An operator-theoretic approach to graph rigidity

Derek Kitson

Lancaster University

8th May 2019
AGA, Trinity College Dublin

Outline

- L. Euler (1766): Conjectured that all “closed spatial figures" are continuously rigid.
- A. Cauchy (1813): Proved all convex polyhedra are continuously rigid.
- M. Dehn (1916): Proved all convex polyhedra are infinitesimally rigid.
- H. Gluck (1975): Proved almost all simply connected closed surfaces are infinitesimally rigid.
- R. Connelly (1977): Constructed a continuously flexible non-convex polyhedron.

Let G be the 1-skeleton of a triangulated sphere.

Combinatorial part: (Induction) Each edge in G is either contractible, or, is contained in a non-facial 3-cycle.

Suppose G contains no contractible edges.

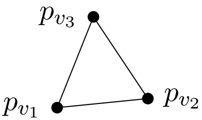
Pick an edge, and extract the subgraph bounded by its non-facial 3-cycle. This is a smaller triangulated sphere.

Geometric part: The base graph K_3 is minimally 3-rigid.

The reverse graph moves, called vertex splitting and isostatic block substitution, both preserve minimal 3-rigidity.

Example

Let $G = K_3$ and $p : V \to \mathbb{R}^2$, $v \mapsto p_v = (p_v^x, p_v^y)$.



The rigidity matrix is a $|E| \times 2|V|$-matrix:

$$
\begin{array}{cc}
 & \begin{array}{cccccc} (v_1;x) & (v_1;y) & (v_2;x) & (v_2;y) & (v_3;x) & (v_3;y) \end{array} \\
\begin{array}{c} v_1v_2 \\ v_1v_3 \\ v_2v_3 \end{array} &
\begin{pmatrix}
p_{v_1}^x - p_{v_2}^x & p_{v_1}^y - p_{v_2}^y & p_{v_2}^x - p_{v_1}^x & p_{v_2}^y - p_{v_1}^y & 0 & 0 \\
p_{v_1}^x - p_{v_3}^x & p_{v_1}^y - p_{v_3}^y & 0 & 0 & p_{v_3}^x - p_{v_1}^x & p_{v_3}^y - p_{v_1}^y \\
0 & 0 & p_{v_2}^x - p_{v_3}^x & p_{v_2}^y - p_{v_3}^y & p_{v_3}^x - p_{v_2}^x & p_{v_3}^y - p_{v_2}^y
\end{pmatrix}
\end{array}
$$

A bar-joint framework in a normed space X consists of a graph $G = (V, E)$ and map $p : V \to X$, $v \mapsto p_v$.

Suppose that, for each edge $vw \in E$, $p_v - p_w$ is a smooth point in X.

The (vw, v)-entry of the rigidity matrix is the unique support functional for $p_v - p_w$.

Varying the norm on X gives rise to different rigidity matrices.

Let $G = (V, E)$ be a simple graph and let X and Y be linear spaces over $\mathbb{K}$.

To each ordered pair $(v, w) \in V \times V$ assign a linear map $q(v, w) : X \to Y$ such that

- $q(v, w) = -q(w, v)$,
- $q(v, w) = 0$ whenever $vw \notin E$.

We will call the pair (G, q) a framework.

A co-boundary matrix for a framework (G, q) has rows indexed by E, columns indexed by V, and entries

$$c_{e,v} = \begin{cases} q(v, w) & \text{if } e = vw, \\ 0 & \text{otherwise.} \end{cases}$$

Example

- Incidence matrices for directed graphs.
- Rigidity matrices for geometric constraint systems.

The co-boundary matrix for (G, q) takes the form,

$$\begin{array}{cc} & \begin{array}{ccccccccccc} & & & v & & & & w & & \end{array} \\ vw & \left(\begin{array}{cccccccccc} & & & \vdots & & & & \vdots & & \\ 0 & \cdots & 0 & q(v,w) & 0 & \cdots & 0 & -q(v,w) & 0 & \cdots \\ & & & \vdots & & & & \vdots & & \end{array}\right) \end{array}$$

Goal: To understand co-boundary matrices for infinite frameworks.

For an index set I and normed space X, we will consider the following spaces:

- $\ell^\infty(I;X) = \{(x_i)_{i\in I} : \sup_{i\in I} \|x_i\| < \infty\}$.
- $c_0(I;X) = \{(x_i)_{i\in I} : \forall\, \epsilon > 0,\ \exists\, I_0$ fin s.t. $\sup_{i\in I\setminus I_0} \|x_i\| < \epsilon\}$.
- $\ell^p(I;X) = \{(x_i)_{i\in I} : \sum_{i\in I} \|x_i\|^p < \infty\}$, $p \in [1,\infty)$.

$C(G,q)$ gives rise to the following linear maps:

- $C(G,q) : \ell^\infty(V;X) \to \ell^\infty(E;Y)$.
- $C(G,q) : c_0(V;X) \to c_0(E;Y)$, assuming G is locally finite.
- $C(G,q) : \ell^p(V;X) \to \ell^p(E;Y)$, $p \in [1,\infty)$, assuming G has bounded degree.

Questions:

- When is $C(G, q)$ a bounded operator?
- When is it a compact operator?
- When is it bounded below?
- Can we compute its operator norm?

Related work:

- Maddox, Infinite matrices of operators. Lecture Notes in Mathematics, 786. Springer, Berlin, 1980.
- Mohar and Woess, A survey of spectra of infinite graphs. Bull. London Math. Soc. 1989.
- Agrawal, Berge, Colbert-Pollack, Martinez-Avenano, Sliheet, Norms, kernels and eigenvalues of some infinite graphs. 2018. arXiv:1812.08276v1

Proposition

Let Z *be a subspace of* $\ell^\infty(V;X)$ *which contains* $c_{00}(V;X)$*. TFAE:*

(i) $q : V \times V \to L(X,Y)$ *is a bounded function.*

(ii) $C(G,q) : Z \to \ell^\infty(E;Y)$ *is a bounded operator.*

Moreover, $\|C(G,q)\|_{op} = 2\|q\|_\infty$.

