The cb-norm approximations by two-sided multiplications and elementary operators

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based on the joint work with Richard M. Timoney

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Preliminaries

Let A be a C^*-algebra. An attractive and fairly large class of bounded linear maps $\phi : A \to A$ that preserve all ideals of A is the class of **elementary operators**, that is, those that can be expressed as a finite sum

$$\phi = \sum_i M_{a_i,b_i}$$

of **two-sided multiplications** $M_{a_i,b_i} : x \mapsto a_i x b_i$, where $a_i, b_i \in A$.

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In fact, elementary operators are completely bounded (cb), i.e.

$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi_n\| < \infty,$$

where for each n, ϕ_n is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$.

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where for each n, ϕ_n is an induced map on $M_n(A)$; $\phi_n([a_{ij}]) = [\phi(a_{ij})]$.
Moreover, we have the the following estimate for their cb-norm:

$$\left\|\sum_i M_{a_i,b_i}\right\|_{cb} \leq \left\|\sum_i a_i \otimes b_i\right\|_h , \tag{1}$$

where $\|\cdot\|_h$ is the Haagerup tensor norm on $A \otimes A$, i.e.

$$\|u\|_h = \inf \left\{ \left\| \sum_i a_i a_i^* \right\|^{\frac{1}{2}} \left\| \sum_i b_i^* b_i \right\|^{\frac{1}{2}} \ : \ u = \sum_i a_i \otimes b_i \right\}.$$

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Hence, if by $\mathcal{E}\ell(A)$ we denote the set of all elementary operators on A and by $\mathrm{ICB}(A)$ the set of all completely bounded operators on A that preserve all ideals of A, inequality (1) implies that the mapping

$$(A \otimes A, \| \cdot \|_h) \to (\mathcal{E}\ell(A), \| \cdot \|_{cb})$$

given by

$$\sum_i a_i \otimes b_i \mapsto \sum_i M_{a_i, b_i}.$$

is a well-defined contraction. Its continuous extension to the completed Haagerup tensor product $A \otimes_h A$ is known as a **canonical contraction** from $A \otimes_h A$ to $\mathrm{ICB}(A)$ and is denoted by θ_A.

Theorem (Haagerup 1980, Chatterjee-Sinclair 1992, Mathieu 2003)

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We have the following general questions:

Question

Which operators $\phi \in \mathrm{ICB}(A)$ admit a cb-norm approximation by

(a) two-sided multiplications, or

(b) elementary operators?

The closure of two-sided multiplications

Let $\mathrm{TM}(A) = \{M_{a,b} : \ a, b \in A\}$ and let $\overline{\overline{\mathrm{TM}(A)}}$ be its operator norm closure. It is well-known that the cb-norm on $\mathrm{TM}(A)$ coincides with the usual operator norm.

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Example

If X is a LCH space, the C^*-algebra $C_0(X, \mathbb{M}_n)$ is n-homogeneous.

Theorem (Fell & Tomiyama-Takesaki)

If A is an n-homogeneous C^*-algebra, then its (primitive) spectrum X is a LCH space and there is a locally trivial bundle $\mathcal{E}$ over X with fibre $\mathbb{M}_n$ and structure group $\mathrm{Aut}(\mathbb{M}_n) = PU(n) = U(n)/\mathbb{S}^1$ such that A is isomorphic to the algebra $\Gamma_0(\mathcal{E})$ of sections of $\mathcal{E}$ that vanish at infinity.

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Moreover, any two such algebras $A_i = \Gamma_0(\mathcal{E}_i)$ with spectra X_i are isomorphic if and only if there is a homeomorphism $f : X_1 \to X_2$ such that $\mathcal{E}_1 \cong f^*(\mathcal{E}_2)$ as bundles over X_1.

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Throughout this section A will denote an n-homogeneous C^*-algebra A with the primitive spectrum X and the associate $\mathbb{M}_n$-bundle $\mathcal{E}$. In this case every bounded linear map on A is completely bounded, so $\mathrm{ICB}(A) = \mathrm{IB}(A)$.

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Notation

For $x \in X$ let $A_x := A/x \cong \mathbb{M}_n$. For $\phi \in \mathrm{IB}(A)$ and $x \in X$ we write ϕ_x for the induced map on A_x (i.e. $\phi_x(a_x) = \phi(a)_x$).

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$\mathrm{FTM}(A) := \{\phi \in \mathrm{IB}(A) : \phi_x \in \mathrm{TM}(A_x)\ \forall x \in X\ \&\ (x \mapsto \|\phi_x\|) \in C_0(X)\}$.

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$\mathrm{FTM}(A)$ may seem as the most obvious candidate for the norm closure of $\mathrm{TM}(A)$ due to the following fact:

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Auxiliary notation

- $\mathrm{TM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{TM}(A) \ : \ \phi_x \neq 0 \ \forall x \in X\}$;
- $\mathrm{FTM}_{\mathrm{nv}}(A) = \{\phi \in \mathrm{FTM}(A) \ : \ \mathrm{TM}(A_x) \ni \phi_x \neq 0 \ \forall x \in X\}$.

