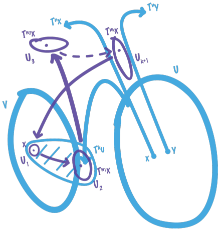
Distributionally Chaotic Functions

Clifford Gilmore
(With F. Martínez-Giménez and A. Peris)

University College Cork

AGA
In memory of Richard Timoney
May 2019

IRISH RESEARCH COUNCIL
An Chomhairle um Thaighde in Éirinn



Dynamical Systems

Typical setting

A pair (X, T), where

- X a topological space,
- $T\colon X \to X$ a continuous map.

Interested in the long term evolution of iterates of T

$$T^n = \underbrace{T \circ \cdots \circ T}_{n-\text{fold}}$$

Setting

- X separable Hilbert, Banach, or Fréchet space.
- $T: X \to X$ a continuous linear operator.

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Recall

A *Fréchet space* is a vector space X, endowed with an increasing sequence $(\|\cdot\|_k)_{k\in\mathbb{N}}$ of seminorms that define the metric

$$d(x,y) := \sum_{k=1}^{\infty} 2^{-k} \frac{\|x-y\|_k}{1+\|x-y\|_k}, \qquad (x, y \in X)$$

under which X is complete.

Linear Dynamics

Definition

If there exists $x \in X$ such that

$$\overline{\{x, Tx, T^2x, T^3x, \dots\}} = X$$

then T is a *hypercyclic operator*.

- Such an $x \in X$ called a *hypercyclic vector* for T.
- Interesting dynamics? Only in the infinite dimensional setting.

Examples

- Birkhoff (1929): translation operator

$$f(z) \mapsto f(z+a)$$

  for $a \neq 0$ on the space of entire functions $H(\mathbb{C})$.
- MacLane (1952): differentiation operator on $H(\mathbb{C})$

$$D : f \mapsto f'.$$

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F. Bayart and É. Matheron.
Dynamics of linear operators.
Cambridge Tracts in Mathematics, vol. 179. Cambridge University Press, Cambridge, 2009.

K.-G. Grosse-Erdmann and A. Peris Manguillot.
Linear chaos.
Universitext. Springer, London, 2011.

Theorem (Ansari-Bernal (Bonet and Peris))

Every infinite-dimensional, separable Banach (Fréchet) space supports a hypercyclic operator.

Some motivation (Functional Analysis)

- Counter examples to the invariant subspace problem.
- Read (1988): $\exists T : \ell^1 \to \ell^1$ such that every nonzero $x \in \ell^1$ is a hypercyclic vector for T.
  $\Longrightarrow T$ does not possess a non-trivial, closed invariant subset.

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Linear Chaos (in the sense of Devaney):

$T\colon X \to X$ a continuous linear operator.

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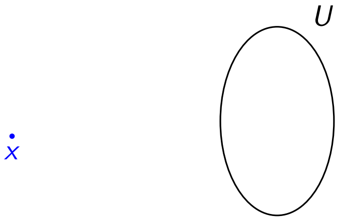
Recurrence in Linear Dynamics

$T\colon X \to X$

Given any nonempty, open $U \subset X$,



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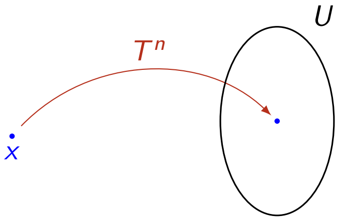

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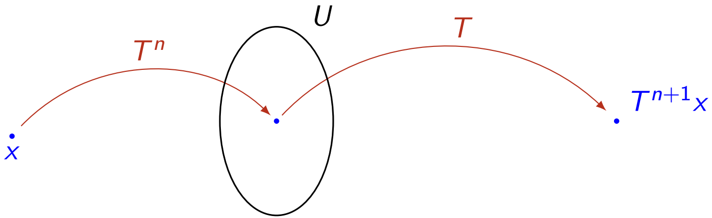

