It is with great sadness that we mark the death of our friend and colleague, Richard Timoney, who passed away on New Year's Day.
The AGA conference is dedicated to his memory.

Analysis, Geometry and Algebra

AGA

Evolutionary phenomena and non-associative algebras

M. Victoria Velasco

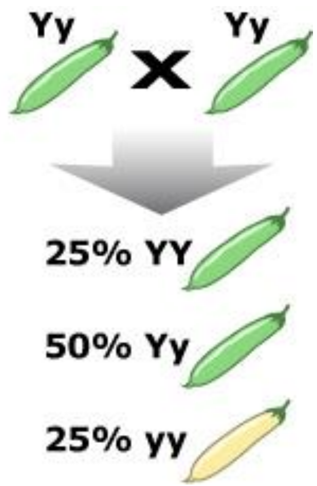
ugr Universidad de Granada

8-10th May 2019

TRINITY COLLEGE DUBLIN

Non-associative algebras in Genetics

Gregor Johann Mendel (1822 -1884)



Gregor Mendel is usually considered to be the founder of modern Genetics.

Summary of Mendel's laws

LAW	PARENT CROSS	OFFSPRING
DOMINANCE	TT x tt tall x short	100% Tt tall
SEGREGATION	Tt x Tt tall x tall	75% tall 25% short
INDEPENDENT ASSORTMENT	RrGg x RrGg round & green x round & green	9/16 round seeds & green pods 3/16 round seeds & yellow pods 3/16 wrinkled seeds & green pods 1/16 wrinkled seeds & yellow pods

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- Mendel's pea plant experiments conducted between 1856 and 1863 (testing thousands of pea plants) established many of the rules of heredity. Nowadays they are referred to as the laws of Mendelian inheritance.

- Mendel described these principles in a two-part paper, presented in 1865 to the Natural History Society of Brno (Czech Republic), and published one year later.

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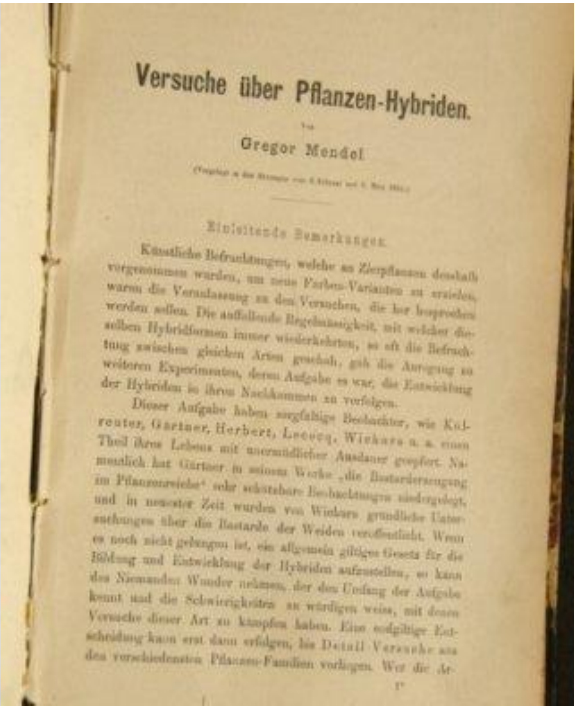
Non-associative algebras in Genetics

Proceedings of the Natural History Society of Brünn (1866).

- Few favorable reports in local newspapers, but ignored by the scientific community (only cited about three times over the next thirty-five years).
- Even the paper received plenty of criticism. Nevertheless, it is considered a germinal work today.
- The profound significance of Mendel's work was not recognized until 1900 more than three decades later with the rediscovery of his laws.



Versuche über Pflanzen-Hybriden.

von

Gregor Mendel

Einleitende Bemerkungen.

Künstliche Befruchtungen, welche an Zierpflanzen deshalb vorgenommen wurden, um neue Farben-Varianten zu erzielen, waren die Veranlassung zu den Versuchen, die hier besprochen werden sollen. Die auffallende Regelmässigkeit, mit welcher dieselben Hybridformen immer wiederkehrten, so oft die Befruchtung zwischen gleichen Arten geschah, gab die Anregung zu weiteren Experimenten, deren Aufgabe es war, die Entwicklung der Hybriden in ihren Nachkommen zu verfolgen.

Dieser Aufgabe haben sorgfältige Beobachter, wie Kölreuter, Gärtner, Herbert, Lecocq, Wichura u. a. einen Theil ihres Lebens mit unermüdlicher Ausdauer geopfert. Namentlich hat Gärtner in seinem Werke „die Bastarderzeugung im Pflanzenreiche" sehr schätzbare Beobachtungen niedergelegt, und in neuester Zeit wurden von Wichura gründliche Untersuchungen über die Bastarde der Weiden veröffentlicht. Wenn es noch nicht gelungen ist, ein allgemein giltiges Gesetz für die Bildung und Entwicklung der Hybriden aufzustellen, so kann das Niemanden Wunder nehmen, der den Umfang der Aufgabe kennt und die Schwierigkeiten zu würdigen weiss, mit denen Versuche dieser Art zu kämpfen haben. Eine endgiltige Entscheidung kann erst dann erfolgen, bis Detail-Versuche aus den verschiedensten Pflanzen-Familien vorliegen. Wer die Ar-

Erich von Tschermak, Hugo de Vries, Carl Correns, and William Jasper Spillman independently verified several of Mendel's experimental findings, ushering in the modern age of genetics.

> With only two publications in Verhandlungen des naturforschenden Vereins Brünn, a journal with a nano-scale JIF, Mendel would not have been given tenure in today's academic world.
>
> John Fleischman (2016)
>
> Celebration of the 150 aniversary of the paper

Non-associative algebras in Genetics

- G. Mendel himself was a pioneer in expressing the laws of genetics making use of mathematical notation.

