Complex Contact Manifolds

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Short Presentation

In this lecture the complex contact manifolds from a Riemannian geometric point of view, comparing the ideas with those of real contact metric geometry, are discussed. One important notion is that of a *normal complex contact metric structure*.

In real contact geometry the question of locally symmetric contact metric manifolds has a **long history** and a **short answer**.

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Boeckx and Cho, 2006: a locally symmetric contact metric manifold is locally isometric to $S^{2n+1}(1)$ or to $E^{n+1} \times S^n(4)$, the tangent sphere bundle of Euclidean space.

Various studies and generalizations of this question were made in the intervening years. Perhaps most importantly, since the locally symmetric condition is very restrictive, Takahashi, 1977, introduced the notion of a locally ϕ-symmetric space for Sasakian manifolds by restricting the locally symmetric condition to the contact subbundle and showed that these manifolds locally fiber over Hermitian symmetric spaces.

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Blair and Vanhecke, 1987: this condition is equivalent to reflections in the integral curves of the characteristic (Reeb) vector field being isometries.

Subsequently, to extend the notion to contact metric manifolds, Boeckx and Vanhecke, 1997, took this reflection idea as the definition of a strongly locally ϕ-symmetric space; a contact metric manifold satisfying the condition of restricting local symmetric to the contact subbundle is called a weakly locally ϕ-symmetric space.

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• We show that a locally symmetric normal complex contact metric manifolds is **locally isometric** to the complex projective space, $\mathbb{C}P^{2n+1}(4)$, of constant holomorphic curvature $+4$.

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• When such reflections are **isometries** we show that **the manifold fibers locally over a locally symmetric space**.

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• If the complex contact structure is given by a **global holomorphic contact form**, then the **manifold fibers over a locally symmetric complex symplectic manifold**.

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The complex contact structure determines a non-integrable subbundle $\mathcal{H}$ by the equation $\theta = 0$; $\mathcal{H}$ is called the *complex contact subbundle* or the *horizontal subbundle*.

Already in 1959 Kobayashi (also Boothby, 1961, 1962) observed that for a compact complex contact manifold a complex contact structure is given by a global 1-form if and only if the first Chern class vanishess. It is for this reason that we do not require global contact forms. Even for the canonical example of a complex contact manifold, $\mathbf{C}P^{2n+1}$, the structure is not given by a global form.

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In fact since a holomorphic differential form on a compact Kaehler manifold is not closed, no compact Kaehler manifold has a complex contact structure given by a global contact form.

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In particular, Foreman (2000) gave a complex Boothby-Wang fibration with global complex contact form and vertical fibres $S^1 \times S^1$.

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1) In each $\mathcal{O}$ there exist 1-forms u and $v = u \circ J$ with dual vector fields U and $V = -JU$ and (1,1) tensor fields G and $H = GJ$ such that

$$G^2 = H^2 = -I + u \otimes U + v \otimes V,$$

$$GJ = -JG, \quad GU = 0, \quad g(X, GY) = -g(GX, Y),$$

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2) On $\mathcal{O} \cap \mathcal{O}' \neq \emptyset$,

$$u' = Au - Bv, \quad v' = Bu + Av,$$

$$G' = AG - BH, \quad H' = BG + AH$$

where A and B are functions with $A^2 + B^2 = 1$.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is $u - iv$ to within a non-vanishing complex-valued function multiple. The local tensor fields G and H are related to du and dv by

$$du(X, Y) = \widehat{G}(X, Y) + (\sigma \wedge v)(X, Y),$$
$$dv(X, Y) = \widehat{H}(X, Y) - (\sigma \wedge u)(X, Y)$$

for some 1-form σ and where $\widehat{G}(X, Y) = g(X, GY)$ and $\widehat{H}(X, Y) = g(X, HY)$.

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for some 1-form σ and where $\widehat{G}(X,Y) = g(X,GY)$ and $\widehat{H}(X,Y) = g(X,HY)$.

