A Filtering Laplace Transform Integration Scheme For Numerical Weather Prediction

by

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Abstract

A filtering time integration scheme is developed and tested for use in atmospheric models. The method uses a modified inversion of the Laplace transform (LT) and is designed to eliminate spurious high frequency components while faithfully simulating low frequency modes. The method is examined both analytically and numerically.

For the numerical studies, two atmospheric models are developed, based on the shallow water equations. The first uses an Eulerian form of the governing equations and is based on the reference Spectral Transform Shallow Water Model (STSWM). The second uses a semi-Lagrangian trajectory approach. The LT method is implemented in both models. The models are tested against reference semi-implicit models using standard test cases and perform competitively in terms of accuracy and efficiency. Like semi-implicit schemes, the LT method has attractive stability properties. In particular, the semi-Lagrangian LT discretisation permits simulations with high timesteps, exceeding the CFL cutoff of Eulerian models.

There are a number of additional benefits. The LT scheme is proven to simulate accurately the phase speed of gravity waves. This is in contrast to semi-implicit methods, which maintain stability by slowing down fast-moving waves. This improved representation is shown both analytically and numerically in the case of dynamically significant Kelvin waves.

In addition, the semi-Lagrangian LT method has advantages for the treatment of orography. Semi-Lagrangian semi-implicit discretisations have been shown to generate a spurious resonance where there is flow over a mountain at high Courant number. It is demonstrated, with both a linear analysis and numerical simulations with the fully nonlinear shallow water equations, that the LT discretisation does not suffer from this problem.
Acknowledgments

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My interest in research stemmed from my undergraduate days in the School of Mathematical Sciences at University College Cork. My thanks to my former lecturers there and in particular to Dr. Michael O Callaghan who was instrumental in my choice to pursue a PhD in UCD.

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Frequently-Used Notation

\( a \)  \hspace{1cm} \text{Radius of the Earth}

\( f \)  \hspace{1cm} \text{Coriolis parameter}

\( g \)  \hspace{1cm} \text{Acceleration due to gravity}

\( h \)  \hspace{1cm} \text{Height of a fluid surface in a shallow water model}

\( i \)  \hspace{1cm} \text{Imaginary unit}

\( N \)  \hspace{1cm} \text{Number of points for inversion integral for Laplace transform scheme}

\( t \)  \hspace{1cm} \text{Time}

\( v \)  \hspace{1cm} \text{Two-dimensional horizontal velocity \((u, v)\)}

\( \beta \)  \hspace{1cm} \text{Rate of change of Coriolis parameter \( f \) with latitude}

\( \gamma \)  \hspace{1cm} \text{Cutoff frequency for Laplace transform scheme}

\( \delta \)  \hspace{1cm} \text{Two-dimensional divergence}

\( \zeta \)  \hspace{1cm} \text{Vertical component of relative vorticity}

\( \eta \)  \hspace{1cm} \text{Vertical component of absolute vorticity}

\( \lambda \)  \hspace{1cm} \text{Longitude}

\( \mu \)  \hspace{1cm} \sin \lambda

\( \tau_c \)  \hspace{1cm} \text{Cutoff period for Laplace transform scheme}

\( \Phi \)  \hspace{1cm} \text{Geopotential of fluid surface}

\( \Phi_s \)  \hspace{1cm} \text{Geopotential of underlying surface}

\( \phi \)  \hspace{1cm} \text{Latitude}

\( \chi \)  \hspace{1cm} \text{Two-dimensional velocity potential}

\( \psi \)  \hspace{1cm} \text{Two-dimensional stream function}

\( \Omega \)  \hspace{1cm} \text{Rotation rate of Earth}
Chapter 1

Introduction

1.1 Numerical Schemes for Weather Prediction

The aim of this research is to investigate a novel integration method for application to a range of physical problems, in particular to numerical weather prediction (NWP). In the field of NWP we seek to solve numerically the evolution equations of the atmosphere, in order to forecast future conditions. In this work we focus on the time discretisation of the governing partial differential equations.