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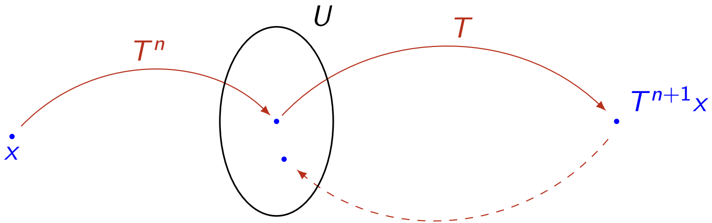

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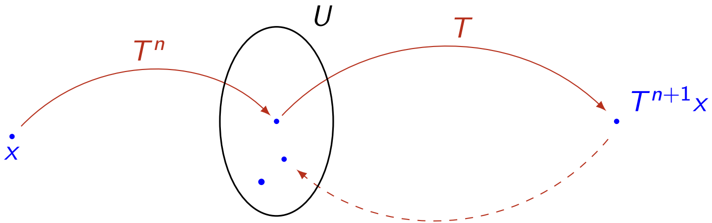

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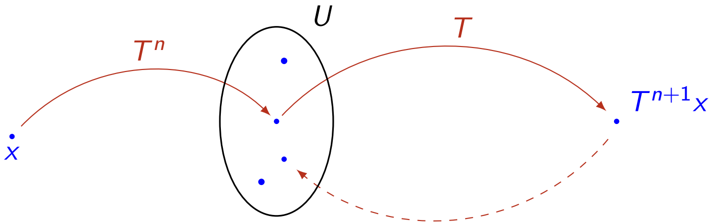

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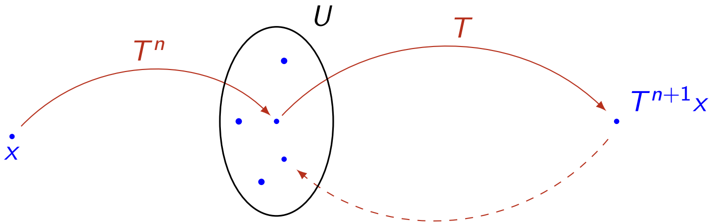

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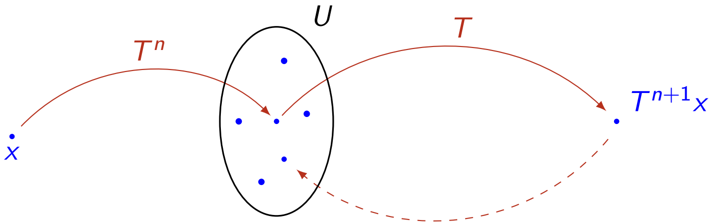

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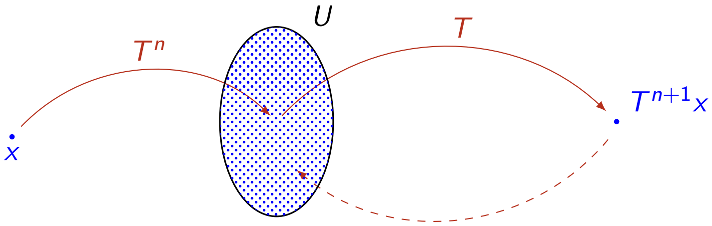

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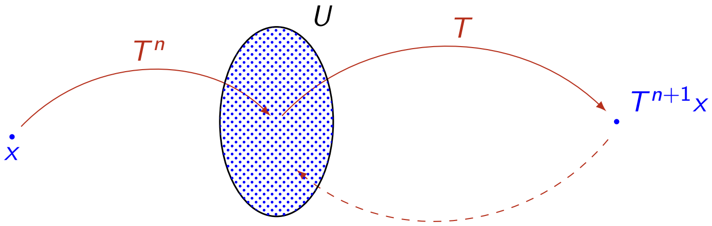

Density of the T-orbit in U?

$$\frac{\{n : T^n x \in U, 1 \leq n \leq N\}}{}$$

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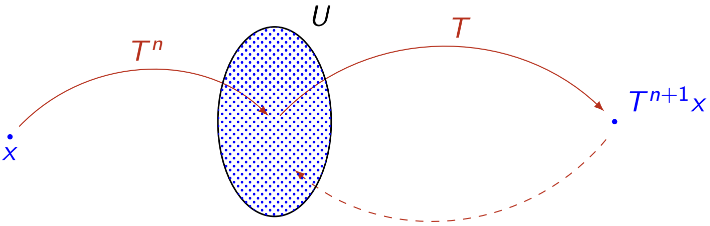

