Chapter 6. The origins of the differential and integral calculus

We now sketch the origins of the differential and integral calculus, probably the most powerful technique introduced into mathematics since the golden age of Greek geometry. Archimedes, in the 3rd century BCE, had been able to calculate areas under curves and volumes of certain solids by a method of approximation, called the method of exhaustion, based on using known areas and volumes of rectangles, discs, etc. His results were usually expressed, not in absolute terms, but in terms of comparisons of volumes. For instance, suppose we have a sphere of radius $r$ which is surrounded exactly by a circular cylinder of radius $r$ and height $2r$. Then Archimedes showed that the volume of the sphere is two thirds that of the cylinder. This is of course well known to us, as if $V$ is the volume of the sphere and $V_1$ is the volume of the cylinder, then

$$V = \frac{4}{3}\pi r^3, \quad V_1 = 2\pi r^3$$

and the result follows.

Each case of his area and volume calculation was worked out on its own merits, and no algorithm for handling general problems emerged. It was also important for Archimedes to maintain Greek standards of rigour in proof, for no clear idea of limiting processes or of infinity was known at the time. This made the method rather laborious by modern standards. The integral calculus eventually provided the necessary algorithm for calculating areas, volumes, centres of gravity, and so on. Some of Archimedes’s ideas were known in the Renaissance, as his work On the Sphere and Cylinder was available in Latin translations. Interestingly enough, another important contribution of Archimedes, called simply The Method, which contained further volume calculations, was not known until 1906. A palimpsest (which is a parchment that has been partly erased and then re-used), on which a copy of The Method had been written by a 10th century scribe, was discovered in Constantinople by the historian Joseph Heiberg. It is difficult to read directly, but under ultra-violet light with modern image-enhancing technology it may easily be deciphered. It is the most important Archimedes manuscript known, and is the sole source of some of his work. The manuscript was sold at auction in New York in 1998 for around 2 million dollars (probably to a computer millionaire), to the dismay of some historians of mathematics and amidst disputes about its rightful ownership.

The famous German astronomer Johann Kepler (1571-1630) knew of Archimedes’s work on volume calculation. He was able to put his knowledge to practical use when he
was asked to find the best proportions for making wine casks. His response to this practical problem was the book he wrote in 1615: *Nova stereometria doliorum* (New solid geometry of wine casks). In this work, Kepler considered both theoretical and applicable volume calculations. He broke away from the methods of Archimedes, abandoning the extreme rigour previously considered appropriate, and introduced new approximation techniques. One of his main contributions was the volume of the solid obtained by rotating a segment of a conic section about an axis in its own plane. His book gives the volumes of almost 100 solids. He is considered the great precursor of the infinitesimal method of the integral calculus.

A further important addition to the foundations of the integral calculus was made by the Italian priest Bonaventura Cavalieri (1598-1647). Cavalieri was a follower of the great physicist Galileo Galilei, who was himself interested in problems involving area and volume. Cavalieri’s ideas are contained in his book *Geometria indivisibilibus continuorum* of 1635. In this work, he expounded his method of *indivisibles*, an infinitesimal technique influenced by Archimedes’s method of exhaustion. It is difficult to explain exactly what indivisibles are, but we can think of them as in some sense the material out of which continuous substances are constructed. Cavalieri considered a line as composed of an infinite number of points, a surface as composed of an infinite number of lines, and so on. The idea of an indivisible was not entirely new, as it had occurred already in medieval scholastic philosophy.

The main advantage of the method of indivisibles was that it was more systematic compared with the method of exhaustion, although perhaps more subject to criticism from lacking extreme rigour. The method made it easy to find the area of an ellipse or the volume of a sphere. Cavalieri found, in effect, a result equivalent to evaluating the integral

\[
\int_0^a x^m \, dx \quad \text{as} \quad \frac{a^{m+1}}{m+1}
\]

when \( m \) is a positive integer. Of course, he did not express his ideas in this form, and it should be noted that his knowledge of the new algebraic notation (as say compared with Descartes) was weak. The main thrust of his arguments was to establish inequalities for estimating the sums

\[
1^m + 2^m + \cdots + n^m
\]

when \( n \) and \( m \) are positive integers. In effect, by looking at special cases and generalizing,
he established inequalities of the form

$$1^m + 2^m + \cdots + n^m > \frac{n^{m+1}}{m+1} > 1^m + 2^m + \cdots + (n-1)^m.$$  

The same type of analysis had in fact already been considered by the French mathematicians Fermat and Roberval a little earlier.