- After Mendel, researchers such as Jennings (1917), Serebrovskij (1934) and Glivenko (1936) provided a more precise mathematical approach of Mendel's laws. This culminated in the algebraic formulation of Mendel's laws in the papers:

Etherington, I. M. H., *Genetic algebras., Proc. Royal Soc. Edinburgh 59,* (1939), *242-258.*

Etherington, I. M. H., Non-associative algebra and the symbolism of genetics, *Proceedings of the Royal Society of Edinburgh* vol. **61**, (1941) , 24-42.

Ivor Malcolm Haddon Etherington (1908-1994)

- After this, many other authors have made relevant contributions to Genetics, in the context of non-associative algebras: Schafer, Gonshor, Haldane, Holgate, Heuch, Reiersöl, Abraham, Lyubich and Wörz Busekros, among others. Therefore many non-associative algebras with genetic meaning arose such as Mendelian, gametic, zygotic, baric, train algebras, Bernstein algebras or evolution algebras, etc. All of them are named **genetic algebras**.

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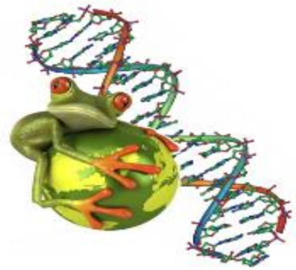
Non-associative algebras in Genetics

- Thus, *non-associative algebras become an appropriate mathematical framework for studying Mendelian genetics*.

Lyubich Y. I., *Mathematical structures in populations genetics*, Springer-Verlag 1992.

Reed, M. L, Algebraic structure of genetic inheritance, *Amer. Math. Soc. Bulletin*. New Series 34 (1997), 107–130.

Wörz-Busekros, A. *Algebras in genetics*, Lecture Notes in Biomathematics 36, Springer-Verlag, 1980.



- M. Reed (1997): Non-associative algebra, in general, is currently a very active field of mathematical research. However, in comparison with the body of literature **the study of the algebras associated with the problems of genetic inheritance is still in its infancy** of other classes of nonassociative algebras (e.g., Lie algebras or Jordan algebras), **…**
For whatever reason, these "genetic algebras" are not widely discussed or studied presently by American mathematicians. Hopefully, this article will open an avenue for future discussion and research into this fascinating class of nonassociative algebras and their relationship to the science of genetic inheritance.

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The necessary language from Mathematics

Definition: An **algebra** is a linear space A together with a bilinear map $(a, b) \to ab$ (the **product**), $A \times A \to A$.

If A is an algebra then it is said that:

- A is **associative** if $(ab)c = a(bc)$, for every $a, b, c \in A$.
- A is **commutative** if $ab = ba$, for every $a, b \in A$.

Unless otherwise specified, A does not need to be associative or commutative.

Thus, from now on: **Non-associative means non necessarily associative.**

A finite-dimensional algebra A is determined by means a basis $B = \{e_1, \cdots, e_n\}$ and a multiplication table given by the expresions

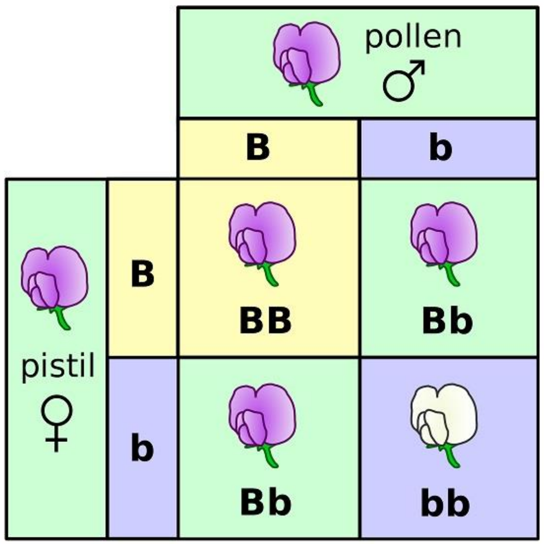
$$e_i e_j = \sum_{k=1}^{n} \alpha_{ijk}\, e_k.$$

Multiplication	e_1	e_2	$\cdots$	e_n
e_1				
e_2				
$\vdots$				
e_2				

The gametic algebra

- When gametes fuse (or reproduce) to form diploid zygotes a natural “multiplication“ operation occurs (which indicates how the hereditary information will be passed down to the next generation).

- Example: Gametic multiplication. Simple Mendelian inheritance for a single gene with two alleles A and a.



Punnett square diagram (tabular summary of possible combinations of maternal alleles with paternal ones).

Diploid zygotes

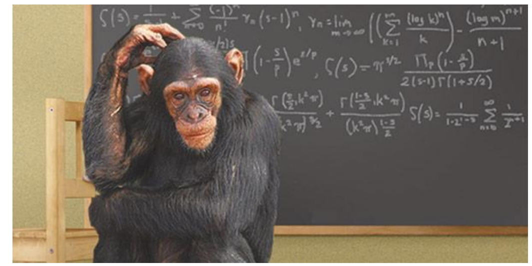
Multiplication	**B**	**b**
B	**B**	$\frac{1}{2}B + \frac{1}{2}\mathbf{b}$
b	$\frac{1}{2}B + \frac{1}{2}\mathbf{b}$	**b**

This models how hereditary information passed by reproduction of sperm and eggs, for instance.

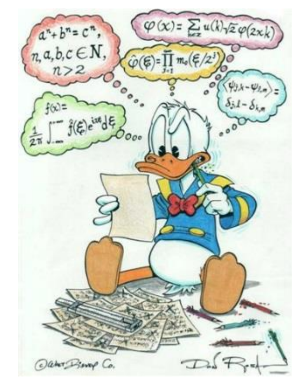
The gametic algebra

- Let A be the gametic algebra for a single gene with two alleles **B** and **b**. Remember that:

Multiplication	**B**	**b**
B	**B**	$\frac{1}{2}B + \frac{1}{2}\mathbf{b}$
b	$\frac{1}{2}B + \frac{1}{2}\mathbf{b}$	**b**



- Let us write it in the way that mathematicians usually do it.