Moreover on $\mathcal{O} \cap \mathcal{O}'$ it is easy to check that $U' \wedge V' = U \wedge V$ and hence we have a global vertical bundle $\mathcal{V}$ orthogonal to $\mathcal{H}$ which is generally assumed to be integrable; in this case σ takes the form $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi-Civita connection of g. The subbundle $\mathcal{V}$ can be thought of as the analogue of the characteristic or Reeb vector field of real contact geometry.

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In this setting Foreman proved a converse to his construction as a complex Boothby-Wang theorem.

Theorem. *Let P be a $(2n+1)$-dimensional compact complex contact manifold with a global contact form $\theta = u - iv$ such that the corresponding vertical vector fields U and V are regular. Then θ generates a free $S^1 \times S^1$ action on P and $p : P \to M$ is a principal $S^1 \times S^1$-bundle over a complex symplectic manifold M such that θ is a connection form for this fibration and the complex symplectic form Φ on M is given by $p^\star \Phi = d\theta$.*

Examples of Complex Contact Manifolds

- Complex Heisenberg group
- Odd-dimensional complex projective space
- Twistor spaces
- The complex Boothby-Wang fibration
- $\mathbf{C}^{n+1} \times \mathbf{C}P^n(16)$

Normal Complex Contact Manifolds

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Their notion is the vanishing of the two tensor fields S and T given by

$$S(X,Y) = [G,G](X,Y) + 2\widehat{G}(X,Y)U - 2\widehat{H}(X,Y)V + 2(v(Y)HX$$

$$-v(X)HY) + \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,$$

$$T(X,Y) = [H,H](X,Y) - 2\widehat{G}(X,Y)U + 2\widehat{H}(X,Y)V + 2(u(Y)GX$$

$$-u(X)GY) + \sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX$$

where $[G,G]$ and $[H,H]$ denote the Nijenhuis tensors of G and H respectively.

However this notion seems to be too strong; among its implications is that the underlying Hermitian manifold (M, g) is Kähler. Thus while indeed one of the canonical examples of a complex contact manifold, the odd-dimensional complex projective space, is normal in this sense, the complex Heisenberg group, is not.

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A complex contact metric structure is *normal* if

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$$S(U, X) = T(V, X) = 0, \text{ for every } X.$$

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Even though the definition appears to depend on the special nature of U and V, it respects the change in overlaps, $\mathcal{O} \cap \mathcal{O}'$, and is a global notion. With this notion of normality both odd-dimensional complex projective space and the complex Heisenberg group with their standard complex contact metric structures are normal.

We now give expressions for the covariant derivatives of the structures tensors on a normal complex contact metric manifold:

$$\nabla_X U = -GX + \sigma(X)V,$$

$$\nabla_X V = -HX - \sigma(X)U.$$

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$$\begin{gathered} g((\nabla_X G)Y, Z) = \sigma(X)g(HY, Z) + v(X)d\sigma(GZ, GY) \\ -2v(X)g(HGY, Z) - u(Y)g(X, Z) - v(Y)g(JX, Z) \\ +u(Z)g(X, Y) + v(Z)g(JX, Y), \\ g((\nabla_X H)Y, Z) = -\sigma(X)g(GY, Z) - u(X)d\sigma(HZ, HY) \\ -2u(X)g(GHY, Z) + u(Y)g(JX, Z) - v(Y)g(X, Z) \\ +u(Z)g(X, JY) + v(Z)g(X, Y). \end{gathered}$$

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-2u(X)g(GHY, Z) + u(Y)g(JX, Z) - v(Y)g(X, Z) \\
+u(Z)g(X, JY) + v(Z)g(X, Y).
\end{gathered}$$

For the underlying Hermitian structure we have

$$\begin{gathered}
g((\nabla_X J)Y, Z) = u(X)\big(d\sigma(Z, GY) - 2g(HY, Z)\big) \\
+v(X)\big(d\sigma(Z, HY) + 2g(GY, Z)\big).
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$$d\sigma(JX, Y) = -d\sigma(X, JY),$$

$$d\sigma(GY, GX) = d\sigma(X, Y) - 2u \wedge v(X, Y) d\sigma(U, V),$$

$$d\sigma(HY, HX) = d\sigma(X, Y) - 2u \wedge v(X, Y) d\sigma(U, V),$$

$$d\sigma(U, X) = v(X) d\sigma(U, V), \quad d\sigma(V, X) = -u(X) d\sigma(U, V).$$

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$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]},$$

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First of all we have $R(U,V)V = -2d\sigma(U,V)U$ and a similar expression for $R(V,U)U$, either of which gives the sectional curvature $R(U,V,V,U) = -2d\sigma(U,V)$.