The next significant contribution to the methods of the calculus that we wish to describe is that made by the English mathematician John Wallis (1616-1703). Wallis was professor of geometry at Oxford and he wrote a number of influential books. We will concentrate on his *Arithmetica infinitorum*, published in 1655. Wallis knew of the existence of Cavalieri’s work on indivisibles, but probably had never seen his *Geometria indivisibilis continuorum*. Wallis obtained results, amounting to the calculation of definite integrals, rather similar to those of Cavalieri. His approach was, however, more arithmetical and less laborious than Cavalieri’s. Indeed, Wallis was probably more of an algebraist than a geometer, although geometric considerations still were important to him. What is more, on the basis of his preliminary findings, he was able to infer the general pattern for evaluating, in effect,

$$\int_0^1 x^k \, dx.$$  

He used, in essence, a form of induction, or more correctly, an argument by analogy, in which he did not rigorously justify all his steps, and this led to criticism from continental mathematicians. Another point to observe is that Wallis was the first mathematician to make use of fractional powers, such as $x^{1/2}$ or $x^{2/3}$ (recall that Descartes, twenty or so years earlier, had popularized the use of integer powers or exponents). Wallis even went so far as to integrate such fractional powers, obtaining

$$\int_0^1 x^{p/q} \, dx = \frac{1}{p/q + 1}.$$  

His argument was really by analogy with the whole number case, an idea that is still used in introductory courses in integration. Despite the lack of mathematical rigour, his results were correct and provided a powerful impetus for sweeping generalization by Newton a few years later.

Wallis also showed the relevance of infinite processes to problems of mathematical analysis. A noteworthy example is his infinite product formula

$$\frac{\pi}{4} = \frac{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \cdots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \cdots}.$$  

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In our modern notation, he obtained this result by considering integrals of the form
\[ \int_0^{\pi/2} \sin^m \theta \, d\theta. \]
His arguments in obtaining this formula were not entirely convincing but they proved to be correct, and he had thus extended the range of theorems accessible by limiting processes. The formula for \( \frac{\pi}{4} \) is remarkable, as \( \pi \) is transcendental (it is not the root of any polynomial with rational coefficients) but is shown to be a limit of simple fractions. We can say that Wallis prepared the way for the study of infinite series, rather than polynomials. Such series have been a key feature in the calculus and much of all subsequent analysis.

A person who may have played a significant role in introducing Newton to the concepts of the calculus is the English mathematician Isaac Barrow (1630-77). He was professor of mathematics at Cambridge from 1663 until 1669. His successor in the professorship was Newton. Barrow was primarily a geometer following the traditions of the ancient Greeks. However, around 1664 he became interested in the problem of finding tangents to curves and he developed an approach involving moving points and lines. In his university lectures at Cambridge, which were subsequently published, he gave his own generalization of tangent and area procedures based on his extensive reading of the works of such notable contemporary mathematicians as Descartes, Wallis, Fermat and especially the Scottish mathematician James Gregory, who is considered to be an important fore-runner of Newton. The lectures contained ideas that could have been exploited but they were probably not studied outside Cambridge.

It has been a matter of much conjecture whether Newton was in any sense a student of Barrow’s. It was always assumed that he was influenced by Barrow, as he was working at Cambridge at the time of Barrow’s lectures on tangent and area problems. Furthermore, Newton’s first great advances in the foundations of the calculus date from 1664-65, which is the time when Barrow first studied the problems that underlie the calculus. It is clear that Barrow’s notion of generating curves by the motion of points was important to Newton’s foundation of the differential calculus, but on the whole Newton denied any direct influence from Barrow.

