Spectra of weighted composition operators on analytic function spaces

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based on joint work with

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Throughout this talk E stands for a complex Banach space of arbitrary dimension and $B_E = \{x \in E : \|x\| < 1\}$ for its open unit ball. Moreover, let $\varphi : B_E \to B_E$ be an analytic mapping and $u \in H(B_E)$, where $H(B_E)$ is the space of analytic functions on B_E.

Recall that a mapping is analytic if it is Fréchet differentiable at every point in its domain.

Each such pair (φ, u) induces via composition and multiplication a weighted composition operator

$$uC_\varphi(f)(x) = u(x)\big((f \circ \varphi)(x)\big),\ x \in B_E$$

which preserves $H(B_E)$.

Our object of study is the operator uC_φ acting on a Banach space, $X(B_E)$, of analytic functions on B_E; specifically, its spectrum $\sigma(uC_\varphi)$. We focus in the case of $\varphi(0) = 0$.

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In earlier research on the spectrum there were some common elements: the appearance of the essential spectral radius, the decomposition of the space provided by the Taylor series of the function variable and a crucial Lemma that is a nice sharpening by C. Cowen and B. MacCluer [CM] of H. Kamowitz technique [Ka] that has been further exploited by many other authors [AL],[BGL],[GGL], [GM1],[MS], [Ze]. And also the availability of interpolating sequences.

In order that this pieces fit together, we need to know some information about the space $X(B_E)$. Such is collected in a number of conditions that we check is satisfied by very natural and common Banach spaces of analytic functions like the weighted Bergman spaces, $A^p_\alpha(\mathbb{B}_N)$, the Hardy spaces, $H^p(\mathbb{B}_N)$, $1 \leq p < \infty$, and, even in the infinite dimensional setting, the weighted spaces of analytic functions $H^\infty_v(B_E)$ as we will see.

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Recall that

$$H_b(B_E) := \{f : B_E \to \mathbb{C} : f \text{ anal. \& bdd. on balls of radius } < 1\}$$

is a Fréchet algebra when endowed with the topology of uniform convergence on balls of radius less than 1.

By $H^\infty(B_E)$ we denote the subspace of $H_b(B_E)$ of bounded functions endowed with the topology of uniform convergence on B_E.

We deal with a vector space $X(B_E)$ of analytic functions on B_E and a norm on it $\|\cdot\|$, that renders $X(B_E)$ a Banach space. As usual, for each $x \in B_E$, δ_x is the evaluation functional defined by $\delta_x(f) = f(x)$ for all $f \in X(B_E)$. We assume that $X(B_E)$ contains the constant functions, so then all δ_x are non-zero.

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Conditions on $X(B_E)$

(I) *For every* $x \in B_E$, $\delta_x : X(B_E) \to \mathbb{C}$ *is a linear bounded functional, and the closed unit ball* $\mathbf{B} = \{f \in X(B_E) : ||f|| \leq 1\}$ *of* $X(B_E)$ *is compact with respect to the compact-open topology* τ_0.

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In particular, for each $x \in B_E$ there is a $f_x \in X(B_E)$ with $||f_x|| \leq 1$ such that $||\delta_x||_X = f_x(x)$. Moreover, by the Dixmier-Ng theorem, there is a Banach space ${}^*X(B_E)$ whose dual space is isometrically isomorphic to $X(B_E)$ and further, the mapping $x \in B_E \mapsto \delta_x \in {}^*X(B_E)$ is holomorphic because it is weakly holomorphic. Actually, ${}^*X(B_E)$ is the subspace of $X(B_E)^*$ of the elements that are τ_0-continuous on bounded sets.

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If both **(I)** and **(II)** hold, the multiplication operator $M_g(f) = fg$ is continuous on $X(B_E)$, thanks to the closed graph theorem. A subsequent application of the closed graph theorem shows the existence of a constant $M_X > 0$ such that $\|M_g\| \leq M_X \|g\|_\infty$.

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This inclusion mapping is a continuous embedding thanks to the closed graph theorem.

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Denote by $P_n f$ the n-th term of the Taylor series at 0 of the analytic function $f \in X(B_E)$. For $m \in \mathbb{N}$, let

$$X_m(B_E) = \{f \in X(B_E) : P_n f = 0 \text{ for } n = 0, 1, \ldots, m-1\}.$$

That is, a function in $X(B_E)$ belongs to $X_m(B_E)$ if the first $m-1$ terms of its Taylor series at 0 vanish. Equivalently, $f \in X(B_E)$ belongs to $X_m(B_E)$ if, and only if, $\frac{f(x)}{\|x\|^m}$ is bounded in some punctured ball centered at 0.