Proposition

If $C(G,q)$ *maps* $\ell^p(V;X)$ *into* $\ell^\infty(E;Y)$*, for some* $p \in [1,\infty)$*, then* $q : V \times V \to L(X,Y)$ *is a bounded function.*

Proposition

Let $p \in [1, \infty)$. If G has bounded degree then TFAE:

(i) $q : V \times V \to L(X, Y)$ *is a bounded function.*

(ii) $C(G, q) : \ell^p(V; X) \to \ell^p(E; Y)$ *is a bounded operator.*

Moreover,

$$2^{1-\frac{1}{p}}\|q\|_\infty \leq \|C(G,q)\|_{op} \leq 2^{1-\frac{1}{p}}\|q\|_\infty \Delta(G)^{\frac{1}{p}},$$

where $\Delta(G) = \sup_{v \in V} \deg(v)$.

Proposition

Suppose $\Delta(G) < \infty$ *and* $q : V \times V \to L(X, Y)$ *is bounded.*

(i) *If* $G_k \to G$ *as* $k \to \infty$ *then,*

$$\|C(G, q)\|_{op} = \lim_{k\to\infty} \|C(G_k, q)\|_{op}.$$

(ii) *If* $\mathcal{S}$ *and* $\mathcal{S}'$ *denote respectively the set of all subgraphs and the set of all finite subgraphs of* G *then,*

$$\|C(G, q)\|_{op} = \sup_{G_0\in\mathcal{S}} \|C(G_0, q)\|_{op} = \sup_{G_0\in\mathcal{S}'} \|C(G_0, q)\|_{op}.$$

The operator norms refer to the cases:

(a) $C(G, q) \in B(c_0(V; X), c_0(E; Y))$, *and,*

(b) $C(G, q) \in B(\ell^p(V; X), \ell^p(E; Y))$, *where* $p \in [1, \infty)$.

Proposition

Let Z be a subspace of $\ell^\infty(V;X)$ which contains $c_{00}(V;X)$. If $q : V \times V \to L(X,Y)$ vanishes at infinity then the operator $C(G,q) : Z \to \ell^\infty(E;Y)$ is compact.

Proposition

Suppose one of the following conditions holds.

(i) $C(G,q) \in K(c_0(V;X), c_0(E;Y))$.

(ii) $C(G,q) \in K(\ell^p(V;X), \ell^p(E;Y))$, *where* $p \in [1,\infty)$.

Then $q : V \times V \to L(X,Y)$ vanishes at infinity.

Proposition

If G has bounded degree then TFAE:

(i) $q : V \times V \to L(X, Y)$ *vanishes at infinity.*

(ii) $C(G, q) \in K(c_0(V; X), c_0(E; Y))$.

(iii) $C(G, q) \in K(\ell^p(V; X), \ell^p(E; Y))$, *where* $p \in [1, \infty)$.

Given framework (G,q) , define

$$h : V \to \mathbb{R}, \quad h(v) = \sup_{vw\in E} \|q(v,w)\|_{op}.$$

Proposition

Suppose one of the following conditions holds.

(a) $C(G,q)$ *maps* Z *into* $c_0(E;Y)$*, where* Z *is a subspace of* $\ell^\infty(V;X)$ *which contains* $c_{00}(V;X)$*, and* $C(G,q) : Z \to c_0(E;Y)$ *is bounded below.*

(b) G *has bounded degree and* $C(G,q) : \ell^p(V;X) \to \ell^p(E;Y)$ *is bounded below, where* $p \in [1,\infty)$*.*

Then the function $h : V \to \mathbb{R}$ *is bounded away from zero.*

If $V_0 \subset V$ then denote by ∂V_0 the set of edges of G with exactly one vertex in V_0.

The isoperimetric constant for G is the value,

$$i(G) = \inf_{V_0 \text{ finite}} \frac{|\partial V_0|}{|V_0|},$$

where the infimum is taken over all finite subsets V_0 of V.

Denote by $\chi(V;X)$ the set of finitely supported vectors with constant non-zero entries.

Proposition

Let G be a locally finite graph and let $p \in [1,\infty)$.
If $q : V \times V \to L(X,Y)$ is bounded then,

$$\inf\{\|C(G,q)z\|_p : z \in \chi(V;X),\ \|z\|_p = 1\} \leq i(G)^{\frac{1}{p}}\,\|q\|_\infty.$$

In particular, if $\Delta(G) < \infty$ and $i(G) = 0$ then the operator $C(G,q) : \ell^p(V;X) \to \ell^p(E;Y)$ is not bounded below.

- J. Cruickshank, D.K., S. C. Power. The generic rigidity of a partially triangulated torus. *Proceedings of the London Mathematical Society* (2019). https://doi.org/10.1112/plms.12215
- J. Cruickshank, D.K., S. C. Power. The generic rigidity of triangulated spheres with blocks and holes. *Journal of Combinatorial Theory, Series B.,* (2017), no. 122, 550–577.
- E. Kastis, D.K., S. C. Power. Co-boundary matrices for infinite frameworks. *In preparation.*
- D. K., R. L. Levene. Graph rigidity for unitarily invariant matrix norms. 29p. *arXiv:1709.08967*
- D.K., A. Nixon, B. Schulze. Rigidity of symmetric frameworks in normed spaces. 31p. *arXiv:1808.04484*

Thank you