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$$R(X,U)U = X, \quad R(X,V)V = X,$$

$$R(X,Y)U = 2(g(X,JY) + d\sigma(X,Y))V,$$

$$R(X,Y)V = -2(g(X,JY) + d\sigma(X,Y))U,$$

$$R(X,U)V = \sigma(U)GX + (\nabla_U H)X - JX,$$

$$R(X,V)U = -\sigma(V)HX + (\nabla_V G)X + JX,$$

Locally Symmetric Normal Complex Contact Manifolds

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We give a characterization in complex contact geometry of complex projective space of constant holomorphic curvature $+4$, $\mathbb{C}P^{2n+1}(4)$:

Theorem

Theorem 1. *[BM-Rocky] Let M^{2n+1} be a locally symmetric normal complex contact metric manifold. Then M^{2n+1} is locally isometric to $\mathbb{C}P^{2n+1}(4)$. Thus in the complete, simply connected case the manifold is globally isometric to $\mathbb{C}P^{2n+1}(4)$.*

Proof's sketch: We begin with the observation that since our manifold is locally symmetric it is semi-symmetric, i. e. $R \cdot R = 0$, so that

$$R(R(X,Y)X_1, X_2, X_3, X_4) + R(X_1, R(X,Y)X_2, X_3, X_4)$$

$$+R(X_1, X_2, R(X,Y)X_3, X_4) + R(X_1, X_2, X_3, R(X,Y)X_4) = 0.$$

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Take $X_4 = U$, $X_1 = X_3 = Y = V$, and X_2 and X horizontal
This gives us two cases to consider,

$$2 + d\sigma(U,V) = 0 \text{ and } g(X_2, JX) + d\sigma(X_2, X) = 0.$$

In the first case first note that since $R(U,V)V = -2d\sigma(U,V)U$, $d\sigma(U,V) = -2$ implies that

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Also we can prove that

$$R(Y,JY,JY,Y) = 4.$$

We compute the holomorphic sectional curvature for a general vector $X = X' + u(X)U + v(X)V$. Suppose both the horizontal and vertical holomorphic sectional curvatures have value μ. Then a long computation using normality gives

$$R(X, JX, JX, X) = \mu(|X'|^4 + (u(X)^2 + v(X)^2)^2) - 4|X'|^2(u(X)^2 + v(X)^2)$$
$$+6(u(X)^2 + v(X)^2)d\sigma(X', JX'),$$

but for us $\mu = 4$ and $d\sigma = -2\Omega$, where Ω is the fundamental 2-form of Hermitian structure, giving $R(X, JX, JX, X) = 4$ for all X.

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Thus the complex contact metric manifold M is locally isometric to $\mathbb{C}P^{2n+1}(4)$.

To eliminate the second case, note that $R(Z, U, V, U) = 0$ for horizontal Z and from this a contradiction occurs.

Reflections in the Vertical Foliation

As we have seen, the condition of local symmetry for a normal complex contact metric manifold is extremely strong. We therefore consider a weaker condition in terms of local reflections in the integral submanifolds of the vertical subbundle of a normal complex contact metric manifold. To do this we first recall the notion of a *local reflection* in a submanifold.

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Given a Riemannian manifold (M, g) and a submanifold N, *local reflection* in N, φ_N, is defined as follows. For $m \in M$ consider the minimal geodesic from m to N meeting N orthogonally at p. Let X be the unit vector at p tangent to the geodesic in the direction toward m. Then φ_N maps $m = \exp_p(tX) \longrightarrow \exp_p(-tX)$.

Chen and Vanhecke, 1989: necessary and sufficient conditions for a reflection to be isometric.