Multiplication	e_1	e_2
e_1	e_1	$\frac{1}{2}e_1 + \frac{1}{2}e_2$
e_2	$\frac{1}{2}e_1 + \frac{1}{2}e_2$	e_2

This algebra is not associative!!

$$e_1(e_1\ e_2) \neq {e_1}^2\ e_2.$$

The zygotic algebra

- Multiplication table of the zygotic algebra for Simple Mendelian inheritance for a single gene with two alleles A and a. (Zygotic multiplication)

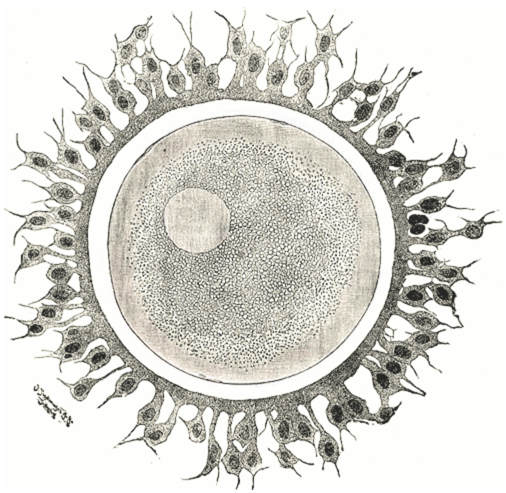
By multiplication of gametes we obtain the zygotes. Since, for instance,
$\mathbf{Bb} = \mathbf{B} \times \mathbf{b} = \frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{b}$ we have that $\mathbf{Bb} \times \mathbf{Bb} = (\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{b})(\frac{1}{2}\mathbf{B} + \frac{1}{2}\mathbf{b})$.

Therefore we obtain the following multiplication table:

×	BB	Bb	bb
BB	BB	$\frac{1}{2}(\mathbf{BB} + \mathbf{Bb})$	—Bb)
Bb	$\frac{1}{2}(\mathbf{BB} + \mathbf{Bb})$	$\frac{1}{4}\mathbf{BB} + \frac{1}{2}\mathbf{Bb} + \frac{1}{4}\mathbf{bb}$	$\frac{1}{2}(\mathbf{Bb} + \mathbf{bb})$
bb	Bb	$\frac{1}{2}(\mathbf{Bb} + \mathbf{bb})$	bb

If both gametes carry the same allele, then the offspring will inherit it.

If gametes carrying Bb and bb a reproduce, half of the time the offspring will inherit BB and the other half of the time it will inherit Bb.



Human zygote

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The zygotic algebra

- The zygotic algebra (for Simple Mendelian inheritance for a single gene with two alleles **B** and **b**) is the 3-dimensional real algebra A with basis $B = \{\mathbf{BB}, \mathbf{Bb}, \mathbf{bb}\}$ and multiplication given by

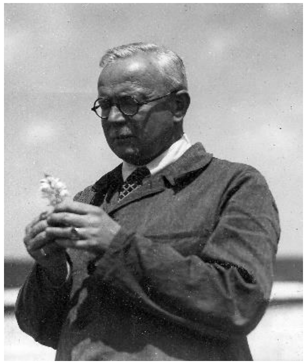
×	BB	Bb	bb
BB	$\mathbf{BB}$	$\frac{1}{2}(\mathbf{BB}+\mathbf{Bb})$	**Bb**
Bb	$\frac{1}{2}(\mathbf{BB}+\mathbf{Bb})$	$\frac{1}{4}\mathbf{BB}+\frac{1}{2}\mathbf{Bb}+\frac{1}{4}\mathbf{bb}$	$\frac{1}{2}(\mathbf{Bb}+\mathbf{bb})$
bb	**Bb**	$\frac{1}{2}(\mathbf{Bb}+\mathbf{bb})$	**bb**

- In standard mathematical notation, this is the algebra A with basis $B = \{e_1, e_2, e_3\}$ and multiplication table:

×	e_1	e_2	e_2
e_1	e_1	$\frac{1}{2}(e_1+e_2)$	e_2
e_2	$\frac{1}{2}(e_1+e_2)$	$\frac{1}{4}e_1+\frac{1}{2}e_2+\frac{1}{4}e_3$	$\frac{1}{2}(e_2+e_3)$
e_3	e_2	$\frac{1}{2}(e_2+e_3)$	e_3

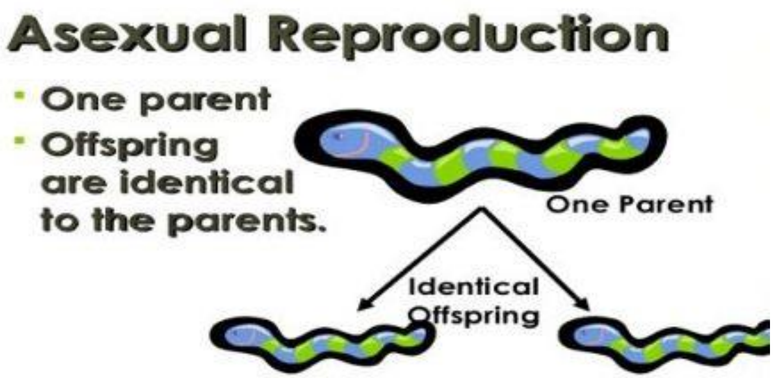
Non-Mendelian Genetics

After the works of E. Baur and C. Correns, it was clear that many hereditary mechanisms do not follow Mendel's laws. This is the case of incomplete dominance, systems of polygenes or multiple alleles, or the asexual inheritance. Therefore, non-Mendelian genetics arose.