The first serious attempt at numerical forecasting was made by Lewis Fry Richardson. In his 1922 book “Weather Prediction by Numerical Process”, he attempted a forecast by hand. A detailed description of this work may be found in Lynch (2006). The next milestone in the development of NWP was in 1950, when Charney, Fjørtoft and von Neumann first used an electronic computer to solve the barotropic vorticity equation (Charney et al., 1950). This led, within a few years, to operational computer weather forecasting.

In the subsequent decades, much research was carried out on suitable numerical techniques. Key concerns were, and still are, accuracy and stability. The stability of explicit timestepping schemes is governed by the Courant-Friedrichs-Lewy (CFL) condition, which limits the maximum allowable timestep (Courant et al., 1928). In operational NWP, efficiency is crucial, as we need to be able to produce regular and timely forecasts. Clearly we would like to use the longest timestep possible while still retaining acceptable accuracy. Initially explicit finite difference schemes were employed (some of these are discussed in Kalnay, 2006). The CFL criterion proved
restrictive for these, as computational stability was governed by the fastest wave present in the system. A timestep shorter than necessary for purely accuracy concerns had to be used. While fully implicit methods offered potentially unconditional stability, their implementation led to complicated coupled nonlinear systems, which seemed impractical to solve in an operational context.

The development of the semi-implicit method by Robert (1969) was a major breakthrough. By averaging terms leading to fast moving gravity waves, Robert was able to achieve accuracy comparable to explicit methods with a timestep which was six times longer. The discretisation led to a more tractable Helmholtz equation which could be efficiently solved. The introduction of the semi-Lagrangian technique for advection also led to large efficiency gains. While the semi-implicit method had brought great advantages, Robert (1981) showed that the timesteps required for stability were still larger than was strictly necessary to achieve reasonable accuracy. By combining the semi-implicit averaging with a semi-Lagrangian treatment of advection, Robert (1981, 1982) was able to perform stable and accurate integrations with even longer timesteps. Bates and McDonald (1982) showed that there was no CFL restriction with the semi-Lagrangian advection scheme, and they were the first to implement it in an operational forecast model. Further details on the development of semi-Lagrangian methods may be found in the review of Staniforth and Côté (1991).

Despite these advances, there remain a number of issues with these schemes. The semi-implicit averaging, for example, maintains stability by slowing down the fast moving waves in the system. This may be problematic if we need to simulate a physical phenomenon which is influenced by such waves. Traditional semi-Lagrangian methods are known not to conserve mass. Every discretisation technique invariably has its own strengths and weaknesses and there is still no ‘perfect’ scheme. It is important, therefore, that research into numerical methods for atmospheric models is continued. With this motivation we investigate in this thesis a numerical scheme which offers some advantages over existing schemes.

1.2 The Laplace Transform Method

High frequency noise has been a problem throughout the history of NWP. As outlined in Lynch (2006), this was the main cause of the failure of Richardson’s fore-
cast. High frequency waves could be avoided by the use of a filtered set of equations. However, it became clear that the filtered equations were inadequate for producing accurate weather forecasts and so some means of controlling noise in primitive equation models was needed.

Various initialisation techniques were developed to this end. One such method, first presented in Lynch (1985a, 1985b), used a modified inversion of the Laplace transform (LT) to remove high frequency components from the initial conditions. The LT initialisation scheme was reviewed in Daley (1991). In Van Isacker and Struyelaert (1985) and Lynch (1986) this method was extended beyond initialisation and a filtering timestepping scheme was developed from the idea. This was furthered by Lynch (1991).

The work presented in this thesis picks up from here. We seek to develop further the LT discretisation as a viable numerical scheme for NWP. The aim is to explore the benefits of the technique over existing schemes, as well to investigate the potential drawbacks.

### 1.3 Thesis Outline

In Chapter 2 the background theory and mathematical formulation of the LT integration scheme is presented. The effect of the discretisation on individual wave components is analysed and the stability properties are discussed. Some examples of the LT scheme for solving simple ordinary differential equations are given.