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Theorem. *Let (M, g) be a Riemannian manifold and N a submanifold. Then the reflection φ_N is a local isometry if and only if:*

N is totally geodesic;

$(\nabla^{2k}_{X \cdots X} R)(X, Y)X$ *is normal to N;*

$(\nabla^{2k+1}_{X \cdots X} R)(X, Y)X$ *is tangent to N;*

$(\nabla^{2k+1}_{X \cdots X} R)(X, V)X$ *is normal to N*

for all vectors X, Y normal to N and vectors V tangent to N and all $k \in \mathbb{N}$.

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Proposition. *Let γ be a geodesic on a normal complex contact metric manifold. If $\gamma'(0)$ is a horizontal vector , then $\gamma'(s)$ is horizontal for all s.*

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An easy computation shows that the horizontal parts of the Lie derivatives $\mathcal{L}_U g$ and $\mathcal{L}_V g$ vanish. Thus the metric is projectable and we denote by g' the metric on the base, ∇' its Levi-Civita connection and R' its curvature. For vectors fields X, Y, etc. on the base we denote by X^*, etc. their horizontal lifts to M.

Theorem

Theorem 2. *[BM-Rocky] Let* M^{2n+1} *be a normal complex contact metric manifold and suppose that the foliation induced by vertical subbundle is regular. If reflections in the integral submanifolds of the vertical subbundle are isometries, then the manifold fibers over a locally symmetric space.*

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First note that from the fundamental equations of a Riemannian submersion,

$$\nabla_{X^*} Y^* = (\nabla'_X Y)^* + u(\nabla_{X^*} Y^*)U + v(\nabla_{X^*} Y^*)V.$$

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$$\nabla_{X^*} Y^* = (\nabla'_X Y)^* + u(\nabla_{X^*} Y^*)U + v(\nabla_{X^*} Y^*)V.$$

Consequently from the equations for the curvature of a Riemannian submersion

$$\begin{aligned} & R(X^*, Y^*, Z^*, W^*) \\ = & R'(X, Y, Z, W) + 2(u(\nabla_{X^*} Y^*)u(\nabla_{Z^*} W^*) + v(\nabla_{X^*} Y^*)v(\nabla_{Z^*} W^*)) \\ & -u(\nabla_{Y^*} Z^*)u(\nabla_{X^*} W^*) + v(\nabla_{Y^*} Z^*)v(\nabla_{X^*} W^*) \\ & +u(\nabla_{X^*} Z^*)u(\nabla_{Y^*} W^*) - v(\nabla_{X^*} Z^*)v(\nabla_{Y^*} W^*). \end{aligned}$$

From this, using the normality, we have

$$R(X^*, Y^*)X^* = (R'(X, Y)X)^* - 3(g(GX^*, Y^*)GX^* + g(HX^*, Y^*)HX^*).$$

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We finally obtained

$$0 = g((\nabla_{X^*}R)(X^*, Y^*)X^*, Y^*) = g'((\nabla'_X R')(X, Y)X, Y)$$

and hence that the base manifold M' is locally symmetric

Theorem

Theorem 3. *[BM-Rocky] Let M^{2n+1} be a normal complex contact metric manifold whose vertical foliation is regular and whose underlying Hermitian structure is Kähler. If reflections in the integral submanifolds of the vertical subbundle are isometries, then M^{2n+1} fibers over a quaternionic symmetric space.*

Proof's sketch: Since M^{2n+1} is Kähler, we have that $d\sigma(Z, GY) = 2g(HY, Z)$ and hence replacing Y by $-GY$ with Y horizontal we see that $d\sigma$ is equal to minus twice the fundamental 2-form when restricted to horizontal vectors. Thus for X and Y horizontal we have $d\sigma(X, Y) = -2\Omega(X, Y)$.

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The 4-form $\Lambda = \widehat{G} \wedge \widehat{G} + \widehat{H} \wedge \widehat{H} + \Omega \wedge \Omega$ is projectable giving an almost quaternionic structure Λ' on the base manifold M'.

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Then

$$(\nabla_X \Lambda)(Y_1, Y_2, Y_3, Y_4) = 2\sigma(X)(\widehat{H}\wedge\widehat{G}-\widehat{G}\wedge\widehat{H})(Y_1, Y_2, Y_3, Y_4) = 0.$$

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That the base manifold is symmetric follows from Theorem 2.