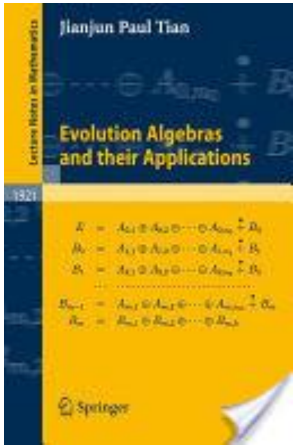
Erwin Baur (1875-1933)

Carl Correns (1864-1933)



Evolution algebras

J. P. Tian (2008): During the early days in this area, it appeared that **the general genetic algebras could be developed into a field of independent mathematical interest, because these algebras are in general not associative and do not belong to any of the well-known classes of nonassociative algebras such as Lie algebras, alternative algebras, or Jordan algebras….** They possess some distinguishing properties that lead to many interesting mathematical results.



J. P. Tian, *Evolution Algebras and their Applications. Series: Lecture Notes in Mathematics, Springer* (2008).

J. P. Tian (2008): Logically, we can ask **what Non-Mendelian Genetics offers to mathematics**. The answer is "evolution algebras"… It is in the interplay between purely mathematical structures and the corresponding genetic properties that makes this area so fascinating…..

- Thus, in this book, the so-called **evolution algebras** were introduced as a new and outstanding type of genetic algebras to model the non-Mendelian genetics.

Evolution algebras

Definition: An evolution algebra is an algebra A provided with a natural basis B. This is a basis $B = \{e_i : i \in \Lambda\}$ such that $e_i e_j = 0$, if $i \neq j$.

- If $e_i^2 = \sum_{k \in \Lambda} \omega_{ki} e_k$ then, the structure matrix is defined as $M(A)_B = (\omega_{ki})$ and encodes the **dynamic nature** of A.

$M(A)_B \in CFM_\Lambda(\mathbb{K})$ matrices of size $\Lambda \times \Lambda$ for which every column has a finite number of non-zero entries

- In general, evolution algebras are not power associative, Jordan, Lie etc. Nevertheless they are commutative and hence flexibles:

$$a(ba) = (ab)a, \quad \forall a, b \in A.$$

- The systematic study of **infinite-dimensional evolution algebras** from an algebraic point of view was achieved in the following paper.

Y. Cabrera, & M. Siles & M.V. V., Evolution algebras of arbitrary dimension and their decompositions. *Linear Algebra and its Applications* **495**, (2016) 122-162.

Finite-dimensional evolution algebras

A **finite-dimensional evolution algebra** is an algebra A provided with a basis $B = \{e_1, \dots, e_n\}$ respect to which the multiplication table is like

$\times$	e_1	e_2	$\cdots$	e_n
e_1	$e_1^2 = \sum_{k\in\Lambda} \omega_{k1} e_k$	0		0
e_2	0	$e_2^2 = \sum_{k\in\Lambda} \omega_{k2} e_k$		0
$\vdots$			$\ddots$	
e_n	0	0		$e_n^2 = \sum_{k\in\Lambda} \omega_{kn} e_k$

The product of A, and hence A is determined by the the **stucture matrix**

$$M_A(B) = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix}$$

$$\begin{aligned} e_1^2 &= \omega_{11}e_1 + \omega_{21}e_2 + \cdots + \omega_{n1}e_n \\ e_2^2 &= \omega_{12}e_1 + \omega_{22}e_2 + \cdots + \omega_{n2}e_n \\ &\vdots \\ e_n^2 &= \omega_{1n}e_1 + \omega_{2n}e_2 + \cdots + \omega_{nn}e_n \end{aligned}$$

Finite-dimensional evolution algebras

- Let A be an evolution algebra and $B = \{e_1, \dots, e_n\}$ a natural basis.
  Let $M_A(B)$ the structure matrix of A respect to B.

$$M_A(B) = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix}$$

$$e_k^2 = \omega_{k1}e_1 + \omega_{k2}e_2 + \cdots + \omega_{kn}e_n \qquad (k = 1, 2, \cdots, n)$$
$$e_i\, e_j = 0 \qquad (i \neq j)$$

- This matrix determines the product of A. Indeed, if $a = \sum \alpha_i e_i$ and $b = \sum \beta_i e_i$ then

$$ab = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1\beta_1 \\ \vdots \\ \alpha_n\beta_n \end{pmatrix}$$

- Also $M_A(B)$ determines the evolution operator of A relative to B, namely the unique linear map $L_B : A \to A$ such that

$$e_i \to L_B(e_i) = e_i^2$$
$$a = \sum \alpha_i e_i \to L_B(a) = \sum_{i=1}^n \alpha_i\, {e_i}^2$$

If e_i are gametes then, by self-replication, $e_i \to L_B(e_i) = {e_i}^2$.

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Evolution algebras and dynamics

Let A be an evolution algebra and $B = \{e_1, \dots, e_n\}$ a natural basis.

- The evolution operator is given by

$$a = \sum \alpha_i e_i \rightarrow L_B(a) = \alpha_1 e_1^2 + \dots + \alpha_n e_n^2$$

Fact:

$$L_B(a) = \alpha_1 e_1^2 + \dots + \alpha_n e_n^2 = (e_1 + \dots + e_n)(\alpha_1 e_1 + \dots + \alpha_n e_n)$$