We turn now to study the work of the person generally held to be the first to develop the calculus on a systematic basis and see the connection between the differential and integral processes. Isaac Newton (1642-1727) was born on Christmas Day 1642 in Woolsthorpe, Lincolnshire, England. His father had already died in October 1642 and
the lack of a father is thought to have had an effect on Newton’s personality. His mother remarried after two years and the young Newton was brought up by his grandparents for the next twelve years. He attended a grammar school in Grantham, a few miles from his home, where he resided with an apothecary. He learned Latin and probably the rudiments of arithmetic. In 1656, his mother returned to Woolsthorpe, after the death of her second husband. She intended that her son should take over the running of the family farm, but he showed no aptitude for this task. Her brother was a clergyman and graduate of Cambridge University, and he advised that Newton should attend Cambridge as a student, on account of his interest in mechanical problems.

Newton was admitted a member of Trinity College, Cambridge on June 5, 1661. We know little of his studies in the university. He certainly had already read a book entitled Logic, which excused him from attending lectures on the subject. He bought an edition of Euclid’s Elements and, having glanced at a few of the propositions, he found them so obvious that he wondered why anyone would prove them. Nonetheless, in his scholarship examination of 1664, he was found to be deficient in his knowledge of Euclid’s geometry, and he then set about re-reading Euclid with more diligence. He also began a study of Descartes’s La Géométrie in a Latin translation, which proved more challenging. Indeed, this work seems to have inspired him to investigate advanced mathematics on his own initiative. His notebook of 1664 shows him quoting work of the French mathematician Viète and John Wallis’s Arithmetica infinitorum. He experimented with the binomial theorem to extract roots of numbers, and studied optical lenses. Many years later (in 1699), Newton wrote in this same notebook that between 1664 and 1665, he discovered the method of infinite series and how to find the area under a hyperbola:

In the winter of the years 1664 and 1665 upon reading Dr Wallis’s Arithmetica Infinitorum & trying to interpolate his progressions for squaring the circle I found out another infinite series for squaring the circle and then another for squaring the Hyperbola.

Here, squaring the circle means finding the area of a circle, or, in effect, calculating the integral

$$\int_0^1 \sqrt{1-x^2} \, dx$$

without assuming knowledge of \(\pi\). By reading Wallis’s work on finding areas, Newton was led to understand that the integrand \(\sqrt{1-x^2}\) could be expanded as an infinite series and
the general indefinite integral
\[ \int \sqrt{1 - x^2} \, dx \]
could also be expressed by term by term integration as
\[ x - \frac{1}{2} \frac{x^3}{3} - \frac{1}{5} \frac{x^5}{5} - \frac{1}{7} \frac{x^7}{16} \ldots \]

James Gregory had applied similar ideas in evaluating
\[ \int \frac{1}{1 + x^2} \, dx \]
by expanding \((1 + x^2)^{-1}\) as an infinite geometric series, thereby obtaining an inverse tan series.

Newton also discovered the infinite series representation for the \(\sin^{-1} x\) function, and for
\[ \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \]
He used his formula to calculate special values of the log function, giving areas under a hyperbola, to fifty or more decimal places. The surviving manuscripts show that Newton was a formidable calculator, well able to test the accuracy of his theoretical results.

In January 1665, Newton took his degree of bachelor of arts. The method of examination was by oral questioning, and involved the defence of two or three theses, often taken from the works of Aristotle. The method of fluxions (a form of differential calculus) occurred to him in 1665, and manuscripts of his dating from 1665 and 1666 provide some indication of his early approach to this topic. In the summer of 1665, Newton left Cambridge as a precaution against the plague and returned to his family home for the next 18 months. It was during this time that Newton made four fundamental discoveries:

- the general binomial theorem;
- the connection between the methods used in tangent and area problems (the fundamental theorem of the calculus);
- the law of universal gravitation;
- the theory of the composition of white light.