Theorem

Theorem 4. *[BM-Rocky] Let* M^{2n+1} *be a normal complex contact metric manifold whose vertical foliation is regular and whose complex contact structure is given by a global holomorphic contact form. If reflections in the integral submanifolds of the vertical subbundle are isometries, then* M^{2n+1} *fibers over a locally symmetric complex symplectic manifold.*

Proof's sketch: We noted that when the complex contact structure is given by a global holomorphic 1-form, u and v may be taken globally such that $\theta = u - iv$ and $\sigma = 0$. Thus $\widehat{G}$ and $\widehat{H}$ are closed 2-forms and the Lie derivatives of $\widehat{G}$, $\widehat{H}$ and Ω, as given in the preceding proof, vanish. Therefore each of these 2-forms projects to a closed 2-form on M', say $\widehat{G}'$, $\widehat{H}'$ and Ω' respectively.

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The projectability of Ω and g from the complex manifold M^{2n+1}, gives M' a complex structure for which g' is a Hermitian metric.
Now let $\Psi = \widehat{G}' - i\widehat{H}'$.

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Since Ψ is the projection of $d\theta$, it is a closed holomorphic 2-form and from the rank of $\widehat{G}'$ and $\widehat{H}'$, $\Psi^n \neq 0$ giving us a complex symplectic structure on M'.

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Again the rest of the result follows from Theorem 2.

Complex (κ, μ)-spaces

In real contact geometry the (κ, μ)-spaces were introduced by Blair, Koufogiorgos, Papantoniu, 1995 and their relation to locally ϕ-symmetric spaces was studied by Boeckx, 1999. As an analogue to (κ, μ)-spaces in complex contact geometry, Korkmaz, 2003, introduced the notion of *complex* (κ, μ)*-spaces.*

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We study the homogeneity and local symmetry of complex (κ, μ)-spaces.

Preliminaries

Let M be a complex contact metric manifold with structure tensors (u, v, U, V, G, H, J, g). For a positive constant α, one defines new tensors by

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$$\widetilde{u} = \alpha u, \quad \widetilde{v} = \alpha v,$$
$$\widetilde{U} = \frac{1}{\alpha} U, \quad \widetilde{V} = \frac{1}{\alpha} V,$$
$$\widetilde{G} = G, \quad \widetilde{H} = H,$$
$$\widetilde{g} = \alpha g + \alpha(\alpha - 1)(u \otimes u + v \otimes v).$$

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$$\widetilde{g} = \alpha g + \alpha(\alpha - 1)(u \otimes u + v \otimes v).$$

This change of structure is called an *$\mathcal{H}$-homothetic deformation.* Under an $\mathcal{H}$-homothetic deformation the 1-form σ does not change, while the symmetric tensor h transforms by $\widetilde{h} = \frac{1}{\alpha} h$.

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In order to get the conditions for a complex (κ, μ)-manifold, one applies an $\mathcal{H}$ -homothetic deformation to this structure and then it is reasonable to work with the assumption $h_U = h_V$ from the outset. Since we now have $h_U = h_V$, we denote these by h and it follows that h is a symmetric operator which anti-commutes with G and H, and commutes with J.

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Korkmaz, 2003, defines a complex (κ, μ)-space, as the complex analogue of a (κ, μ)-space from real contact geometry.

Definition. A *complex* (κ, μ)*-space* is a complex contact metric manifold (M, u, v, U, V, G, H, g) with $h_U = h_V =: h$ whose curvature tensor and 2-form $d\sigma$ satisfy

$$
\begin{aligned}
R(X, Y)U = \kappa[u(Y)X - u(X)Y] + \mu[u(Y)hX - u(X)hY] \\
+(\kappa - \mu)[v(Y)JX - v(X)JY] \\
+2[(k - \mu)g(JX, Y) + (4\kappa - 3\mu)u \wedge v(X, Y)]V, \\
R(X, Y)V = \kappa[v(Y)X - v(X)Y] + \mu[v(Y)hX - v(X)hY] \\
-(\kappa - \mu)[u(Y)JX - u(X)JY] \\
-2[(\kappa - \mu)g(JX, Y) + (4\kappa - 3\mu)u \wedge v(X, Y)]U, \\
d\sigma(X, Y) = (2 - \mu)g(JX, Y) + 2g(JhX, Y) + 2(2 - \mu)u \wedge v(X, Y),
\end{aligned}
$$

for some constants κ and μ.