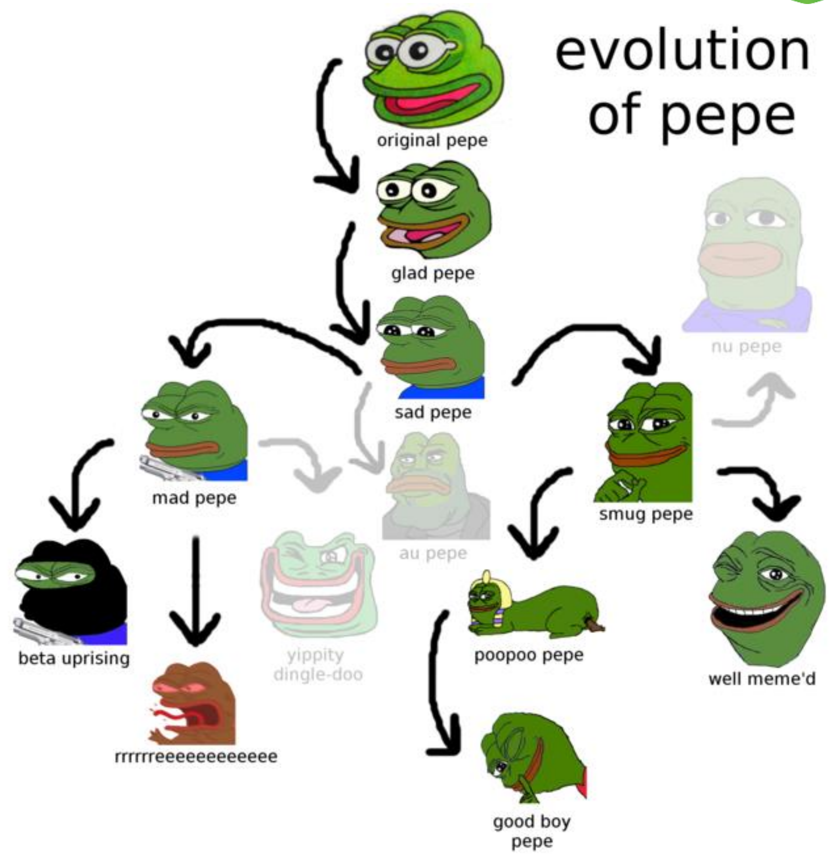
Consequently, if $e = e_1 + \dots + e_n$ then,

$$L_B(a) = L_e(a) = ea.$$

Thus,

$$L_B(a) = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

(evolution operator)



- If dim $A = \infty$ still makes sense somehow that $L_B(a) = L_e(a) = ea$ where $e = \sum_{i \in I} e_i$

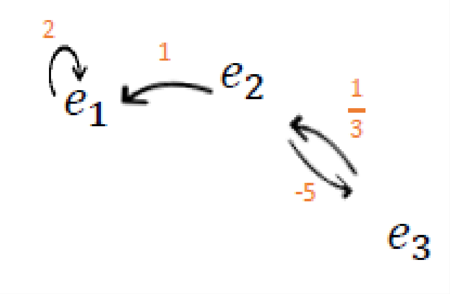
Evolution algebras and graphs

- Evolution operator: $L_e(a) = ea = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ (for $a = \sum \alpha_i e_i$).

Fact: By looking at $M_A(B)$ as the adjacency matrix of a graph we have that evolution algebras are deeply connected with weighted digraphs.

Example: $B = \{e_1, e_2, e_3\}$, natural basis of A with product

$e_1{}^2 = 2e_1$; $e_2{}^2 = e_1 - 5e_3$, and $e_3{}^2 = \frac{1}{3} e_2$.
$(e_i e_j = 0,$ if $i \neq j)$.



Structure matrix: $M(A)_B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & -5 & 0 \end{pmatrix}$

Evolution algebras and graphs

- Evolution operator: $L_e(a) = ea = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$ (for $a = \sum \alpha_i e_i$).

$L_e(e_i) = e_i^2 = \omega_{i1}e_1 + \omega_{i2}e_2 + \cdots + \omega_{in}e_n \equiv$ evolution of e_i in one generation.

$L_e(a)$ describes how the state a develops into another state in one generation.
$L_e^2(a) := L_e\left(L_e(a)\right)$ describes de evolution of a in two generations and similarly,
$L_e^n(a)$ describes de evolution of a after n-generations.

Facts:

- An evolution algebra is a discrete dynamical system.
- Moreover, whenever $M_A(B)$ is a stochastic matrix then, the above expression defines a discrete Markov process.

Evolution algebras

Graph Theory

Dynamical systems & Markov chains

Evolution algebras and their connections

Fact: Evolution algebras have direct and strong connections with many branches of the mathematics and other Sciences !!!

Markov Processes

Dinamical systems

Graph theory

Non-associative algebras

Biology

Phisycs

Evolution algebras

- How are the ideals of an evolution algebra?
- How characterize basic algebraic properties such as semisimplicity or non degeneracy?
- When an evolution algebra is the direct sum of some others?
- For a reducible evolution algebra, there exists an optimal decomposition? If there exists, have to be unique?

Definition: An algebra A is simple if $A^2 \neq 0$ and 0 is the only proper ideal.

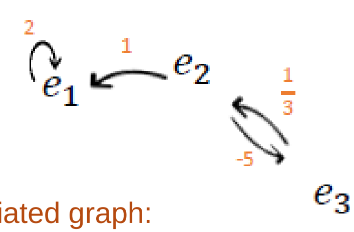
Theorem (characterization of simplicity): Let A be an evolution algebra and $B = \{e_i : i \in \Lambda\}$ a natural basis. Then,

(i) A is simple $\Leftrightarrow$ $A = lin\{e_i^2 : i \in \Lambda\}$ and the associated graph is connected

(ii) If $\dim A < \infty$ then, A is simple $\Leftrightarrow$ $|M_B(A)| \neq 0$ and $\Lambda = D(i)$ for every $i \in \Lambda$.

Example: If $B = \{e_1, e_2, e_3\}$ and $M(A)_B = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & -5 & 0 \end{pmatrix}$ then, A has nonzero proper ideals.

Associated graph:



Evolution algebras: descomposability

Answer to Problem of irreducibility in the finite-dimensional case:

Theorem: Let A be a non-degenerate evolution algebra and $B = \{e_1, \dots, e_n\}$ a natural basis. Then $A = I \oplus J$ for some ideals I and J if, and only if, for some reordering of B, the corresponding structure matrix is

$$\begin{pmatrix} W_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & \widetilde{W}_{(n-m)\times(n-m)} \end{pmatrix}.$$

In this case, $I = lin\{e_i : i = 1, \dots, m\}$ and $J = lin\{e_i : i = m+1, \dots, n\}$.

