x), \quad n \geq 1,$$

$$a_0 > 0, \quad a_n, c_n > 0.$$

Conversely, a sequence of polynomials satisfying the above is orthogonal with respect to some measure μ. (Favard 1935)

Christoffel-Darboux formula $\qquad H_n = (p_n, p_n)$

$$\sum_{n=0}^{N} \frac{p_n(x)p_n(y)}{H_n} = \frac{p_{N+1}(x)p_N(y) - p_N(x)p_{N+1}(y)}{a_N H_N (x-y)}$$

Theorem The zeros of p_n are real and simple and interlace with those of p_{n+1}.

Alternation points

Chebyshev points $h_n = \cos(\frac{n\pi}{m})$, $n = 0, 1, \dots, m$

$T_m(h_n) = (-1)^n$

Def. Numbers $h_0 > h_1 > \cdots > h_m$ are alternation points for orthogonal polynomials $p_0, p_1, \dots, p_m$ if

$$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad j, n = 0, \dots, m$$

The h_n's are the roots of $\pi_m = p_1 p_m - p_{m-1}$

Theorem Any $m + 1$ numbers $h_0 > h_1 > \cdots > h_m$ are alternation points for some orthogonal polynomials $p_0, p_1, \dots, p_m$. (2018)

By the alternation property, the polys below vanish on $\mathcal{N}_k$:

$X_j(s,t) = p_{m-j}(s)\tilde{p}_j(t) - (-1)^k p_j(s)\tilde{p}_{m-j}(t), \quad 0 \le j \le m,$

$Y_0(s,t) = p_1(s)p_m(s) - p_{m-1}(s) = \pi_m(s),$

$Y_j(s,t) = p_{m-j+1}(s)\tilde{p}_j(t) - (-1)^k p_{j-1}(s)\tilde{p}_{m-j}(t), \quad 1 \le j \le m$

Construction of Lagrange polynomials

There exist orthogonal polys $p_0, \ldots, p_m$ and $\tilde{p}_0, \ldots, \tilde{p}_m$ with
$p_{m-j}(h_n) = (-1)^n p_j(h_n), \quad \tilde{p}_{m-j}(\tilde{h}_n) = (-1)^n \tilde{p}_j(\tilde{h}_n)$
Choose $c_j = 1$, so $H_j = a_0 H_0 / a_j$ and similarly for $\tilde{c}_j$ and $\tilde{H}_j$

$$K_n(s,t,u,v) = \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{p_{i-j}(s)\tilde{p}_j(t)p_{i-j}(u)\tilde{p}_j(v)}{H_{i-j}\tilde{H}_j}, \quad 0 \le n \le m$$

$$\text{Define } \; 2G_m(s,t,u,v) = K_{m-1}(s,t,u,v) + K_m(s,t,u,v) + A_m p_m(s) p_m(u) + \tilde{A}_m \tilde{p}_m(t) \tilde{p}_m(v),$$

$$\text{where } A_m = \frac{1}{\tilde{H}_0}\left(\frac{1}{H_0} - \frac{1}{H_m}\right), \quad \tilde{A}_m = \frac{1}{H_0}\left(\frac{1}{\tilde{H}_0} - \frac{1}{\tilde{H}_m}\right)$$

$$P_{n,q}(s,t) = \lambda_{n,q} G_m(s,t,h_n,\tilde{h}_q), \quad \lambda_{n,q} = G_m(h_n,\tilde{h}_q,h_n,\tilde{h}_q)^{-1}.$$

Theorem (2018) Let $k = 0$ or $k = 1$ and let $(n,q) \in Q_k$. Then $P_{n,q}$ is a polynomial of degree m satisfying $P_{n,q}(h_n, \tilde{h}_q) = 1$ and $P_{n,q}(x) = 0$ for all $x \in \mathcal{N}_k$ with $x \neq (h_n, \tilde{h}_q)$.