The curvature tensor of a complex (κ, μ)-space with $\kappa < 1$ is completely determined (Korkmaz, 2003):

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Theorem. *Let* (M, u, v, U, V, G, H, g) *be a complex* (κ, μ)*-space. Then* $\kappa \leq 1$*. If* $\kappa = 1$*, then* $h = 0$ *and* M *is normal. If* $\kappa < 1$*, then* M *admits three mutually orthogonal and integrable distributions* $[0]$, $[\lambda]$ *and* $[-\lambda]$*, defined by the eigenspaces of* h*, where* $\lambda = \sqrt{1-\kappa}$*. Moreover,*

$$R(X_\lambda, Y_\lambda)Z_{-\lambda} = (\kappa - \mu)[g(X_\lambda, GZ_{-\lambda})GY_\lambda - g(Y_\lambda, GZ_{-\lambda})GX_\lambda$$

$$+g(X_\lambda, HZ_{-\lambda})HY_\lambda - g(Y_\lambda, HZ_{-\lambda})HX_\lambda],$$

$$R(X_{-\lambda}, Y_{-\lambda})Z_{\lambda} = (\kappa - \mu)[g(X_{-\lambda}, GZ_{\lambda})GY_{-\lambda} - g(Y_{-\lambda}, GZ_{\lambda})GX_{-\lambda}$$

$$+g(X_{-\lambda}, HZ_{\lambda})HY_{-\lambda} - g(Y_{-\lambda}, HZ_{\lambda})HX_{-\lambda}],$$

$$R(X_{\lambda}, Y_{-\lambda})Z_{-\lambda} = -\kappa[g(X_{\lambda}, GZ_{-\lambda})GY_{-\lambda} + g(X_{\lambda}, HZ_{-\lambda})HY_{-\lambda}]$$

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$$R(X_\lambda, Y_\lambda)Z_\lambda = (2-\mu+2\lambda)[g(Y_\lambda, Z_\lambda)X_\lambda - g(Y_\lambda, JZ_\lambda)JX_\lambda$$
$$-g(X_\lambda, Z_\lambda)Y_\lambda + g(X_\lambda, JZ_\lambda)JY_\lambda - 2g(JX_\lambda, Y_\lambda)JZ_\lambda],$$

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where X_λ, Y_λ *and* Z_λ *are in* $[\lambda]$ *and* $X_{-\lambda}$, $Y_{-\lambda}$ *and* $Z_{-\lambda}$ *are in* $[-\lambda]$.

Homogeneity

Because $\kappa \leq 1$ (from the previous Theorem), we have 2 cases to consider:

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2) $\kappa < 1$, $\lambda = \sqrt{1 - \kappa}$.

Remark. The complex projective space $\mathbf{CP}^{2n+1}$ is not a complex (κ, μ)-space, because, for a complex (κ, μ)-space it is easy to see that we have $d\sigma(U, V) = 0$, but this is not true in the case of the complex projectiv space $\mathbf{CP}^{2n+1}$.

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Further, examples of complex (κ, μ)-spaces can be obtained via $\mathcal{H}$-homothetic deformations, where for the new structure one has $\bar{\kappa} = \frac{\kappa+\alpha^2-1}{\alpha^2}$ and $\bar{\mu} = \frac{\mu+2(\alpha-1)}{\alpha}$.

A *homogeneous structure* on a Riemannian manifold (M, g) is a $(1, 2)$-tensor field T satisfying
$\widetilde{\nabla} g = 0$,
$\widetilde{\nabla} R = 0$,
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The existence of a homogeneous structure on a manifold (M, g) implies that (M, g) is locally homogeneous. Under additional topological conditions (complete, connected and simply connected) the manifold is homogeneous.

In the present case, we have a Riemannian manifold (M, g) with additional structure tensors u, v, U, V, G, H.

