11 Path-Connected Sets

11.1 Definition
Suppose that $A$ is a non-empty subset of $M$ and that $x$ and $y \in A$. Then a continuous function $f : [0, 1] \rightarrow A$ where $f(0) = x$ and $f(1) = y$ is called a path in $A$ from $x$ to $y$.

11.2 If $f$ is a path in $A$ from $x$ to $y$ then
$$f([0, 1]) = \{f(t) \in A \mid 0 \leq t \leq 1\}$$
is called an arc in $A$ that joins $x$ to $y$.

11.3 Suppose that $\alpha$ and $\beta \in \mathbb{R}$ where $\alpha < \beta$ and that $f : [\alpha, \beta] \rightarrow M$ is continuous. Then
$$g : [0, 1] \rightarrow M : t \rightarrow f((1-t)\alpha + t\beta)$$is continuous, $g(0) = f(\alpha)$, and $g(1) = f(\beta)$. Thus we could use any non-trivial compact interval as the domain of a path.

11.5 For more information about equivalence relations and the proof of Part (iii) of Theorem 11.4 see Appendix B.

11.6 Definition
A subset $A$ of $M$ is said to be path-connected if and only if, for all $x, y \in A$, there is a path in $A$ from $x$ to $y$.

11.7 A set $A$ is path-connected if and only if any two points in $A$ can be joined by an arc in $A$.

11.8 The expressions “pathwise-connected” and “arcwise-connected” are often used instead of “path-connected”.

11.9 Throughout this chapter we shall take “$x \sim y$ in $A$” to mean “there is a path in $A$ from $x$ to $y$”. Thus $A$ is path-connected if and only if, for all $x, y \in A$, $x \sim y$ in $A$.

11.10 Theorem
Suppose that $A$ is a subset of $M$. Then if $A$ is path-connected then $A$ is connected.

Proof
Suppose that $A$ is a path-connected subset of $M$. We shall prove that $A$ is not disconnected.

Suppose that
$$A \text{ is disconnected.} \quad (1)$$
Since $A$ is disconnected, by Corollary 10.12, there is a continuous two-valued function $f : A \rightarrow E_1$.

Since $f(A) = \{0, 1\}$, there exist $x$ and $y \in A$ such that $f(x) = 0$ and $f(y) = 1$.

11.11 Lemma
Every open ball in $E_n$ is path-connected.

Proof
Suppose that $a \in E_n$ and that $p > 0$. Suppose that $x \in B(a, p)$. The function
$$f : [0, 1] \rightarrow E_n : t \rightarrow (1-t)a + tx$$is continuous, $f(0) = a$, and $f(1) = x$.
is easily shown to be continuous. Also, for all $t \in [0,1]$,
\[ \|f(t) - a\| = \|v - a\| = |t|\|x - a\| < \rho. \]
Therefore, for all $t \in [0,1]$, $f(t) \in B(a,\rho)$. Furthermore, $f(0) = a$ and $f(1) = x$. Therefore $f$ is a path in $B(a,\rho)$ from $a$ to $x$. Thus we have proved that
\[ \text{for all } x \in B(a,\rho), a \sim x \text{ in } B(a,\rho) \quad (1) \]
Now suppose that $x$ and $y \in B(a,\rho)$. Then (1) implies that $a \sim x$ and $a \sim y$ in $B(a,\rho)$.
Therefore, Theorem 11.4 implies that $x \sim y$ and $a \sim y$ in $B(a,\rho)$.

Therefore, the formula in 11.14 describes a line segment. We use the
constructed in the proof of Lemma 11.11 is the line segment from $a$ to $x$.

11.12 In $\mathbb{E}_2$ and $\mathbb{E}_3$ the arc associated with the path $f$
constructed in the proof of Lemma 11.11 is the line segment from $a$ to $x$.

11.13 Theorem
Suppose that $A$ is an open connected subset of $\mathbb{E}_n$. Then $A$ is path-connected.
Proof Since any empty set is path-connected we can assume that $A \neq \emptyset$. We choose $a \in A$ and then let
\[ U = \{ x \in A \mid x \sim a \text{ in } A \} \]
and
\[ V = A \setminus U. \]
Then
\[ U \cup V = A \text{ and } U \cap V = \emptyset. \quad (1) \]
Suppose that $u \in U$. Since $u \in U \subseteq A$ and $A$ is open, there
exists $\rho > 0$ such that $B(u,\rho) \subseteq A$. Let $x \in B(u,\rho)$. By Lemma 11.11, $x \sim u$ (in $A$). Since $u \in U$, $u \sim a$. Therefore, since $\sim$ is an equivalence relation, $x \sim a$, that is, $x \in U$. Therefore $B(u,\rho) \subseteq U$. Therefore
\[ U \text{ is open} \quad (2) \]
Suppose that $v \in V$, that is, that $v \not\sim a$.
Since $v \in V \subseteq A$ and $A$ is open, there exists $\rho > 0$ such that $B(v,\rho) \subseteq A$.
Let $x \in B(v,\rho)$. By Lemma 11.11, $x \sim v$. Suppose that \[ x \sim a \quad (3) \]
Then, since $\sim$ is an equivalence relation, $v \sim a$, that is, $v \in U$.
But $v \not\in U$. Therefore (3) is false, that is, $x \in V$. Therefore $B(v,\rho) \subseteq V$. Therefore
\[ V \text{ is open} \quad (4) \]
Since $A$ is connected, statements (1), (2), and (4) imply that
either $U = \emptyset$ or $V = \emptyset$. But, since $a \sim a$, $a \in U$. Therefore $U \neq \emptyset$ and thus $V = \emptyset$. Therefore
\[ \text{for all } x \in A, x \sim a \quad (5) \]
Since $\sim$ is an equivalence relation, it is easy to see that (5)
implies that, for all $x$ and $y \in A, x \sim y$. Therefore $A$ is
path-connected. \[ \square \]