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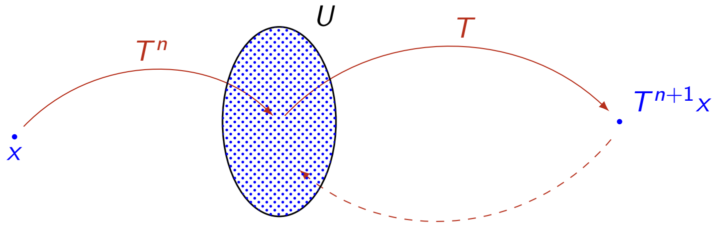

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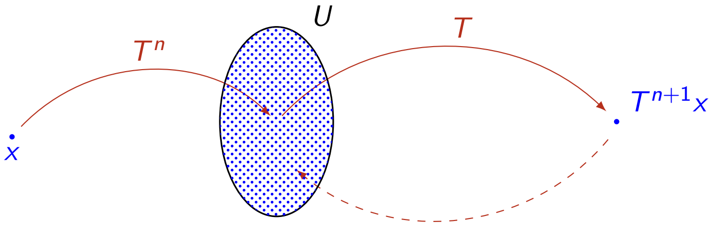

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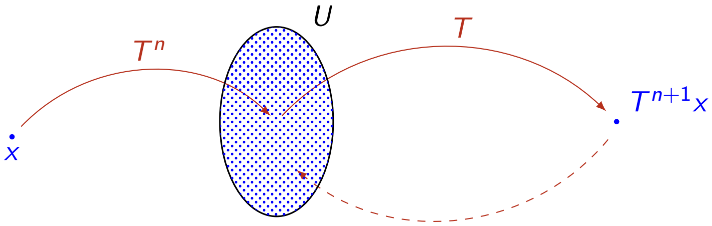

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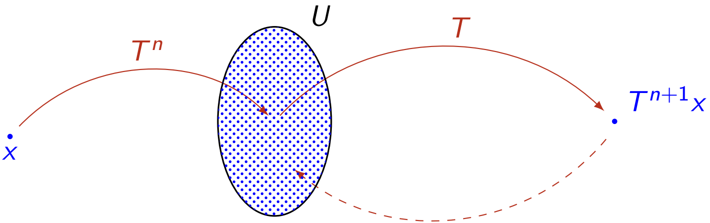

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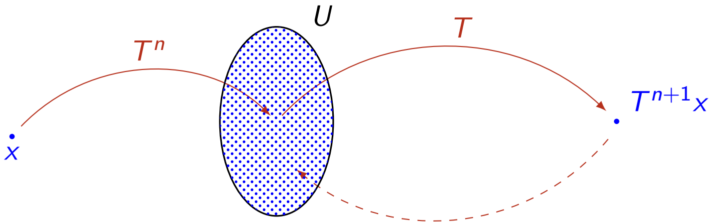

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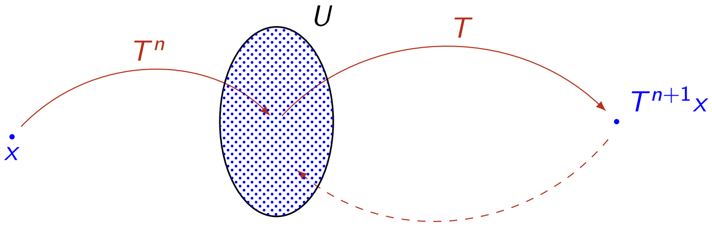

Density of the T-orbit in U?

T is *frequently hypercyclic* if there exists $x \in X$ such that for any nonempty, open $U \subset X$ we have

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Frequently Hypercyclic Operators

- Bayart and Grivaux (2004).
- $x \in X$ a *frequently hypercyclic vector* for T.

Examples

- Translation $f(z) \mapsto f(z+a)$ on $H(\mathbb{C})$, $a \neq 0$.
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Some properties

- Hypercyclic $\not\Longrightarrow$ frequently hypercyclic.
- There exist separable Fréchet spaces with no frequently hypercyclic operators.