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(IV) *For each* $m \in \mathbb{N}$ *there is a constant* $c(m) > 0$ *(depending also on the norm of* $X(B_E)$*) such that for all* $x \in B_E$ *we have*

$$||\delta_x||_{X_m} \leq c(m)||x||^m||\delta_x||,$$

where $X_m(B_E)$ is endowed with norm of $X(B_E)$ and $\|\delta_x\|_{X_m}$ denotes the norm of δ_x restricted to X_m.

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(V) For every $0 < r < 1$, consider $K_r(f)(x) = f(rx)$. *The operator* $K_r : X(B_E) \to X(B_E)$ *is well-defined and* $\|K_r\| \leq 1$. In case $\dim E < \infty$, the operator K_r is compact.

Examples

(a) The weighted space of analytic functions

$$H_v^\infty(B_E) := \{f : B_E \to \mathbb{C} : f \text{ anal. } \& \ \|f\|_v = \sup_{x \in B_E} v(x)|f(x)| < \infty\}$$

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is a Banach space when endowed with the $\|\cdot\|_v$ norm.

Here $v : B_E \to (0,\infty)$ is a *weight*, that is, a continuous, bounded and norm non-increasing function, in particular, $v(x) = v(y)$ if $\|x\| = \|y\|$. For example, $v_\alpha(x) = (1 - ||x||^2)^\alpha$ with $\alpha > 0$ is such a weight. Notice that for the constant weight $v(x) = 1$, $H_v^\infty(B_E) = H^\infty(B_E)$.

Examples

(b) The standard weighted Bergman space $A^p_\alpha(\mathbb{B}_N)$, $\alpha > -1, p \geq 1$, is the set of all analytic functions on $\mathbb{B}_N$ such that

$$||f||^p_{A^p_\alpha} = \int_{\mathbb{B}_N} |f(z)|^p c_\alpha (1 - |z|^2)^\alpha dv(z) < \infty,$$

where $dv(z)$ is the normalized volume measure on $\mathbb{B}_N$ and $c_\alpha = \frac{\Gamma(N+\alpha+1)}{N!\Gamma(\alpha+1)}$.

Examples

(c) The Hardy spaces , $1 \leq p < \infty$, are defined by

$$H^p(\mathbb{B}_N) = \{f \in H(\mathbb{B}_N) : ||f||_{H^p}^p = \sup_{0<r<1} \int_{\mathbb{S}_N} |f(r\zeta)|^p d\sigma(\zeta) < \infty\},$$

where $\mathbb{S}_N$ denotes the unit sphere in $\mathbb{C}^N$ and σ is the normalized surface measure on it.

Examples

(d) The *weighted Hardy spaces of bounded type* $\mathcal{H}(\mathbb{B}_N)$ introduced by Cowen and MacCluer in [CM]. These are Hilbert spaces of analytic functions that include the classical Hardy space $H^2(\mathbb{B}_N)$ and the classical Bergman space $A_0^2(\mathbb{B}_N)$.

Recall that for the essential spectral radius of an operator T we have that $r_e(T) = \inf_n \sqrt[n]{\|T^n\|_e}$.
One reason for the appearance of the essential spectral radius in studying $\sigma(T)$ is that if $\lambda \in \sigma(T)$ and $|\lambda| > r_e(T)$, then λ is an eigenvalue.

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By φ_n we denote the n-fold iterate of φ, so that $\varphi_n = \varphi \circ \varphi \circ \dots \circ \varphi$ (n times).

Proposition

Assume that $\varphi(B_E)$ is a relatively compact subset of E. For the weighted composition operator $uC_\varphi : X(B_E) \to H_v^\infty(B_E)$, we have that

$$\|uC_\varphi\|_e \leq 2 \lim_{s\to 1} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|}.$$

Proposition

Assume that $\varphi(B_E)$ *is a relatively compact subset of* E. *For the weighted composition operator* $uC_\varphi : X(B_E) \to H_v^\infty(B_E)$, *we have that*

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Proposition

Assume that $\varphi(B_E)$ *is a relatively compact subset of* E. *There exists* $M_X > 0$ *such that for* $uC_\varphi : H_v^\infty(B_E) \to X(B_E)$, *we have*

$$\|uC_\varphi\|_e \geq M_X^{-1} \lim_{s\to 1} \sup_{\|\varphi(x)\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|}.$$

Corollary

Assume that $\varphi(B_E)$ *is a relatively compact subset of* E. *For the weighted composition operator* uC_φ *acting on* $H_v^\infty(B_E)$, *we have that*

$$r_e(uC_\varphi) = \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{\|\varphi_n(x)\|\geq s} \frac{\|\delta_{\varphi_n(x)}\||u(x)\cdot \ldots \cdot u(\varphi_n(x))|}{\|\delta_x\|}}.$$

Corollary

Assume that $\varphi(B_E)$ is a relatively compact subset of E. For the weighted composition operator uC_φ acting on $H_v^\infty(B_E)$, we have that

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This extends [Le, Theorem 2.5] to the weighted spaces case.

