Bohr's phenomenon for functions on the Boolean cube

8-10th May 2019, Trinity College Dublin

A talk dedicated to the memory of Richard Timoney

Bohr's power series theorem, 1914

For every $f \in H_\infty(\mathbb{D})$

$$\sum_{n=0}^{\infty} \left| \frac{f^{(n)}(0)}{n!} \right| \boldsymbol{\frac{1}{3^n}} \leq \|f\|_{\mathbb{D}}\,,$$

and here the so-called Bohr radius $\boldsymbol{r = \frac{1}{3}}$ is optimal.

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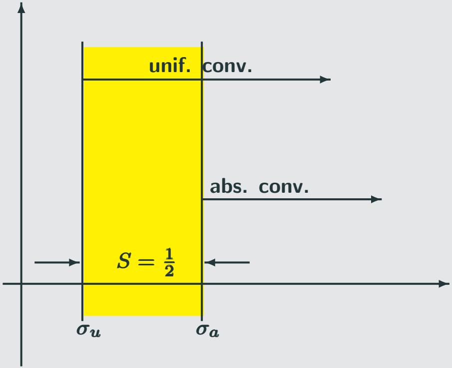
In terms of Fourier analysis ...

For every $f \in H_\infty(\mathbb{T})$

$$\sum_{n=0}^{\infty} |\widehat{f}(n)| \frac{\mathbf{1}}{\mathbf{3^n}} \leq \|f\|_{\mathbb{T}}$$

Harald Bohr, 1887-1951

Bohr-Bohnenblust-Hille Theorem, 1931:



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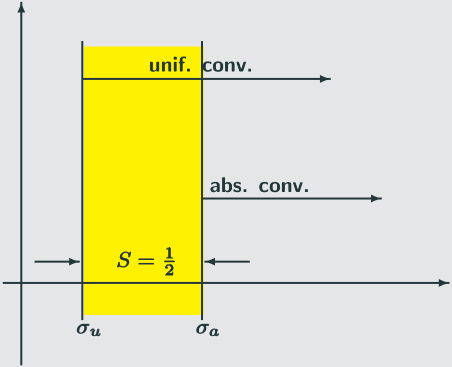

Bohr's vision . . .

Two important papers of Richard and Sean

- Absolute bases, tensor products, and a theorem of Bohr, Studia Math. 1989
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Problem of Richard and Sean

For each N, describe the set of all $\boldsymbol{r} \in [0,1]^N$ such that for any $f \in H_\infty(\mathbb{T}^N)$

$$\sum_{\alpha \in \mathbb{N}_0^N} |\widehat{f}(\alpha)| \boldsymbol{r}^\alpha \leq \|f\|_{\mathbb{T}^N} .$$

Definition – Harald Bohr radius:

$$K^N := \sup\left\{0 < \boldsymbol{r} < 1 : \sum_{\alpha \in \mathbb{N}_0^N} |\widehat{f}(\alpha)| \boldsymbol{r}^{|\alpha|} \leq \|f\|_{\mathbb{T}^N},\ f \in H_\infty(\mathbb{T}^N)\right\}$$

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Bohr radius of $\mathcal{F}_N \subset H_\infty(\mathbb{T}^N)$

$$K(\mathcal{F}_N) := \sup\left\{0 < \boldsymbol{r} < 1 : \sum_{\alpha\in\mathbb{N}_0^N} |\widehat{f}(\alpha)|\,\boldsymbol{r}^{|\alpha|} \le \|f\|_{\mathbb{T}^N},\ f \in \mathcal{F}_N\right\}$$

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Bohr's power series theorem

$$K^1 = K(H_\infty(\mathbb{T})) = \frac{1}{3}$$

Definition – Niels Bohr radius

The Bohr radius is a physical constant, approximately equal to the most probable distance between the center of a nuclide and the electron in an atom in its ground state.

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Harald and Niels

Multidimensional challenges

functions	Bohr radius
$H_\infty(\mathbb{T}^N)$	K^N
$\mathcal{P}(\mathbb{T}^N)$	K^N_{pol}
$\mathcal{P}_{\mathsf{hom}}(\mathbb{T}^N)$	K^N_{hom}
$\mathcal{P}_{\leq d}(\mathbb{T}^N)$	$K^N_{\leq d}$
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Asymtotics?