Answer to Problem of irreducibility in the general case:

Theorem: Let A be a non-degenerate evolution algebra and $B = \{e_i : i \in \Lambda\}$ a natural basis. Then, $A = \oplus_{\gamma\in\Gamma} I_\gamma$ (where each I_γ is an ideal of A) if, and only if, there exists a disjoint decomposition $\Lambda = \sqcup \Lambda_\gamma$ such that $I_\gamma = lin\{e_i : i \in \Lambda_\gamma\}$.

In this situation: For every $e_i \in B$, there exists a unique $\gamma \in \Gamma$ such that $e_i \in I_\gamma$.

Y. Cabrera, & M. Siles & M. V. V., Evolution algebras of arbitrary dimension and their decompositions. *Linear Algebra and its Applications* **495**, (2016) 122-162.

Evolution algebras: the Jacobson radical

- How is the Jacobson radical of an evolution algebra A?
  By the way, when A has a unit?
- When an evolution algebra A is semisimple? When A is radical?

Definition: The Jacobson radical a commutative algebra the intersection of all maximal modular ideals (M ideal such that $A(1-u) \subseteq M$).

Theorem: If $\dim(A)=\infty$ then A has not a unit.
If If $\dim(A)<\infty$ then A has a unit if $\Leftrightarrow$ $M(A)_B$ is diagonal with non-zero entries.

Definition: $i_0 \in \Lambda$ is a modular index if $\omega_{i_0 i_0} \neq 0$ and $\omega_{j i_0} = 0$ if $i_0 \neq$j.
$\Lambda_m = \{i \in \Lambda : \ i \text{ is modular}\}$. (Self-loops or bockles)

Theorem: $Rad(A) = lin\{e_i : i \in \Lambda \setminus \Lambda_m\}$. Moreover:
(i) A is a radical algebra $\Leftrightarrow$ $\Lambda_m = \emptyset$
(ii) A is semisimple $\Leftrightarrow \Lambda_m = \Lambda$

M.V. V., About the Jacobson radical of an evolution algebra, *to appear in J. Spectral Theory EMS.*

Evolution algebras as Banach algebras

- When an evolution algebra is a normed algebra?
  When it is Banach algebra?
- The evolution operator of a normed evolution algebra is continuous?

Theorem: Every evolution algebra is a normed algebra.

Proof: Let A be an evolution algebra and $\|\cdot\|_1$ the l_1-norm associated to a natural basis $B = \{e_i : i \in \Lambda\}$. Then,
$\|\cdot\|_1$ is an algebra norm $\Leftrightarrow$ $\|{e_i}^2\|_1 \leq 1, \quad \forall\, i \in \Lambda \Leftrightarrow$ L_B is continuous with $\|L_B\|_1 \leq 1$.

Theorem: Let $(A, \|\cdot\|)$ be an evolution algebra. Then:
A is Banach algebra $\Leftrightarrow A = A_0 \oplus A_1$ where A_1 is a finite-dimensional evolution algebra and A_0 is a zero product Banach algebra.

Corollary: Non-degenerate Banach evolution algebras are finite-dimensional.

Corollary: The completion of a non-degeneratre infinite-dimensional normed evolution algebra is not an evolution algebra.

P. Mellon, & M.V. V., The evolution operator and its dynamics on an evolution algebra, *Banach J. Math. Anal*.

Evolution algebras and equilibrium states

Fact: Let A be an evolution algebra.

(i) If $\dim(A) < \infty$ and if B is a normalized natural basis then, the l_1-norm respect to B is a Banach-algebra norm on A (and of course all norms on A are equivalent to it).

(i) If $\dim(A) = \infty$ then,
(a) The evolution operator L_B **does not** need to be continuous.
(b) $\|\cdot\|$ **is not** complete (if A is non-degenerate).
(c) The completion of A **is not** an evolution algebra.

Let us stay in the finite-dimensional case to study the behaviour of the evolution operator.

- As a discrete time dynamical system, the limit point of $\{L_e^n(b)\}$ describes the long-term evolutionary state of b.

- The role of e is not so important, because if $B = \{e_1, \ldots, e_n\}$ natural basis with evolution element $e = e_1 + \cdots + e_n$, then, $\tilde{B} = \{\alpha_1 e_1, \ldots, \alpha_n e_n\}$ is another natural basis (for non-zero α_i) with evolution element $\tilde{e} = \alpha_1 e_1 + \cdots + \alpha_n e_n$.

Evolution algebras and equilibrium states

- How are the equilibrium generators of a finite-dimensional evolution algebra A? When A reaches equilibrium?

Definition: Let A be a finite dimensional evolution algebra We say that $a \in \mathrm{A}$ is an equilibrium generator if $L_a^n(b)$ converges $\forall b \in \mathrm{A}$.
We say that A reaches equilibrium if e is a equilibrium generator.

Theorem: Let A be a finite dimensional evolution algebra and B be a natural basis.
For every $a \in \mathrm{A}$.

(i) If $\sigma_m(a) \cap \mathbb{C} \setminus \overline{\Delta} \neq \emptyset$ then, a is not an equilibrium generator.

(ii) If $\sigma_m(a) \subseteq \Delta$ then, a is trivially an equilibrium generator and $\lim_n L_a^n = 0$.

(iii) If $\sigma_m(a) \subseteq \overline{\Delta}$ and $\sigma_m(a) \cap \partial\overline{\Delta} \neq \emptyset$ then,

$$a \text{ is an equilibrium generator} \Leftrightarrow \sigma_m(a) \cap \partial\overline{\Delta} = \{1\} \text{ and 1 has index one.}$$

In this case, $P_a := \lim_n L_a^n$ is a projection that commutes with L_a. Moreover, the equilibrium points $\lim_n L_a^n(b)$ lie in $P_a(A)$, for all $b \in \mathrm{A}$.