(nv signifies nowhere-vanishing)

Theorem (G.-Timoney 2018)

To each operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ we can (canonically) associate a complex line subbundle $\mathcal{L}_\phi$ of $\mathcal{E}$ with the property that

$$\phi \in \mathrm{TM}_{\mathrm{nv}}(A) \quad \Longleftrightarrow \quad \mathcal{L}_\phi \text{ is a trivial bundle.}$$

Further, is X is σ-compact, then for every complex line subbundle $\mathcal{L}$ of $\mathcal{E}$ we can find an operator $\phi_{\mathcal{L}} \in \mathrm{FTM}_{\mathrm{nv}}(A)$ such that $\mathcal{L}_{\phi_{\mathcal{L}}} = \mathcal{L}$.

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If X is paracompact, the locally trivial complex line bundles over X are classified by the homotopy classes from X to $\mathbb{C}P^\infty$, and/or by the elements of the second integral Čech cohomology $\check{H}^2(X;\mathbb{Z})$

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Hence, for a homogeneous C^*-algebra A we can define a map

$$\psi : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$$

which sends an operator $\phi \in \mathrm{FTM}_{\mathrm{nv}}(A)$ to the corresponding class of the bundle $\mathcal{L}_\phi$ in $\check{H}^2(X;\mathbb{Z})$. Then $\psi^{-1}(0) = \mathrm{TM}_{\mathrm{nv}}(A)$ (by the latter theorem).

Corollary

If X is paracompact and $\check{H}^2(X;\mathbb{Z}) = 0$, then $\mathrm{FTM_{nv}}(A) = \mathrm{TM_{nv}}(A)$.

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Theorem (G.-Timoney 2018)

Let X be a CH space with $\dim X \leq d < \infty$. For each $n \geq 1$ let $A_n = C(X, \mathbb{M}_n)$. If $p = \left\lceil \sqrt{(d+1)/2} \right\rceil$, then for every $n \geq p$ the mapping $\psi : \mathrm{FTM}_{\mathrm{nv}}(A) \to \check{H}^2(X;\mathbb{Z})$ is surjective. In particular, if $\check{H}^2(X;\mathbb{Z}) \neq 0$, then $\mathrm{TM}_{\mathrm{nv}}(A_n)$ is a proper subset of $\mathrm{FTM}_{\mathrm{nv}}(A_n)$ for all $n \geq p$.

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Example

If $X = \mathbb{S}^2$ or $X = \mathbb{S}^1 \times \mathbb{S}^1$, then for $A = C(X, \mathbb{M}_n)$ we have $\mathrm{TM}_{\mathrm{nv}}(A) \subsetneq \mathrm{FTM}_{\mathrm{nv}}(A)$ for all $n \ge 2$.

Theorem (G.-Timoney 2018)

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. Consider the following two conditions:

(a) *$\forall U \subset X$ open, each complex line subbundle of $\mathcal{E}|_U$ is trivial.*

(b) $\mathrm{FTM}(A) = \mathrm{TM}(A)$.

Then (a) $\Rightarrow$ (b). If A is separable, then (a) and (b) are equivalent.

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Corollary

Let A be an n-homogeneous C^*-algebra with $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$, or if X is homeomorphic to a subset of a non-compact connected 2-manifold, then* $\mathrm{FTM}(A) = \mathrm{TM}(A)$.

(b) *If X is σ-compact and contains a nonempty open subset homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{TM}(A)$ is a proper subset of $\mathrm{FTM}(A)$.*

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(a) $\phi \in \overline{\overline{\mathrm{TM}(A)}}$.

(b) $\phi \in \mathrm{FTM}(A)$ *and for* $\mathrm{coz}(\phi) := \{x \in X : \phi_x \neq 0\}$ *the bundle* $\mathcal{L}_\phi$ *is trivial on each compact subset of* $\mathrm{coz}(\phi)$.

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Definition

A locally trivial fibre bundle $\mathcal{F}$ over a locally compact Hausdorff space X is said to be a **phantom bundle** *if $\mathcal{F}$ is not globally trivial, but is trivial on each compact subset of X.*

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Corollary

Let $A = \Gamma_0(\mathcal{E})$ be a homogeneous C^*-algebra. The set $\mathrm{TM}(A)$ is not norm closed if and only if there exists a σ-compact open subset U of X and a phantom complex line subbundle of $\mathcal{E}|_U$.

Remark

Let G be a group and n a positive integer. Recall that a space X is called an **Eilenberg-MacLane** space of type $K(G, n)$, if it's n-th homotopy group $\pi_n(X)$ is isomorphic to G and all other homotopy groups trivial. If $n > 1$ then G must be abelian (since for all $n > 1$, the homotopy groups $\pi_n(X)$ are abelian). We state some basic facts about Eilenberg-MacLane spaces:

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- There exists a CW-complex $K(G, n)$ for any group G at $n = 1$, and abelian group G at $n > 1$. Moreover such a CW-complex is unique up to homotopy type. Hence, by abuse of notation, it is common to denote any such space by $K(G, n)$.