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Chaos (Li-Yorke, 1975)

First mention of *chaos* in mathematical literature

- X a separable Banach space.
- $T: X \to X$ a continuous linear operator.

Definition (Li and Yorke)

T is *Li-Yorke chaotic* if there exists an uncountable $\Gamma \subset X$ such that for every pair $(x, y) \in \Gamma \times \Gamma$ of distinct points we have

$$\liminf_{n\to\infty} \|T^n x - T^n y\| = 0 \quad \text{and} \quad \limsup_{n\to\infty} \|T^n x - T^n y\| > 0.$$

- Pairs are *proximal* but not *asymptotic*.

Li-Yorke Chaos	Hypercyclicity
Local aspects of dynamics of pairs of vectors.	Complex global behaviour.

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Irregularity

Setting

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Definition (Beauzamy, 1988)

We say $x \in X$ is an *irregular vector* for T if there exist increasing sequences (j_k) and (n_k) of positive integers such that

$$\lim_{k\to\infty} T^{j_k}x = 0 \quad \text{and} \quad \lim_{k\to\infty} \|T^{n_k}x\| = \infty.$$

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Distributional Chaos

- The *upper density* of a set $A \subset \mathbb{N}$ is defined as

$$\overline{\text{dens}}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \dots, n\}|}{n}.$$

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The vector $x \in X$ is *distributionally irregular* for T if there exist $A, B \subset \mathbb{N}$ with

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such that

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$$\lim_{n\in A} T^n x = 0 \text{ and } \quad \lim_{n\in B} \|T^n x\| = \infty.$$

- The orbit has a more complicated statistical dependence.
- The usual suspects: Translation and differentiation on $H(\mathbb{C})$.

Distributional Chaos

- The *upper density* of a set $A \subset \mathbb{N}$ is defined as
$$\overline{\text{dens}}(A) := \limsup_{n\to\infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$
- X a Banach space.
- $T\colon X \to X$ a continuous linear operator.

Definition

The vector $x \in X$ is *distributionally irregular* for T if there exist $A, B \subset \mathbb{N}$ with

$$\overline{\text{dens}}(A) = 1 = \overline{\text{dens}}(B)$$

such that

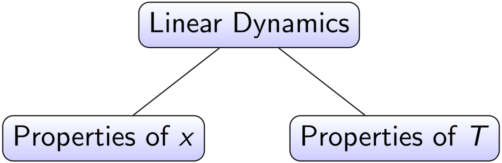
$$\lim_{n\in A} T^n x = 0 \text{ and } \quad \lim_{n\in B} \|T^n x\| = \infty.$$

- The orbit has a more complicated statistical dependence.
- The usual suspects: Translation and differentiation on $H(\mathbb{C})$.

Chaotic Relationships

- T distributionally irregular $\iff$ T *distributionally chaotic*. (Bermúdez, Bonilla, Martínez-Giménezz, Peris, 2011; & Bernardes, Bonilla, Múller, Peris, 2013)
- *Distributional chaos* $\not\Rightarrow$ hypercyclic. (Martínez-Giménez, Oprocha, Peris 2009)
- *Distributionally chaos* $\not\Rightarrow$ frequently hypercyclic. (Bermúdez, Bonilla, Martínez-Giménez, Peris, 2011)
- *Distributionally chaos* $\not\Rightarrow$ Devaney chaos. (Bermúdez, Bonilla, Martínez-Giménezz, Peris, 2011)
- Hypercyclic $\not\Rightarrow$ *distributional chaos*. (Martínez-Giménez, Oprocha, Peris 2013)
- Frequent hypercyclicity $\not\Rightarrow$ *distributional chaos*. (Bayart and Ruzsa, 2015)
- Devaney chaos $\not\Rightarrow$ *distributional chaos*. (Menet, 2017)
- Devaney chaos $\not\Rightarrow$ frequently hypercyclic. (Menet, 2017)

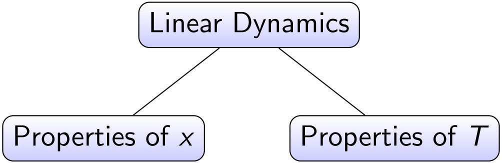
Aspects of Linear Dynamics



Today's Question:

What are the permissible *growth rates* of entire functions that are *'chaotic'* with respect to differentiation?