The following lower estimate is easier to prove

Proposition

Assume that $||\delta_x|| \to \infty$ *when* $||x|| \to 1$ *and that the continuous polynomials are dense in* $X(B_E)$. *Then for the weighted composition operator* uC_φ *acting on* $X(B_E)$, *we have that*

$$\|uC_\varphi\|_e \geq \lim_{s\to 1} \sup_{\|x\|\geq s} \frac{|u(x)|\|\delta_{\varphi(x)}\|}{\|\delta_x\|}.$$

Let S be an operator on a direct sum of Banach spaces $X = X_1 \oplus ... \oplus X_m$. Such an operator leaves invariant each direct subsum $X_k \oplus ... \oplus X_m$ if and only if it has a lower triangular matrix representation

$$S = \begin{pmatrix} S_{11} & 0 & 0 & \dots & 0 \\ S_{21} & S_{22} & 0 & \dots & 0 \\ \vdots & & & & \\ \dots & \dots & \dots & S_{m-1,m-1} & 0 \\ S_{m1} & S_{m2} & \dots & S_{m,m-1} & S_{mm} \end{pmatrix},$$

where $S_{jk} : X_j \to X_k$.

Theorem [GGL, Corollary 2.4]

Let $X = X_1 \oplus ... \oplus X_m$ be a direct sum of Banach spaces, and let S be an operator on X with a lower triangular matrix representation. If X is infinite dimensional, and the operators $S_{11}, .., S_{m-1,m-1}$ are Riesz operators, then $\sigma(S) = \sigma(S_{11}) \cup ... \cup \sigma(S_{mm})$.

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Recall that an operator S is called a Riesz operator if $r_e(S) = 0$.

The Taylor series expansion at 0 of each element f in $X(B_E)$ yields a direct sum decomposition of $X(B_E)$,

$$X(B_E) = P_0 \oplus \ldots \oplus P_{m-1} \oplus X_m(B_E)$$

because the mapping $f \in X(B_E) \mapsto P_k(f) \in P_k$ is a continuous projection of $X(B_E)$ thanks to conditions (II) and (III).

Consequently, uC_φ has a lower triangular matrix representation

$$uC_\varphi = \begin{pmatrix} C_{11} & 0 & 0 & \ldots & 0 \\ C_{21} & C_{22} & 0 & \ldots & 0 \\ \vdots & & & & \\ \ldots & \ldots & \ldots & C_{m-1,m-1} & 0 \\ C_{m1} & C_{m2} & \ldots & C_{m,m-1} & C_m, \end{pmatrix}$$

where the operator C_m is the restriction of uC_φ to $X_m(B_E)$.

Proposition

For every $f \in P_k$,

$$C_{kk}(f)(x) = u(0)\hat{f}\big(\varphi'(0)(x), \cdots, \varphi'(0)(x)\big),$$

where $\hat{f}$ *is the* k*-linear symmetric mapping determining* f.

Now we apply Lemma 3.1 in [GGL] to obtain

Lemma

$$\sigma\big(C_{kk}\big) = \{u(0) \cdot \lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \le j \le k\}.$$

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Now we apply Lemma 3.1 in [GGL] to obtain

Lemma

$$\sigma\big(C_{kk}\big) = \{u(0)\cdot\lambda_1\cdots\lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \le j \le k\}.$$

Lemma ([GM])

Let E be a complex Banach space and let $\varphi : B_E \to B_E$ be analytic such that $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that there exist $W \subset B_E$, with $\varphi(W) \subset W$, $\delta > 0$ and $\epsilon > 0$ such that

$$\frac{1 - \|\varphi(x)\|}{1 - \|x\|} \geq 1 + \epsilon, \quad \text{for all } x \in W \text{ such that } \|x\| \geq \delta \ . \tag{3.1}$$

Then, there exists a constant $M \geq 1$ which depends only on ϵ, such that any finite iteration sequence $\{x_0, x_1, ..., x_N\}$ satisfying $x_0 \in W$ and $\|x_N\| \geq \delta$ is an interpolating sequence for $H^\infty(B_E)$ with interpolation constant not greater than M.