2-dim Christoffel-Darboux identity

Define

$$U_m(s,t,u,v) = \frac{Y_m(s,t)\tilde{p}_m(v)}{\tilde{H}_0} + \sum_{j=0}^{m-1} \frac{[X_j(s,t)p_{m-j+1}(u) + Y_j(s,t)p_{m-j}(u)]\tilde{p}_j(v)}{\tilde{H}_j}$$

Then $U_m(s,t,u,v) = 0$ when $(s,t) \in \mathcal{N}_k$ and

$$2a_0 H_0(s-u)G_m(s,t,u,v) = U_m(s,t,u,v) - U_m(u,v,s,t)$$

The RHS vanishes when $(s,t) = (h_{n'}, \tilde{h}_{q'})$ and $(u,v) = (h_n, \tilde{h}_q)$

Thus if $h_{n'} \neq h_n$ then $G_m(h_{n'}, \tilde{h}_{q'}, h_n, \tilde{h}_q) = 0$

The same conclusion holds when $\tilde{h}_{q'} \neq \tilde{h}_q$ by symmetry.

Cubature lemma

Define orthogonal polynomials of degree m by

$$\mathcal{S}_m = \{p \in \mathcal{P}_m(\mathbb{R}^k) : \ (p, q) = 0 \text{ for all } q \in \mathcal{P}_{m-1}(\mathbb{R}^k)\}$$

Lemma (2015) Let $\{x_i\}_{i=1}^n$ be n distinct points of $\mathbb{R}^k$ having Lagrange polys $\{P_i\}_{i=1}^n$ in $\mathcal{P}_m(\mathbb{R}^k)$.

Conditions (a) and (b) below are equivalent.

a) If $p \in \mathcal{P}_m(\mathbb{R}^k)$ and $p(x_i) = 0$ for all i then $p \in \mathcal{S}_m$.
For each i, there is an $S_i \in \mathcal{S}_m$ with

$$P_i(x) = \lambda_i K_{m-1}(x, x_i) + S_i(x), \quad x \in \mathbb{R}^k.$$

b)

$$\int_{\mathbb{R}^k} p(x)\, d\mu(x) = \sum_{i=1}^{n} \lambda_i p(x_i)$$

for all $p \in \mathcal{P}_{2m-1}(\mathbb{R}^k)$.

Cubature theorem

This representation of Lagrange polys implies

Theorem (2018) Let μ and $\tilde{\mu}$ be measures corresponding to the two decreasing sequences used to define the even-odd nodes. Then

$$\iint_{\mathbb{R}^2} p(s,t)\, d(\mu \times \tilde{\mu})(s,t) = \sum_{(n,q)\in Q_k} \lambda_{n,q}\, p(h_n, \tilde{h}_q)$$

for all bivariate polys p of degree at most $2m-1$ and for $k = 0, 1$.

Applies to T_n, U_n, V_n, W_n, k_n Unusual

Theorem $\lambda_{n,q} = 2\lambda_n \tilde{\lambda}_q \quad 0 \le n, q \le m$

$$\lambda_n = \frac{H_0 w_n}{\sum_{n=0}^{m} w_n}, \quad w_n = \frac{(-1)^n}{\prod_{j\neq n}(h_n - h_j)}$$

Discrete measure ν in which $p_0, \ldots, p_m$ are orthogonal

$$\nu = \sum_{n=0}^{m} w_n \delta_{h_n}, \quad \delta_x = \text{ Dirac measure at } x$$

Algorithm

$P := \prod_{n \text{ even}}(x - h_n)$; $Q := \prod_{n \text{ odd}}(x - h_n)$;
$S := \sum_{n=0}^{m}(-1)^n h_n$;
if m is even then $s_0 := 1$; $t_0 := 0$; $s_1 := x - S$; $t_1 := 1$
        else $s_0 := 1$; $t_0 := -1$; $s_1 := 1$; $t_1 := 1$ end(if)
$k := [(m+1)/2]$;
$p_k := s_0 Q - t_0 P$; $p_{k-1} := s_1 Q - t_1 P$;
if m is even then $p_{k+1} := s_1 Q + t_1 P$ end(if)

\# Main loop
for $j := 2$ to k do
    $q := \mathrm{quotient}(p_{k-j+2}, p_{k-j+1})$;   \# linear
    $s_j := (q s_{j-1} - s_{j-2})/c_{k-j+1}$;
    $t_j := (q t_{j-1} - t_{j-2})/c_{k-j+1}$;
    $p_{k-j} := s_j Q - t_j P$;
    $p_{m-k+j} := s_j Q + t_j P$
end(do)
for $j = 0$ to m do $p_j := p_j/p_0$ end(do)

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