The episode of the falling apple, which is said to have occasioned the notion of gravitation, occurred at his Lincolnshire home, Newton later declared.
Newton was elected a Fellow of Trinity College, Cambridge on October 1, 1667. Between 1666 and 1669, he studied varied subjects, including the transmission of light, alchemy and further investigations of fluxions. In 1669, he became aware of Nicholaus Mercator’s book *Logarithmotechnia*, published in late 1668, in which the infinite series for \( \log(1 + x) \) was given. Newton suddenly realized that other people were discovering the method of infinite series, which he had greatly developed in 1664-65. He therefore wrote a tract, entitled *De analysi per aequationes infinitas*. This tract was circulated to several important mathematicians, to establish Newton’s priority of discovery. The tract was handwritten and was never printed, although a version subsequently appeared many years later, incorporated into a larger work. It was perhaps typical of Newton that he never really gave a proper presentation of his work on the differential calculus, and this was partly to blame for the famous dispute with Leibniz.

Isaac Barrow resigned his professorship of mathematics at Cambridge University in the summer of 1669 and Newton was elected his successor on October 29, 1669, at the age of 26, at Cambridge University. Part of his professorial duties was to lecture once a week during term on some mathematical or scientific topic, and to answer student queries. Much of the work he presented in his lectures was published subsequently and proved immensely influential, although it is reported that the lectures themselves were largely unattended and unappreciated. He first lectured about optics and presented an account of his researches on the subject to the Royal Society of London in the spring of 1672. He had been elected a fellow of the Royal Society in January, 1672 and at that time he had submitted to the Society a description of a revolutionary reflecting telescope that he had invented. In February 1672, he then communicated a paper to the Royal Society on the composition of white light. This paper proved to be highly controversial and provoked a sustained antagonism with Robert Hooke, later to be secretary of the Royal Society. This controversy had a negative effect on Newton, and he resolved not to place his subsequent research on public view and then be forced to defend it. In actual fact, Newton did publish further researches and experimental observations on light.

Kepler, whom we mentioned at the beginning of this chapter, had observed experimentally that each planet moves in an elliptical orbit about the Sun, the Sun being at one of the foci of the ellipse. Newton had come to believe that a gravitational force exerted by the Earth held the Moon in its orbit about the Earth. From Kepler’s laws, Newton was led to propose an inverse square law of gravitational attraction. He calculated to see
if his gravitational theory could describe the orbit of the Moon, but, using the currently
available figure for the radius of the Earth, he was led to a slightly disappointing deviation
of theory from observed behaviour. He laid aside his theory but took it up again in 1679,
for by this time, a revised and more accurate estimation of the Earth’s radius had become
available, and Newton now found that his theory made predictions much more in line with
the observations.

In January 1684, Christopher Wren, Edmund Halley and Robert Hooke discussed
among themselves the law of gravitation. While the inverse square law of attraction was
agreed to be the most likely, confirmation was not available. Hooke claimed to have a
solution to describe the motion of a body moving under the influence of a central force
varying as the inverse square of the distance from the centre, but he never succeeded in
furnishing details. Halley then travelled to Cambridge in August 1684 and asked Newton
what would be the curve described by a planet moving under the gravitational attraction
of the Sun and varying as the inverse square of its distance to the Sun. Newton replied
that it would be an ellipse and that he had calculated this already. Newton sent a copy of
his proof to Halley in London in November 1684. Halley advised Newton to communicate
his findings officially to the Royal Society to secure priority of discovery, and this Newton
did in February 1685. His initial work was contained in a short treatise of 24 pages entitled
De motu, consisting of four theorems and seven problems. De motu was to form the basis
of the first sections of Book 1 of the Principia. Between 1685 and 1686, Newton composed
the Principia. He was frequently in correspondence with John Flamsteed, the Astronomer
Royal, regarding accurate data on planetary orbits, motion of satellites, extent of tides,
etc. This information was required to corroborate the accuracy of Newton’s gravitational
theory.