Highlight

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Crucial case – Defant-Frerick-Ortega-Ounaies-Seip 2011

$$K^N_{\leq d} \sim \begin{cases} \sqrt{\frac{\log N}{N}} & \log N \leq d \\ \left(\frac{d}{N}\right)^{\frac{d-1}{2d}} & d < \log N \end{cases}$$

Crucial tool – Bohneblust-Hille-inequality, 1931

- $\forall\, f \in \mathcal{P}_{\leq d}(\mathbb{T}^N):\ \left(\sum_{|\alpha|\leq d} |\widehat{f}(\alpha)|^{\frac{2d}{d+1}}\right)^{\frac{d+1}{2d}} \leq\ C(d)\, \|f\|_{\mathbb{T}^N}$
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Notation

$$\mathrm{BH}_{\mathbb{T}}^{\leq d} := \text{best constant } C(d)$$

Original estimate from 1931:

$$\mathrm{BH}_{\mathbb{T}}^{\leq d} \leq 2^{\frac{d-1}{2}} \frac{d^{d+\frac{d+1}{2d}}}{(d!)^{\frac{d+1}{2d}}}$$

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Defant-Frerick-Ortega-Ounaies-Seip 2011:

$$\mathrm{BH}_{\mathbb{T}}^{\leq d} \leq \sqrt{2}^{d}$$

Best result so far – Bayart-Pellegrino-Seoane 2014:

$$\mathrm{BH}_{\mathbb{T}}^{\leq d} \leq C^{\sqrt{d \log d}}$$

In particular:

$$\overline{\lim}_{d} \sqrt[d]{\mathrm{BH}_{\mathbb{T}}^{\leq d}} = 1$$

Real Bohr radii?

Is there any $\boldsymbol{r} > 0$ such that for all real polynomials $f = \sum_{n=0}^{d} c_n x^n$

$$\sum_{n=0}^{d} |c_n| \boldsymbol{r^n} \leq \|f\|_{[-1,1]} \,\textbf{?}$$

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Consider

$$f(x) = 1 - x^2$$

This example shows that the Bohr radius of all real polynomials on $[-1, 1]$ is 0. Then also all real polynomials on $[-1, 1]^N$ have no positive Bohr radius. What about smaller classes of such polynomials?

Real BH-inequality ?

Is there some constant $C(d) > 0$ such that for every real degree-d polynomial $f(x) = \sum_{|\alpha|\leq d} c_\alpha x^\alpha$, $x \in \mathbb{R}^N$

$$\Big(\sum_{|\alpha|\leq d} |c_\alpha|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq \; C(d)\, \|f\|_{[-1,1]^N} \; ?$$

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Yes – Klimek 1995:

For every degree d polynomial $P : \mathbb{C}^N \to \mathbb{C}$

$$\|P\|_{\mathbb{D}^N} \leq (1+\sqrt{2})^d \|P\|_{[-1,1]^N}$$

Real BH-inequality ?

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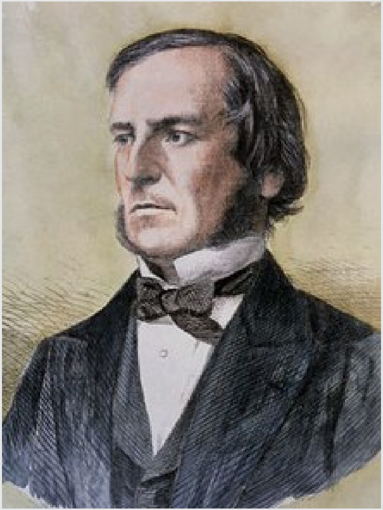
Defant-Mastyło-Pérez 2018

$$\overline{\lim}_d \ \sqrt[d]{\mathrm{BH}^{\le d}_{[-1,1]}} = 1 + \sqrt{2}$$

Boolean radii and the BH-inequality on the Boolean cube

George Boole, 1815-1864



Fourier analysis of functions on the Boolean cube

$$f : \{\pm 1\}^N \to \mathbb{R}\,, \quad N \in \mathbb{N}$$

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Applications

- Theoretical computer sciences
- Combinatorics
- Graph theory
- Social choice theory
- Cryptography
- Quantum computation

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Example – majority function

$$\mathsf{Maj}(x) = \mathsf{sign}(x_1 + \ldots + x_N)$$

Fourier analysis of functions on Boolean cubes $\{\pm 1\}^N$

- $G = \{\pm 1\}^N$ compact abelian group
- Fourier-Walsh expansion:

$$f(x) = \sum_{S \subset \{1,\dots,N\}} \widehat{f}(S) \prod_{n \in S} x_n$$

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degree-d functions

$$\text{degree of } f \; := \max\{|S| \colon \hat{f}(S) \neq 0\} \leq d$$

d-homogeneous functions

$$\hat{f}(S) \neq 0 \text{ only if } |S| = d$$

Finding Fourier coefficients of functions on the Boolean cube may not be easy:

$$\widehat{\mathsf{Maj}}(S) = \begin{cases} 0 & |S| \quad \text{even} \\ (-1)^{\frac{|S|-1}{2}} \frac{1}{2^{N-1}} \binom{N-1}{\frac{N-1}{2}} \binom{\frac{N-1}{2}}{\frac{|S|-1}{2}} \binom{N-1}{|S|-1}^{-1} & |S| \quad \text{odd} \end{cases}$$

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Definition – Boolean radius

$$\rho(\mathcal{F}_N) := \sup \left\{ 0 < \boldsymbol{\rho} < 1 : \sum_S |\widehat{f}(S)| \, \boldsymbol{\rho}^{|S|} \leq \|f\|_{\{\pm 1\}^N} , \ f \in \mathcal{F}_N \right\},$$

where $\mathcal{F}_N$ is a subset of functions $f : \{\pm 1\}^N \to \mathbb{R}$.