Aspects of Linear Dynamics



Today's Question:

What are the permissible *growth rates* of entire functions that are *'chaotic'* with respect to differentiation?

Entire Functions

$D\colon f \mapsto f'$

Growth

- $f \in H(\mathbb{C})$.
- For $1 \leq p < \infty$, the average L^p-norm on a sphere of radius $r > 0$

$$M_p(f, r) := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt \right)^{1/p}$$

- For $p = \infty$

$$M_\infty(f, r) := \sup_{|z|=r} |f(z)|$$

for $r > 0$.

Hypercyclic Case

$D\colon f \mapsto f'$

$1 \leq p \leq \infty$.

- Initial estimates: MacLane (1952).
- Sharp growth: Grosse-Erdmann (1990), Shkarin (1993).
  - Proof: weighted Banach space & a sufficient condition.
- For any function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a D-hypercyclic entire function $f \in H(\mathbb{C})$ with

$$M_p(f,r) \leq \varphi(r)\frac{e^r}{r^{1/2}}$$

  for $r > 0$ sufficiently large.
- There does not exist a D-hypercyclic entire function $f \in H(\mathbb{C})$ with

$$M_p(f,r) \leq C\frac{e^r}{r^{1/2}}$$

  for a constant $C > 0$ and radius $r > 0$.

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Frequently Hypercyclic Case

$D\colon f \mapsto f'$

- Initial estimates: Blasco, Bonilla, Grosse-Erdmann (2010), Bonet and Bonilla (2013).
  - Proof: weighted Banach space & a sufficient condition.
- Optimal growth: Drasin and Saksman (2012).
  - Proof: explicit construction.
- For any $C > 0$ there exists a D-frequently hypercyclic entire function $f \in H(\mathbb{C})$ with

$$M_\infty(f, r) \leq C \frac{e^r}{r^{1/4}}$$

  for all $r > 0$. (Also holds for $1 < p \leq \infty$)
- $p = 1$. For any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a D-frequently hypercyclic entire function $f \in H(\mathbb{C})$ with

$$M_1(f, r) \leq \varphi(r) \frac{e^r}{r^{1/2}}$$

  for all $r > 0$. (Bonet and Bonilla, 2013.)

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Irregular Case

$D\colon f \mapsto f'$

For $1 \leq p \leq \infty$.

- Bernal-González and Bonilla (2016).
  - Proof: weighted Banach space & a sufficient condition.
- For any function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a D-irregular entire function $f \in H(\mathbb{C})$ with

$$M_p(f, r) \leq \varphi(r) \frac{e^r}{r^{1/2}}$$

  for $r > 0$ sufficiently large.
- There does not exist a D-irregular entire function $f \in H(\mathbb{C})$ with

$$M_p(f, r) \leq C \frac{e^r}{r^{1/2}}$$

  for a constant $C > 0$ and radius $r > 0$.

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Distributionally Irregular Case

$D\colon f \mapsto f'$

For $1 \leq p \leq \infty$.

- Initial estimates: Bernal-González and Bonilla (2016).
  - Proof: explicit construction.
- G., Martínez-Giménez and Peris (2019).
  - Proof: weighted Banach space & a sufficient condition.

Theorem (G., Martínez-Giménez and Peris, 2019)

Let $a = (2\max\{2,p\})^{-1}$. For any $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(r) \to \infty$ as $r \to \infty$, there exists a D-distributionally irregular entire function f with

$$M_p(f,r) \leq \varphi(r)\frac{e^r}{r^a}$$

for $r > 0$ sufficiently large.

Distributionally Irregular Case

$D\colon f \mapsto f'$

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for $r > 0$ sufficiently large.

Harmonic Functions

Partial differentiation on the space of harmonic functions on $\mathbb{R}^N$

Hypercyclic

- Aldred and Armitage (1998).

Frequently hypercyclic

- Blasco, Bonilla and Grosse-Erdmann (2010).
- G., Saksman and Tylli (2019).

Distributional chaos

- G., Martínez-Giménez and Peris (2019).

Thank you for your attention! ☺

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