On April 26, 1686, Halley presented Newton’s manuscript Philosophiae naturalis
principia mathematica (Mathematical principles of natural philosophy) to the Royal So-
ciety. This contained only the first book (major part) of the eventually complete work.
Over the next few months, it was suspected that Hooke was delaying the printing of the
manuscript, largely because he felt he deserved some credit for the discovery of the inverse
square law of gravitation. To some extent, Newton’s reluctance to publish accounted for
the confusion and dispute. Nevertheless, the Principia was printed and published by July,
1687. In the 1690’s, after he had published this fundamental work, he apparently lost
interest in mathematics, and eventually left Cambridge to spend the remaining 30 years
of his life in London. He still, however, found time to revise and publish other parts of his
work and engage in scientific correspondence and controversy.

We will briefly describe now Newton’s approach to the differential calculus. He
first developed a differentiation procedure based on the concept of an infinitesimally small
increase \( o \), which ultimately vanishes, of a variable \( x \). Later, he settled on the notion of
a fluxion of the variable, meaning a finite instantaneous speed defined with respect to an
independent time variable. In our terms, the fluxion of \( x \) with respect to the time variable
\( t \) is \( \frac{dx}{dt} \), which is the speed in physical terms. Later, he introduced the notation \( \dot{x} \) for \( \frac{dx}{dt} \),
\( \ddot{x} \) for \( \frac{d^2x}{dt^2} \), and so on, which has been maintained to some extent in dynamics. He also
invented the idea of partial derivatives of a function of two independent variables in 1665,
although again, his notation was very different from that which evolved later.

In his De analysi manuscript, Newton was able to show that the area under the
curve

\[
y = x^{m/n}
\]

is given by the function

\[
x^{1+(m/n)} \over 1 + (m/n)
\]

by introducing an area function and considering small increments in this function. In this
way, he was able to demonstrate that the area problem was the inverse of the tangent
problem for curves, or, in modern terms, that integration is the inverse of differentiation.
This was a decisive step in Newton’s formulation of the calculus. While special cases of
this property were evident in the work of such mathematicians as James Gregory, Newton
was the first to exploit it systematically, especially through his use of infinite series.

In a letter written to John Collins in 1672, Newton described his methods. He
admitted that he had been influenced by the Dutch mathematician Hudde and the Flemish
mathematician Sluse, as well as by Fermat. This letter was important in the priority
dispute arising from the question of who followed whom in the invention of the calculus.
Newton did not properly justify his procedures, but it seems that in his area analysis,
the ordinate \( y \) represents the velocity of the increasing area and the abscissa \( x \) the time.
Newton then found the area by finding its rate of change.

Newton changed his approach to the limiting processes of the calculus over the years.
However, as the limit concept is subtle, and was not properly clarified until the mid-19th
century, Newton was always vulnerable to the accusation that his arguments relied on the manipulation of quantities that are ultimately set equal to 0, and that he was effectively dividing by 0. Of course, in elementary treatments of the calculus, where the limiting process is described more or less intuitively, virtually the same arguments as those of Newton are still used today.

Newton’s fullest account of his version of the calculus was given in his work *De quadratura curvarum*, which was written in 1676 but was not published until 1704, when it appeared as an appendix to his book *Opticks*. In this work, he tried to abandon appeal to infinitesimal quantities. To determine the fluxion of $x^n$, he replaced $x$ by $x + o$, and expanded $(x + o)^n$ by the binomial theorem. Subtracting $x^n$, he obtained the change in $x^n$ corresponding to the change $o$ in $x$. Instead of completing his argument by a dubious neglect of terms, he formed the ratio of the change in $x$ to the change in $x^n$, namely,

$$
1 \to nx^{n-1} + \frac{n(n-1)}{2}ox^{n-2} + \cdots
$$

He then allowed $o$ to approach the value zero. The resulting ratio, 1 to $nx^{n-1}$, Newton called the *ultimate* ratio. This is the clearest account of his limiting process.

In his *Principia*, he made use of some of his ideas of calculating derivatives, although most of the emphasis is given to geometric rather than algebraic methods. (It was once thought that Newton had an aversion to the use of algebra, especially in his published work, but existing manuscripts show that he freely used algebra in his calculations.) The *Principia* is mainly concerned with physics and astronomy, and its intent is to show how the laws of motion and the inverse square law of gravitation enable us to describe the workings of the solar system very accurately. It is of interest to note that, in the first two editions of the *Principia*, Newton acknowledged that Leibniz had independently discovered a version of the calculus, but following the acrimony of their dispute, all reference to Leibniz had been removed from the third edition of 1726.