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then there is a transitive pseudo-group of local automorphisms of complex contact metric structure (u, v, U, V, G, H, g).

The manifold M is then called a *locally homogeneous complex contact metric manifold*

We define a tensor field T on a complex (κ, μ)-space with $\kappa < 1$ by the following formula:

$$T_X Y = [g(GX, Y) + g(GhX, Y)]U + [g(HX, Y) + g(HhX, Y)]V$$

$$-u(Y)(GX + GhX) - v(Y)(HX + HhX) - \frac{\mu}{2}u(X)GY - \frac{\mu}{2}v(X)HY$$

$$-\frac{1}{2}\sigma(X)[JY - u(Y)V + v(Y)U].$$

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$$-\frac{1}{2}\sigma(X)[JY - u(Y)V + v(Y)U].$$

The $(1,1)$-tensor field T_X is skew-symmetric, i.e. $g(T_X Y, Z) + g(Y, T_X Z) = 0$.

[BM-Israel] D.E. Blair, A. Mihai, *Homogeneity and local symmetry of complex* (k, μ)*-spaces*, Israel J. Math. **187** (2012), 451-464:

We prove that the tensor field T defined above satisfies all relations and hence the complex (κ, μ)-space (M, u, v, U, V, G, H, g) with $\kappa < 1$ is a locally homogeneous complex contact metric manifold.

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We prove that the tensor field T defined above satisfies all relations and hence the complex (κ, μ)-space (M, u, v, U, V, G, H, g) with $\kappa < 1$ is a locally homogeneous complex contact metric manifold.

Theorem

Theorem 1. *[BM-Israel] A complex* (κ, μ)*-space* (M, u, v, U, V, G, H, g) *with* $\kappa < 1$ *has a homogeneous structure. Therefore, a complex* (κ, μ)*-space* M *with* $\kappa < 1$ *is locally homogeneous. Moreover, if* M *is complete, connected and simply connected manifold, the complex* (κ, μ)*-space* $(\kappa < 1)$ M *is homogeneous.*

Theorem

Theorem 2. *[BM-Israel] A complex (κ, μ)-space (M, u, v, U, V, G, H, g) with $\kappa < 1$ is a locally homogeneous complex contact metric manifold.*

GH-Local Symmetry

Definition. A complex contact metric manifold is *GH-locally symmetric* if the reflections in the integral submanifolds of the vertical subbundle are isometries.

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Recall that Chen and Vanhecke gave the necessary and sufficient conditions for a reflection in a submanifold to be isometric.

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Theorem

Theorem 3. *[BM-Israel] A complex (k, μ)-space has either $k = 1$ or is GH-locally symmetric.*

M. Belkhelfa, F.Z. Kadi, *Symmetry properties of complex contact space forms*, preprint.

A.T. Vanli, I. Unal, *On complex η-Einstein normal complex contact metric manifolds*, Comm. in Math. and Appl. **8(3)**(2017), 301-313.

Submanifolds in Complex Contact Manifolds

In real contact geometry, the theory of submanifolds plays an important role.

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As **open problems** we propose to find examples of such submanifolds and state good properties for them.

Invariant Submanifolds

Denote by $\widetilde{M}$ a complex contact manifold.

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M is an *invariant submanifold* of $\widetilde{M}$ if $GT_pM \subset T_pM$ and $HT_pM \subset T_pM$, $\forall p \in M$.

We can prove that, in the case when M is invariant, according to the above definition, T_pM is invariant by J, too.

Question: Is M minimal?

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For this reason, an orthonormal basis of M, $\dim M = 2n+2$, can be written as

$$\{E_1, ..., E_n, JE_1, ..., JE_n, U, V\}.$$

• First we proved the **minimality** for an invariant submanifold of a **normal** complex contact manifold, by using the formula of the covariant derivative of J in this case.

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• The second case is to consider an invariant submanifold of a **complex** (k, μ)**-space** and to prove again the **minimality**, by using the formula of the covariant derivative of J given by Korkmaz, 2003.

