2.5 Fair Games, Betting, and Probability

Example 2.5.1 *Ann and Bob are playing a betting game which involves randomly selecting a card from a standard deck (and replacing it before repeating). If the card is a diamond Bob pays Ann €4. If the card is an ace, Ann pays Bob €13. Otherwise no money changes hands.*

Let W_A be Ann's winnings from a game and let W_B be Bob's winnings. Then W_A and W_B are random variables associated to the experiment of playing this game. We can calculate their probability functions and expected values.

x	$P(W_A = x)$
4	12/52
−13	3/52
−9	1/52
0	36/52

x	$P(W_B = x)$
−4	12/52
13	3/52
9	1/52
0	36/52

We have

$$\begin{aligned} E(W_A) &= 4(12/52) + (-13)(3/52) + (-9)(1/52) + 0(36/52) = 0 \\ E(W_B) &= (-4)(12/52) + 13(3/52) + 9(1/52) + 0(36/52) = 0 \end{aligned}$$

So in the long term both Ann and Bob expect to break even, or finish in the position they started in - the rules of the game are not favourable either to Ann or to Bob. We say that this is a *fair game*.

In general in a simple betting game like this between Ann and Bob, if the game is fair we should have

$$E(W_A) = E(W_B).$$

We should also have $E(W_A) + E(W_B) = 0$ since Bob's winnings are Ann's losses (negative winnings). So for a fair game we should have

$$E(W_A) = E(W_B) = 0.$$

Example 2.5.2 *A coin is biased so it shows H (heads) with probability 0.6 and T (tails) with probability 0.4. Bob pays Ann €1 if the coin shows H. For a fair game how much should Ann pay Bob if the coin shows T?*

Solution: Let x be the amount paid by Ann if T shows. Then we want

$$\begin{aligned} E(W_A) &= 0.6(1) + (-x)0.4 = 0 \\ \Longrightarrow 0.6 &= 0.4x \\ \Longrightarrow x &= 1.5 \end{aligned}$$

So we should have $x =$ €1.50 - check that $E(W_B) = 0$ also in this case.

Note on Odds

If a bookmaker quoted odds of $n - m$ (against) some event occurring, this means that a bet of m will be rewarded with a prize of n plus the stake of n returned.
For example if Ann places a bet of €1 on a horse to win a race at odds of 7-1, then the bookmaker will pay Ann €8 (€7 plus her €1 back) if the horse wins.
Note: Odds of 7-1 are read as "7 to 1 (against)".
Odds of 1-3 are read as "3 to 1 on".

If a horse is quoted at n-m to win a race, there is an implication that in $n+m$ runnings of the race the horse would be expected to win m times and lose n times. The horse's "implied probability" of winning the race is $\frac{m}{m+n}$.

Example 2.5.3 *Ireland and Scotland are playing each other in the Six Nations next Saturday. Bob is a bookmaker. He believes that Ireland will win with probability* 0.75*, that Scotland will win with probability* 0.2 *and that the match will be drawn with probability* 0.05*. If Bob wants a fair game for his customers, how should he set the odds for each of these events?*

Solution:

- Suppose Ann places a bet of €1 on Ireland to win at odds of $x - 1$. Then Ann wins x with probability 0.75 and loses €1 with probability 0.25. We have

$$E(W_A) = 0.75x + 0.25(-1)$$

  If $E(W_A) = 0$ we should have $x = 0.25/0.75 = 1/3$, so Bob should offer odds of $\frac{1}{3} - 1$ or $1 - 3$ on Ireland winning.

- Suppose Ann places a bet of €1 on Scotland to win at odds of $y - 1$. Then Ann wins y with probability 0.2 and loses €1 with probability 0.8. We have

$$E(W_A) = 0.2y + 0.8(-1)$$

  If $E(W_A) = 0$ we should have $y = 0.8/0.2 = 4$, so Bob should offer odds of $4 - 1$ on Scotland winning.

- Suppose Ann places a bet of €1 on a draw at odds of $z - 1$. Then Ann wins z with probability 0.05 and loses €1 with probability 0.95 We have

$$E(W_A) = 0.05z + 0.95(-1)$$

  If $E(W_A) = 0$ we should have $z = 0.95/0.05 = 19$, so Bob should offer odds of $19 - 1$ on a draw.