As we have alluded in our description so far, the other person acknowledged to be the main discoverer of the calculus is Gottfried Wilhelm Leibniz (1646-1716). He was born in Leipzig, in Germany. At university, he studied a wide range of topics, including law, theology, philosophy and mathematics. Like Descartes, he was determined to find a universal procedure of reasoning, applicable to virtually all academic disciplines. (He was interested in inventing an algebra of logic, something that George Boole later introduced in his book *The Laws of Thought*, published in 1854.) This abstract approach to reasoning
probably ensured that his version of the calculus was systematic and algorithmic, unlike Newton’s. Much of Leibniz’s career was spent in the diplomatic service of the rulers of Hanover, which enabled him to visit many of the intellectual centres of Europe, including Paris in the early 1670’s. He visited London in 1673 and 1676 and probably saw the manuscript of Newton’s *De analysi* during his stays.

Around 1673, Leibniz realized that the determination of the tangent to a curve depended on the ratio of the difference of the *y*-values (the ordinates) to the difference of the *x*-values (abscissae), as these differences become infinitesimally small. He also realized that the area under a curve is found by taking the sum of the areas of infinitely thin rectangles. Like Newton, he was led to see the role of infinite series (possibly influenced by Newton’s *De analysi*). By 1676, he had essentially obtained all of Newton’s slightly earlier conclusions. He saw furthermore that his procedures could deal not only with simple powers of *x* but also with so-called transcendental functions, such as log *x* and sin *x*.

We owe to Leibniz much of the familiar notation of the calculus, and it is generally agreed that Leibniz’s exposition of the calculus and its fundamental processes was much clearer and more easily used than that of Newton. He eventually decided on the use of *dx* and *dy* for smallest possible *x* and *y* increases (they are sometimes called differentials). Likewise, he introduced

\[
\int y \quad \text{and} \quad \int y \, dx
\]

for finding area, the \(\int\) sign arising as an enlarged *s*, signifying sum. The *calculus differentialis* became the method for finding tangents and the *calculus summatorius* or *calculus integralis* the method for finding areas.

Leibniz was the first person to publish a complete account of the differential calculus. His paper was entitled *Nova methodus pro maximis et minimis, itemque tangentibus*. It appeared in an important journal, *Acta Eruditorum*, published in Leipzig in 1684. This journal was a major source of news in science, where reviews of such books as Newton’s *Principia* kept its readers up to date with the latest ideas. In his paper, Leibniz used the \(d\) notation for derivatives (or differentials). He stated quite clearly the product rule in the form

\[
dv = x \, dv + v \, dx,
\]
where the bar denotes product, and the quotient rule as

\[
\frac{d}{x} \frac{v}{y} = \frac{v dy - y dv}{y^2}.
\]

He also gave the rule for differentiating powers as

\[
dx^n = nx^{n-1}dx.
\]

In 1686, in the same journal, Leibniz gave an account of the integral calculus, emphasizing the inverse nature of the procedures of integration and differentiation. Leibniz’s presentation of the differential calculus proved to be influential, as he soon found followers such as the Bernoulli brothers, who were able to extend the range of its applications significantly.

It may be of interest to examine what communication existed between Newton and Leibniz in the years when Leibniz was perfecting his differential calculus. In 1676, in answer to a request from Leibniz, Newton wrote to him to give brief details of his theory of the calculus. On August 27 of the same year, Leibniz wrote back to him to ask for fuller details on this subject. Newton replied on October 24 1676 in a long letter, describing how he had been led to some of his discoveries. He had interpolated the results of Wallis, and, working by analogy, saw how to expand functions such as

\[
(1 - x^2)^{1/2} \text{ and } (1 - x^2)^{3/2}
\]

into infinite series. He had then proceeded to deduce (or guess) the general binomial theorem. In this same letter, Newton mentioned his method of fluxions, but he gave no details. He also listed some of the functions which he could integrate, enabling him to find corresponding areas. These included

\[
x^m(b + cx^n)^p \text{ and } x^{mn-1}(a + bx^n + cx^{2n})^{\pm 1/2}.
\]