• The third case is of an invariant submanifold of a complex contact manifold with $h_U = h_V = h$; taking this time an orthonormal basis as $\{E_1, ..., E_n, GE_1, ..., GE_n, U, V\}$, the proof of **minimality** follows from the estimation of

$$\left(\widetilde{\nabla}_X G\right) Y + \left(\widetilde{\nabla}_{GX} G\right) GY.$$

• We have finally proved the **minimality** in the general case, by using the formula of the covariant derivative given by Foreman in his Ph.D. Thesis.

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Let $p : TM \longrightarrow \mathcal{H}$ denote the projection to the horizontal subbundle and $J' = pJ$. We then have

$$2g((\nabla_X G)Y, Z) = g([G, G](Y, Z), GX)$$

$$-3v \wedge d\sigma(X, GY, GZ) + 3v \wedge d\sigma(X, Y, Z)$$

$$-2\sigma(X)g(Y, HZ) + 4v(X)g(Y, J'Z) - \sigma(Y)g(Z, HX) + \sigma(GY)g(Z, J'X)$$

$$-2u(Y)g(X, pZ) - 2v(Y)g(Z, J'X) + \sigma(Z)g(Y, HX) - \sigma(GZ)g(Y, J'X)$$

$$+2u(Z)g(X, pY) + 2v(Z)g(Y, J'X).$$

Now taking X horizontal and $Y = X$ we make the following computation

$$2g((\nabla_X G)Y, Z) + 2g((\nabla_{GX} G)GY, Z)$$

$$= g([G, G](X, Z), GX) - 2\sigma(X)g(X, HZ)$$

$$-\sigma(X)g(Z, HX) + \sigma(GX)g(Z, JX)$$

$$+2u(Z)g(X, X) - g([G, G](GX, Z), X) - 2\sigma(GX)g(GX, HZ)$$

$$+\sigma(GX)g(HZ, GX) + \sigma(X)g(Z, HX) + 2u(Z)g(X, X).$$

Expanding the Nijenhuis torsion terms and canceling as appropriate we have

$$g((\nabla_X G)Y, Z) + g((\nabla_{GX} G)GY, Z)$$
$$= \sigma(X)g(HX, Z) + \sigma(GX)g(JX, Z) + 2u(Z)g(X, X).$$

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$$= \sigma(X)g(HX, Z) + \sigma(GX)g(JX, Z) + 2u(Z)g(X, X).$$

Now suppose that Z is normal to the submanifold. Then the previous formula yields

$$g(\alpha(X, GX) - G\alpha(X, X), Z) + g(\alpha(GX, -X) - G\alpha(GX, GX), Z) = 0$$

giving

$$\alpha(X, X) + \alpha(GX, GX) = 0,$$

where α denotes the second fundamental form of the submanifold M.

Example

The Segre embedding:

$$\mathbf{C}P^{2m+1} \times \mathbf{C}P^{2n+1} \to \mathbf{C}P^{(2m+1)(2n+1)+2m+2n+2}$$

$$([z^1, ..., z^{2m+2}], [w^1, ..., w^{2n+2}]) \mapsto [z^1 w^1, ..., z^i w^j, ..., z^{2m+2} w^{2n+2}].$$

Anti-invariant Submanifolds

Let M be a submanifold of a normal complex contact manifold $\tilde{M}$.

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Since the first term of the above equalities is skew-symmetric and the last term is symmetric (in X, Y), then $g(GX, Y) = 0$. Similarly $g(HX, Y) = 0$.

Remark 1. If the vector fields U and V are normal to M, then for any $p \in M$, $G(T_pM) \subset T_p^{\perp}M$ and $H(T_pM) \subset T_p^{\perp}M$.

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Remark 2. The above conditions do not imply $J(T_pM) \subset T_p^{\perp}M$, for all $p \in M$.

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Remark 3. For CC-totally real submanifolds in complex contact space forms we can obtain pinching theorems (estimates of the sectional curvature).

Blair, 2001: study of holomorphic Legendre curves in the complex Heisenberg group.

A. Alarcon, F. Forstneric, *Darboux charts around holomorphic Legendrian curves and applications*, arXiv 2017.

The authors find a holomorphic Darboux chart around any immersed noncompact holomorphic Legendrian curve in a complex contact manifold.