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- Given a CW-complex X, there is a bijection between its cohomology group $H^n(X;G)$ and the homotopy classes $[X, K(G,n)]$ of maps from X to $K(G,n)$.
- $K(\mathbb{Z},2) \cong \mathbb{C}P^\infty$. In particular, for each CW-complex X there is a bijection between $[X, K(\mathbb{Z},2)]$ and isomorphism classes of complex line bundles over X.

Example

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- The standard model of $K(\mathbb{Q},1)$ is the mapping telescope Δ of the sequence

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- Applying $H_1(-;\mathbb{Z})$ to the levels of this mapping telescope gives the system

$$\mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \cdots .$$

  The colimit of this system is $H_1(\Delta;\mathbb{Z}) = \mathbb{Q}$ and all other integral homology groups are trivial. By the universal coefficient theorem for cohomology, each integral cohomology group of Δ is trivial, except for $H^2(\Delta;\mathbb{Z})$ which is isomorphic to $\mathrm{Ext}(\mathbb{Q};\mathbb{Z}) \cong \mathbb{R}$.

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- Therefore, there exist uncountably many mutually nontrivial complex line bundles over Δ. Each such bundle $\mathcal{L}$ is a phantom bundle, since all restrictions of $\mathcal{L}$ over finite subcomplexes of Δ are trivial.

Conclusion

Since Δ is a 2-complex, $\mathcal{L}$ is a direct summand of a trivial bundle $\Delta \times \mathbb{C}^2$. In particular, if $A = C_0(\Delta, \mathbb{M}_2)$, then $\mathrm{TM}(A)$ is not norm closed.

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In private correspondence Prof. Mladen Bestvina (University of Utah) informed us that even inside $\mathbb{R}^3$ there are open subsets of type $K(\mathbb{Q}, 1)$. Using this observation we were able to show the following fact:

Theorem (G.-Timoney 2018)

Let A be an n-homogeneous C^*-algebra with $n \geq 2$.

(a) *If X is second-countable with $\dim X < 2$ or if X is (homeomorphic to) a subset of a non-compact connected 2-manifold, then $\mathrm{TM}(A)$ is not norm closed.*

(b) *If there is a nonempty open subset of X homeomorphic to (an open subset of) $\mathbb{R}^d$ for some $d \geq 3$, then $\mathrm{TM}(A)$ fails to be norm closed.*

Summary

Let A be a separable n-homogeneous C^*-algebra with $n \geq 2$ such that $\dim X = d < \infty$. If X is a CW-complex or a subset of a d-manifold, the following relations between $\mathrm{TM}(A)$, $\overline{\overline{\mathrm{TM}(A)}}$ and $\mathrm{FTM}(A)$ occur:

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(a) If $d < 2$ we always have $\mathrm{TM}(A) = \overline{\overline{\mathrm{TM}(A)}} = \mathrm{FTM}(A)$.

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The cb-norm approximations of derivations and automorphisms by elementary operators

It is well-known that all derivations and $*$-automorphisms of C^*-algebras A are completely bounded. In light of our previous discussion, we may ask which derivations and automorphisms of A lie in the cb-norm closure $\overline{\mathcal{E}\ell(A)}_{cb}$ of $\mathcal{E}\ell(A)$?

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Recall that the **Glimm ideals** of A are the ideals of A generated by the maximal ideals of $Z(A)$. By the Hewitt-Cohen Factorization Theorem, each Glimm ideal of A is of the form mA for some maximal ideal m of $Z(A)$.

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The class of C^*-algebras whose every Glimm ideal is prime includes: prime C^*-algebras, C^*-algebras with Hausdorff primitive spectrum, quotients of AW^*-algebras, and local multiplier algebras.

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In a contrast to the similar result for derivations, the above result cannot be extended even to homogeneous C^*-algebras, which admit only inner derivations by Sproston's Theorem from 1976:

Example

For $n \geq 2$ let $A_n = C(PU(n), \mathbb{M}_n)$. Then A_n admits outer automorphisms that are simultaneously elementary operators.

Moreover, if the primitive spectrum of a C^*-algebra A is rather pathological, it can happen that A admits both outer derivations and outer automorphisms that are simultaneously elementary operators:

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Let A be a C^*-subalgebra of $B = C([1,\infty], \mathbb{M}_2)$ that consists of all $a \in B$ such that If

$$a(n) = \begin{bmatrix} \lambda_n(a) & 0 \\ 0 & \lambda_{n+1}(a) \end{bmatrix} \qquad (n \in \mathbb{N})$$

for some convergent sequence $(\lambda_n(a))$ of complex numbers. Then A admits outer derivations and outer automorphisms that are also elementary operators. In fact, there are outer derivations of A of the form $\delta = M_{a,b} - M_{b,a}$ for suitable $a, b \in A$.

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Problem

If an automorphism σ of a von Neumann algebra A is also an elementary operator, is σ necessarily inner?