11.14 Suppose that $u$ and $v \in \mathbb{E}_n$ where $u \neq v$. Then the set
\[ \{(1-t)u + tv \mid t \in [0,1]\} \]
is the line-segment from $u$ to $v$. Notice that
\[ (1-t)u + tv = u + t(v - u). \]

We denote the line segment from $u$ to $v$ by $[u,v]$.
More generally, suppose that $\alpha$ and $\beta \in \mathbb{R}$ where $\alpha < \beta$. Then
it is easy to show that
\[ [u,v] = \left\{ \frac{1}{\beta - \alpha}[(\beta - t)u + (t - \alpha)v] \mid t \in [\alpha, \beta] \right\}. \]

11.15 In a course on vector-geometry in $\mathbb{R}^2$ or $\mathbb{R}^3$ we prove
that the formula in 11.14 describes a line segment. We use the
formula to define a line-segment in $\mathbb{R}^n$ when $n \geq 4$.

11.16 Suppose that $f : [\alpha, \beta] \to \mathbb{E}_n$ is such that, for all $t \in [\alpha, \beta],$
\[ f(t) = \frac{1}{\beta - \alpha}[(\beta - t)f(\alpha) + (t - \alpha)f(\beta)]. \]
Then
\[ f[\alpha, \beta] = [f(\alpha), f(\beta)] \]
and we say that $f$ is linear on $[\alpha, \beta]$.

11.17 Definition
A function $f : [0,1] \to \mathbb{E}_n$ is said to be a polygonal path in $\mathbb{E}_n$
if and only if

(i) $f$ is continuous on $[0,1]$;

(ii) there exist $0 = x_0 < x_1 < \cdots < x_k = 1$ such that, for all $p = 1, 2, \ldots, k$, $f$ is linear on $[x_{p-1}, x_p]$.

11.18 If $f : [0,1] \to \mathbb{E}_n$ is a polygonal path then $f[0,1]$ is the
union of a finite sequence of line segments,
\[ [u_0, u_1] \cup [u_1, u_2] \cup \cdots \cup [u_{n-1}, u_n], \]
where $u_p = f(x_p)$. 

11.100

11.101
11.19 Definition
A subset \( A \) of \( \mathbb{E}^n \) is said to be polygonally-connected if and only if, for all \( x, y \in A \), there is a polygonal path in \( A \) from \( x \) to \( y \).

11.20 Clearly, if \( A \) is polygonally-connected then it is path-connected. Therefore Theorem 11.10 implies that if \( A \) is polygonally-connected then it is connected.

11.21 Theorem 11.13 is still true and its proof, as given above is still valid if “path-connected” is replaced by “polygonally-connected”. Therefore every open connected subset of \( \mathbb{E}^n \) is polygonally-connected.

Therefore if \( A \) is an open subset of \( \mathbb{E}^n \) then
\[ A \text{ is connected if and only if } A \text{ is path-connected if and only if } A \text{ is polygonally-connected.} \]

11.22 Example
Suppose that
\[ A = \left\{ (x, y) \in \mathbb{E}^2 \mid \| (x, y) - (0, 4) \| \leq 4, \| (x, y) - (0, 2) \| \geq 2, 0 \leq y \leq 2 \right\} \]

Since every open ball centred at \( 0 = (0, 0) \) contains points in \( C \), \( 0 \in B \) is a boundary point of \( C \) and therefore \( A \) is connected.

For all \( \delta > 0 \) – no matter how small – there exists \( 0 < t < \delta \) such that \( \sin(1/t) \) does not belong to the open ball of radius 0.5 centred at \( 0 \). This implies that there is no path in \( A \) from \( (1, \sin(1)) \) to \( 0 \). Therefore \( A \) is not path-connected.

Notice that \( A \) is a closed subset of \( \mathbb{E}^2 \).