P. Mellon, & M.V. V., The evolution operator and its dynamics on an evolution algebra, *Banach J. Math. Anal.*

Marcos J, C, & M.V. V., the multiplicative spectrum and the uniqueness of the complete norm topology, FILOMAT **28** (2014), 473-485.

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Evolution algebras: characterization

- Given a finite-dimensional algebra, A, how can we know if A is an evolution algebra?

Theorem: A be an evolution algebra; $B = \{e_1, \cdots, e_n\}$ a natural basis. Then,
(i) There exists unique bilinear maps $T_1, \ldots, T_n \colon \mathrm{A} \times A \to \mathbb{K}$ such that,

$$ab = \sum_{i=1}^{n} T_K(a, b) e_i \qquad \text{(for all } a, b \in A).$$

Moreover, the matrix associated to T_k respect to B is

$$M_k = \begin{pmatrix} \tau_K^B(e_1, e_1) & \cdots & \tau_K^B(e_1, e_n) \\ \vdots & & \vdots \\ \tau_K^B(e_\mathrm{n}, e_1) & \cdots & \tau_K^B(e_1, e_n) \end{pmatrix} \text{ where } \tau_K^B(a, b) = \alpha_k \text{ if } ab = \sum_{i=1}^{n} \alpha_i e_i.$$

(ii) The algebra A is an evolution algebra $\Leftrightarrow$ there exists and invertible matrix $Q \in M_{n\times n}(\mathbb{K})$ such that $Q^t M_k Q$ is diagonal for every $k = 1, \ldots, n$.

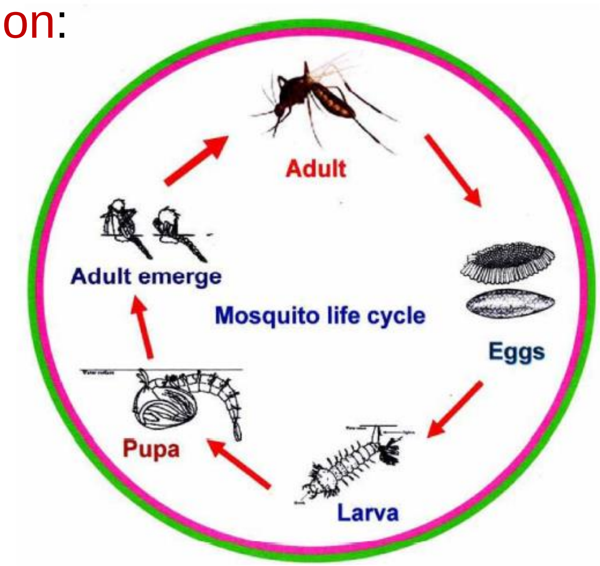
Fact: Therefore we arrive to the *problem of the simultaneously diagonalization via congruence (in fact both problems are equivalent).*

M. Bustamante, P. Mellon, M.V. V., Determining when a finite-dimensional algebra is an evolution algebra: solving the problem of simultaneously diagonalization via congruence.

The evolution algebra of mosquito population

• Evolution algebras are a helpful tool for mathematical modelling?

Evolution algebra of mosquito population:



- At $t \geq 0$ the state of the population of mosquito is given by the density vector

$$(E(t), L(t), P(t), A_h(t), A_r(t), A_o(t))$$

Where:
$E(t)$ = Eggs (hatches when exposed to water)
$L(t)$ = Larvas (hatched egges living in water)
$P(t)$ = Pupa (stage just before emerging as adult)
$A_h(t)$ = host seeking adults
$A_r(t)$ = resting adults
$A_o(t)$ = oviposigtoon seeking adults

- The dynamical behaviour of mosquito population is described by

$$\begin{aligned}
\frac{dE}{dt} &= b\rho_{A_o}A_o - (\mu_E + \rho_E)E,\\
\frac{dL}{dt} &= \rho_E E - (\mu_{1L} + \mu_{2L}L + \rho_L)L,\\
\frac{dP}{dt} &= \rho_L L - (\mu_P + \rho_P)P,\\
\frac{dA_h}{dt} &= \rho_P P + \rho_{A_o}A_o - (\mu_{A_h} + \rho_{A_h})A_h,\\
\frac{dA_r}{dt} &= \rho_{A_h}A_h - (\mu_{A_r} + \rho_{A_r})A_r,\\
\frac{dA_o}{dt} &= \rho_{A_r}A_r - (\mu_{A_o} + \rho_{A_o})A_o,
\end{aligned}$$

Discrete-time version system

$$\begin{aligned}
E_{n+1} &= b\theta O_n + (1 - \hat{e})E_n,\\
L_{n+1} &= eE_n + \left(1 - \hat{l}_1 - \hat{l}_2 L_n\right)L_n,\\
P_{n+1} &= aL_n + (1 - \hat{p})P_n,\\
H_{n+1} &= pP_n + \theta O_n + (1 - \hat{h})H_n,\\
R_{n+1} &= hH_n + (1 - \hat{r})R_n,\\
O_{n+1} &= rR_n + (1 - \hat{\theta})O_n,
\end{aligned}$$

The evolution algebra of the mosquito population

- Therefore the evolution algebra of mosquito population is defined as the evolution algebra A_{mosq} with natural basis $B = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ where

$$
\begin{aligned}
e_1^2 &= (1-\hat{e})e_1 + b\theta e_6,\\
e_2^2 &= ee_1 + (1-\hat{l}_1 - 2\hat{l}_2\epsilon)e_2\\
e_3^2 &= ae_2 + (1-\hat{p})e_3,\\
e_4^2 &= pe_3 + (1-\hat{h})e_4 + \theta e_6,\\
e_5^2 &= he_4 + (1-\hat{r})e_5,\\
e_6^2 &= re_5 + (1-\hat{\theta})e_6,
\end{aligned}
$$

Rozikov U.A., M.V. V., A discrete time dynamical system and an evolution algebra of mosquito population, *Journal of Math. Biology,* (2019) Mar; 78(4):1225-1244.