Newton concluded by expressing regret about controversy arising from his earlier research publications, with the implication that he intended to publish little in future. Leibniz replied on June 21, 1677. He described his method of drawing tangents, which proceeded not by fluxions of lines but by differences of numbers. He also introduced his notation of \(dx\) and \(dy\) for infinitesimal differences or differentials of coordinates.

While Leibniz’s differential calculus seems quite familiar to modern day students of mathematics, the same cannot be said of Newton’s fluxional calculus, which appears unnecessarily complicated. We will briefly attempt to describe how Newton’s calculus
proceeded. He assumed that we may conceive of all geometric magnitudes as generated by continuous motion—for example, a line is generated by the motion of a point, etc. The quantity thus generated is called the fluent or flowing quantity. The velocity of the moving magnitude is the fluxion of the fluent. There then arise two problems. The first is to find the fluxion of a given quantity, or more generally the relation of the fluents being given, to find the relation of their fluxions. This is akin to implicit differentiation. The second method is the inverse method of fluxions: from the fluxion, or some relation involving it, to find the fluent. This amounts to what we call integration, possibly of a differential equation. Newton referred to these procedures as the method of quadrature and the inverse method of tangents. The infinitely small part by which a fluent such as \( x \) increased in a small interval of time \( o \) was called the moment of the fluent and its value was shown to be \( o\dot{x} \). Given \( x \), Newton denoted the fluent whose fluxion was \( x \) by \( x' \) or \([x]\). This is the same as the integral of \( x \) (but not with respect to \( x \)). Subsequently, Newton’s methods were taught at Cambridge for more than 100 years, and the word fluxion was retained. The explanations given for the procedures were usually vague and unconvincing. It was not until almost 1820 that Leibniz’s notation of \( dx \) and the integral sign started to appear in British mathematical textbooks and papers.

Between 1689 and 1693, Newton had fallen under the influence of a little-known Swiss mathematician named Nicholas Fatio de Duillier (1664-1753), who had moved to England in 1687. Later in 1693, Newton suffered what appears to have been a nervous breakdown. After his recovery, Newton decided to abandon his career in Cambridge, devoted to research and scientific discovery, and sought more worldly fame in London as Warden of the Mint in 1696. Nonetheless, Newton was alarmed when informed by John Wallis in 1695 that many continental mathematicians considered the calculus to be the invention of Leibniz alone, perhaps on the basis of his expositions of the subject in Acta Eruditorum. In support of Newton, Duillier published a paper through the Royal Society of London in 1699 in which he implied that Leibniz had plagiarized the ideas of the calculus from Newton. In 1704, Leibniz replied in the Acta Eruditorum that he had priority of publication, and he protested to the Royal Society about the unfairness of the accusation of plagiarism. The Royal Society eventually responded by establishing a committee to investigate the dispute. In 1712, the committee published their conclusions in a report entitled Commercium epistolicum. The report affirmed that Newton had invented the calculus, a point not seriously disputed even by Leibniz. The report also noted that Leibniz had had access in the 1670’s to manuscripts and letters describing Newton’s preliminary
version of the calculus (for example, *De analysi*), so that suspicions of plagiarism were not totally dismissed. It has become apparent that the *Commercium epistolicum* was essentially Newton’s own work—he dictated the conclusions that the committee reported.

The priority dispute was dominated by nationalistic concerns, British mathematicians being especially keen to defend Newton’s honour and proclaim his genius as a reflection of British superiority. As a consequence, the advantages of Leibniz’s notation, subsequently developed by the Bernoullis and Euler, were ignored in Britain, and by paying too much deference to Newton, mathematics stagnated there until the early 19th century. The dispute was led in Britain, albeit secretly, by Newton himself, and he seems to have wanted to remove all record of Leibniz’s achievements.