We will refer to inequalities of the form (3.1) as Julia-type estimates.

Denote

$$\gamma(uC_\varphi; W) := \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{\substack{\|\varphi_n(x)\|\geq s \\ x\in W}} \frac{\|\delta_{\varphi_n(x)}\|_X |u(x)\cdot \ldots \cdot u(\varphi_n(x))|}{\|\delta_x\|_X}}$$

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Crucial Lemma

Consider the weighted composition operator uC_φ acting on $X(B_E)$. Assume that $\varphi(0) = 0$, $\|\varphi'(0)\| < 1$ and that $\varphi(B_E)$ is a relatively compact subset of E. Suppose also that $||\varphi_n|| = 1$ for all $n \in \mathbb{N}$ and that there exists $W \subseteq B_E$ with $\varphi(W) \subseteq W$ and such that a Julia-type estimate holds for some $\epsilon, \delta > 0$. If $\lambda \neq 0$ satifies $|\lambda| < \gamma(uC_\varphi; W)$, then $\lambda \in \sigma(uC_\varphi)$.

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Theorem

Consider the weighted composition operator uC_φ acting on $X(B_E)$. Assume that $\varphi(0)=0$, $\|\varphi'(0)\|<1$ and that $\varphi(B_E)$ is a relatively compact subset of E. Suppose that there exists $W\subseteq B_E$ with $\varphi(W)\subseteq W$ such that a Julia-type estimate holds for some $\epsilon,\delta>0$. Then

$$\{u(0)\}\cup\{u(0)\lambda_1\cdots\lambda_k : \lambda_j\in\sigma(\varphi'(0)), 1\leq j\leq k,\ k\geq 1\}\cup$$

$$\{\lambda : |\lambda|\leq \gamma(uC_\varphi;W)\}\subset\sigma(uC_\varphi).$$

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Corollary

Let $p \geq 1$ *and* $\alpha > -1$. *If* uC_φ *is a bounded operator on* $\mathcal{H}(\mathbb{B}_N)$, $A^p_\alpha(\mathbb{B}_N)$ *and* $H^p(\mathbb{B}_N)$ *respectively, with* $\varphi(0) = 0$ *and* $\|\varphi'(0)\| < 1$, *then*

$$\{u(0)\} \cup \{u(0)\lambda_1 \cdots \lambda_k : \lambda_j \in \sigma(\varphi'(0)), 1 \leq j \leq k,\ k \geq 1\} \cup$$

$$\{\lambda : |\lambda| \leq \gamma\big(uC_\varphi;\mathbb{B}_N\big)\} \subset \sigma(uC_\varphi).$$

Denote

$$\gamma(uC_\varphi; W) := \liminf_n \sqrt[n]{\lim_{s\to 1} \sup_{\substack{\|\varphi_n(x)\|\geq s \\ x\in W}} \frac{\|\delta_{\varphi_n(x)}\|_X |u(x)\cdot \ldots \cdot u(\varphi_n(x))|}{\|\delta_x\|_X}}$$

Corollary

Let E be a Hilbert space or $E = C_0(\mathcal{X})$, $\mathcal{X}$ a locally compact Hausdorff topological space. Assume that $uC_\varphi : H_v^\infty(B_E) \to H_v^\infty(B_E)$ is bounded with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Suppose that $\varphi(B_E)$ is a relatively compact subset of E. Then

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \gamma(uC_\varphi; \varphi(B_E))\} \cup \sigma_p(uC_\varphi) \subset \sigma(uC_\varphi).$$

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Corollary

Let B_E be either the n-ball $\mathbb{B}_N$ or the n-polydisc Δ_N. Assume that $uC_\varphi : H_v^\infty(B_E) \to H_v^\infty(B_E)$ is bounded with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Then

$$\left\{\lambda \in \mathbb{C} : |\lambda| \leq r_e(uC_\varphi)\right\} \cup \sigma_p(uC_\varphi) = \sigma(uC_\varphi).$$

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This holds essentially because on this case,

$$r_e(uC_\varphi) = \gamma\big(uC_\varphi; \varphi(B_E)\big).$$

Since also the equality $r_e(uC_\varphi) = \gamma\big(uC_\varphi; \varphi(B_E)\big)$ holds for $H^\infty(B_E)$, we have

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If $uC_\varphi : H^\infty(B_E) \to H^\infty(B_E)$ is bounded with $\varphi(0) = 0$ and $\|\varphi'(0)\| < 1$. Then

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Thus from here, we recover the main results concerning the spectrum in [GGL], [GM1] and [YZ].

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