Theorem: A_{mosq} is simple if and only if the determinant of the Jacobian matrix is non-zero.

$$
J_{A_{mosq}} = \begin{pmatrix}
1-\hat{e} & 0 & 0 & 0 & 0 & b\theta\\
e & 1-\hat{l}_1-2\hat{l}_2L & 0 & 0 & 0 & 0\\
0 & a & 1-\hat{p} & 0 & 0 & 0\\
0 & 0 & p & 1-\hat{h} & 0 & \theta\\
0 & 0 & 0 & h & 1-\hat{r} & 0\\
0 & 0 & 0 & 0 & r & 1-\hat{\theta}
\end{pmatrix}
$$

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The evolution algebra of mosquito population

Theorem: A_{mosq} reaches equilibrium if, and only $L_e = P + S$ for $P, S \in L(\mathbb{R}^6)$ such that $P^2 = P,\ PS = SP = 0$ and $\rho(S) < 1$.

Theorem: A_{mosq} is a radical algebra.

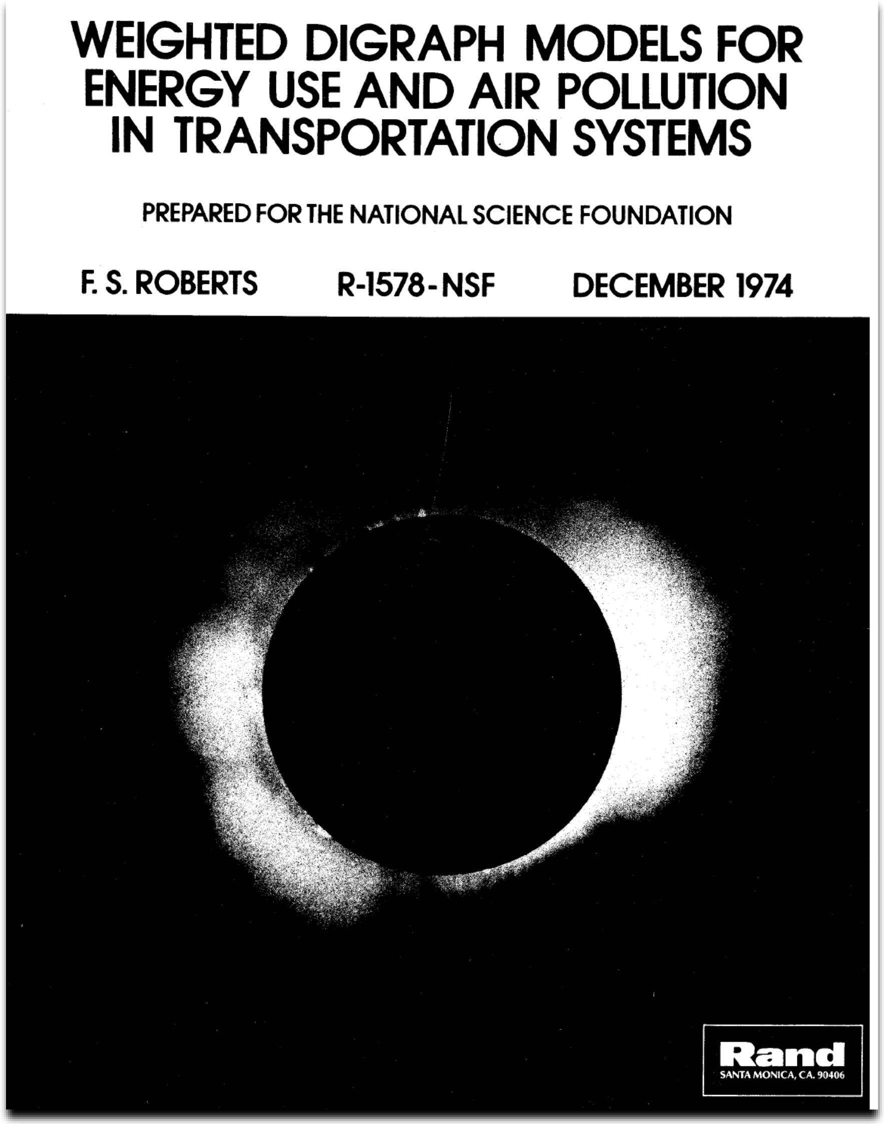
- Ideals on evolution algebras have biological meaning: when the system reach an ideal then all individuals in future generations will stay in it (i.e. an ideal is a subpopulation).
  Mosquito population either has not subpopulations or they have one with 5 genetators.

- Each point $v = (v_1,\ v_2,\ v_3,\ v_4,\ v_5,\ v_6) \in \mathbb{R}^6$ (under a certain rank of the variables) can be considerated a measure of the set $(E, L, P\ A_h\ A_r\ A_o)$, and we have described sets of initial states for which, if the population startes from them, then either
  - tends to extinction or
  - become stable or
  - become arbitrarily large.

Evolution algebras: current work

Current research
(Master Thesis):

To study the evolution algebra associated to these digraphs


WEIGHTED DIGRAPH MODELS FOR ENERGY USE AND AIR POLLUTION IN TRANSPORTATION SYSTEMS

PREPARED FOR THE NATIONAL SCIENCE FOUNDATION

F. S. ROBERTS R-1578-NSF DECEMBER 1974

Rand
SANTA MONICA, CA. 90406

- *The applicability of double weighted digraphs will require the development of mathematical techniques for handing time lags under different technical assumptions …*

They can be constructed relatively inexpensively and can be used at an early stage of a policymaking task to set directions for further research.

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AGA

Thanks for your attention!!