Question

Is there any hope for a positive Boolean radius $\rho(\mathcal{F}_N)$?

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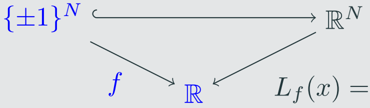
Yes...



$$L_f(x) = \sum_{S \subset \{1,\dots,N\}} \widehat{f}(S) \prod_{k \in S} x_k$$

In this case:

$$\sup_{x \in \{\pm 1\}^N} |f(x)| = \sup_{x \in [-1,1]^N} |L_f(x)|$$

For what?

- Can Bohr's world offer techniques unknown in Boole's world, and vice versa?
- Is there any hope to connect Bohr's world with the topic of quantum information theory?

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Two recent articles ...

- Defant-Mastyło-Pérez: Bohr's phenomenon for functions on the Boolean cube, JFA 2018
- Defant-Mastyło-Pérez: On the Fourier spectrum of functions on the Boolean cube, Math. Ann. 2019

Comparing Bohr and Boolean radii – Defant-Mastyło-Pérez 2018

	Bohr	Boole
all functions	$\sqrt{\frac{\log N}{N}}$	$2^{1/N} - 1$
all degree-d fct.	$\begin{cases} \sqrt{\frac{\log N}{N}} & \log N \leq d \\ \left(\frac{d}{N}\right)^{\frac{d-1}{2d}} & d < \log N \end{cases}$	$\frac{1}{\sqrt{dN}}$
all d-homo. fct.	$\left(\frac{d}{N}\right)^{\frac{d-1}{2d}}$	$\left(\frac{N}{\binom{N}{d}}\right)^{\frac{1}{2d}}$
all homo. fct.	$\sqrt{\frac{\log N}{N}}$	$\sqrt{\frac{\log N}{N}}$

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A Bohnenblust-Hille inequality for functions on the Boolean cube with good control of the constants...

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- $\forall\, f : \{\pm 1\}^N \to \mathbb{R}$ of degree d :

$$\Big(\sum_S |\widehat{f}(S)|^{\frac{2d}{d+1}} \Big)^{\frac{d+1}{2d}} \leq\; C(d)\, \|f\|_{\{\pm 1\}^N}$$

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But the constants are bad ...

$$\mathrm{BH}^{\le d}_{\{\pm 1\}} \ll (d+1)^{\frac{d+1}{2d}} (d!)^{\frac{d-1}{2d}} \sqrt{d}^{\,d+1} (2e)^d$$

What about dth roots ?

Are the dth roots of the constants $\mathrm{BH}^{\leq d}_{\{\pm 1\}}$ subexponential? Recall that this does not hold for the constants $\mathrm{BH}^{\leq d}_{[-1,1]}$.

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Defant-Mastyło-Pérez 2019

There exists a constant $C > 0$ such that for all d

$$\mathrm{BH}_{\{\pm 1\}}^{\leq d} \leq C^{\sqrt{d \log d}}$$

In particular,

$$\overline{\lim}_d \sqrt[d]{\mathrm{BH}_{\{\pm 1\}}^{\leq d}} = 1\,.$$

Problem, Montanaro 2013

$$? \quad \exists\, C \geq 1\ \forall\, d: \quad \mathrm{BH}^{\leq d}_{\{\pm 1\}} \leq d^C \quad ?$$

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Why this problem ?

The need for structure in quantum speedups...

- Quantum computers can offer superpolynomial speedups over classical computers, but only for certain structured problems.
- For every unstructured problem the quantum complexity and the classical complexity are polynomially related**?**

All this is true if the answer to the following so-called AA–conjecture is yes:

Given $f : \{\pm 1\}^N \to [-1, 1]$ of degree d, is there some $1 \leq j \leq N$ such that

$$\left(\frac{\sum_{S \neq \emptyset} \widehat{f}(S)^2}{d} \right)^{O(1)} \leq \sum_{S : j \in S} \widehat{f}(S)^2 \quad ?$$

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O'Donnell: Analysis of Boolean functions

If true, this conjecture would have significant consequences regarding the limitations of efficient quantum computation.

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Hope

There is some hope/evidence that a positive answer to Montanaro's problem on the BH-inequality for functions on the Boolean cube implies the AA-conjecture...