	Ireland	1-3
Fair Prices :	Scotland	4-1
	Draw	19-1

General Fact: If an event will occur with probability p then for a fair price odds against the event should be set at

$$\frac{1-p}{p} - 1.$$

Then if Ann places a €1 bet on the event occurring her expected winnings are

$$p\frac{1-p}{p} + (1-p)(-1) = 1 - p - 1 + p = 0.$$

What Actually Happens

In practice, Bob (the bookmaker) does not really care if his customers get a fair price or not - he cares about his own opportunity for profit and about controlling his risk of loss.

In the above example suppose bets of a_1, a_2 and a_3 are placed on Ireland, Scotland and a draw respectively. Then Bob's expected winnings are

$$\begin{aligned} E(W_B) &= 0.75(a_1 + a + 2 + a_3 - (a_1 + \frac{1}{3}a_1)) + 0.2(a_1 + a_2 + a_3 - (a_2 + 4a_2)) \\ &\quad +0.05(a_1 + a_2 + a_3 - (a_3 + 19a_3)) \\ &= a_1(-0.25 + 0.2 + 0.05) + a_2(0.75 - 0.8 + 0.05) + a_3(0.75 + 0.2 - 0.95) \\ &= 0. \end{aligned}$$

This means that if the "experiment" were repeated a large number of times, Bob would expect to break even "on average". But this is not going to happen - what will happen is one of the following :

1. Ireland wins : Bob's profit is $a_2 + a_3 - \frac{1}{3}a_1$.
2. Scotland wins : Bob's profit is $a_1 + a_3 - 4a_2$.
3. Draw : Bob's profit is $a_1 + a_2 - 19a_3$.

with (estimated) respective probabilities 0.75, 0.2 and 0.05. So a win for Ireland is most likely, and if a_1 is large compared to $a_2 + a_3$, Bob is in danger of making a significant loss. The possible outcomes in terms of profit for Bob depend on the values of a_1, a_2 and a_3. We now consider how Bob can adjust the odds to control his risk.

Example 2.5.4

Suppose Bob quotes odds as above.

(a) Suppose bets of €15, €4 and €1 are placed on Ireland, Scotland and on a draw respectively. We have

	Ireland	Scotland	Draw
Probability	0.75	0.2	0.05
Quoted Odds	1-3	4-1	19-1
Bets Placed	15	4	1

Let W_B be Bob's profit. Then

- Ireland win : $W_B = (15 + 4 + 1) - 15 - \frac{1}{3}(15) = 0$.
- Scotland win : $W_B = (15 + 4 + 1) - 4 - 4(4) = 0$.
- Draw : $W_B = (15 + 4 + 1) - 1 - 19(1) = 0$.

So Bob has no risk (and no profit) regardless of the outcome. Here the combination of bets placed and odds offered make Bob's position riskless.

(b) If bets are placed as above and Bob wants a €2 profit regardless of the outcome, how should the odds be set?
Let the odds on Ireland, Scotland and a draw be x to 1, y to 1 and z to 1 respectively. Bob wants :

- If Ireland win :

$$\begin{aligned} W_B = 20 - 15(1 + x) &= 2 \\ \implies -15x &= -3 \\ \implies x &= 1/5 \end{aligned}$$

Odds for Ireland winning should be set at $\frac{1}{5} - 1$ or $1 - 5$.

- If Scotland win :

$$\begin{aligned} W_B = 20 - 4(1 + y) &= 2 \\ \implies -4y &= -14 \\ \implies y &= 7/2 \end{aligned}$$

Odds for Scotland winning should be set at $\frac{7}{2}$-1 or 7-2.

– If the game is drawn :

$$\begin{aligned} W_B = 20 - 1(1+z) &= 2 \\ \Longrightarrow z &= 17 \end{aligned}$$

Odds for a draw should be set at $17 - 1$.

So we have the following information :

	Ireland	Scotland	Draw
Bets Placed	15	4	1
Believed Probability	0.75	0.3	0.05
Fair Price	1-3	4-1	19-1
Quoted Odds	1-5	7-2	17-1
Implied Probability	5/6	2/9	1/18

The bookmaker makes €2 regardless of the outcome. The bookmaker has no interest in the outcome. Of course this game is no longer fair for the punter.