After these preliminaries, we seek to test the LT method’s efficacy as a numerical solver for the partial differential equations governing the atmosphere. In Chapter 3 we consider the shallow water equations, which are widely used as a testbed for new schemes, in order to avoid many of the complexities of baroclinic models. Based on an existing reference model called STSWM, a spectral model which uses the LT method for its temporal discretisation is developed. This is then tested with various standard test cases and compared with the reference semi-implicit version of STSWM.

An additional comparison is carried out, focussing on the distortion of phase speed by the two schemes. As mentioned already, the semi-implicit averaging slows down fast-moving waves in order to maintain computational stability. A linear analysis in the context of a simple oscillation equation shows that the LT scheme more accurately
Chapter 1: Introduction

simulates the phase speed of a wave. We perform shallow water simulations of Kelvin waves to demonstrate this improvement.

The STSWM model is based on an Eulerian discretisation of the governing equations. As discussed previously, a semi-Lagrangian treatment of advection improves the stability of a scheme. It is therefore interesting to investigate the combination of a semi-Lagrangian and LT method. This shallow water model is formulated in Chapter 4. Again we use a spectral method for the spatial discretisation and test the model against reference Eulerian and semi-Lagrangian semi-implicit versions. The stability and accuracy of the scheme are analysed. We find that the semi-Lagrangian LT scheme allows for stable forecasts with long timesteps. We also examine the need for initialisation when using real data as initial conditions for the model.

Semi-Lagrangian semi-implicit schemes have proven to be very successful for NWP purposes. Nevertheless some issues remain. Chapter 5 explores the problem of orographic resonance. This is a spurious noise which results from the coupling of the two schemes at high Courant number. We investigate the problem analytically with linear models and examine the response for both the semi-Lagrangian semi-implicit and semi-Lagrangian LT methods. We see that, in this simple analytic case, the semi-Lagrangian LT discretisation does not yield any spurious resonant response. We then use the models from Chapter 4 to investigate the problem numerically in the fully nonlinear shallow water equations. Initial conditions consist of 500hPa data from 12 UTC on the 12th February 1979, which has been used as a test case by a number of other authors. These forecasts confirm that the LT scheme is free from orographic resonance.

Finally, a summary of the project is provided in Chapter 6, along with possible future extensions to the work.
Chapter 2

The Laplace Transform Integration Method

2.1 The Laplace Transform as a Filter

2.1.1 Basic Definitions

Given a function \( f(t) \) with \( t \geq 0 \), the Laplace transform (LT) is defined as

\[
\hat{f}(s) \equiv \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) dt
\] (2.1)

The variable \( s \) is complex. The inversion of a transformed function back to the original is given by

\[
f(t) \equiv \mathcal{L}^{-1}\{\hat{f}\} = \frac{1}{2\pi i} \int_{\mathcal{C}} e^{st} \hat{f}(s) ds
\] (2.2)

where the contour \( \mathcal{C} \) is a line parallel to the imaginary axis in the \( s \)-plane, to the right of all the singularities of \( \hat{f} \). Further theory and applications of the Laplace transform may be found in Doetsch (1971).

From the definition in (2.1), it is clear that the LT operator is linear, that is

\[
\mathcal{L}\{af(t) + bg(t)\} = a\hat{f}(s) + b\hat{g}(s)
\]

for constants \( a \) and \( b \). Many other properties can easily be derived from the definition.
Along with linearity, the following will be of most use in this work.

\[
\mathcal{L}\{f'(t)\} = s\hat{f}(s) - f(0) \quad (2.3)
\]
\[
\mathcal{L}\{a\} = \frac{a}{s}, \text{ for constant } a \quad (2.4)
\]
\[
\mathcal{L}\{t\} = \frac{1}{s^2} \quad (2.5)
\]

Property (2.3) tells us that by taking a Laplace transform, a derivative is transformed into an algebraic expression. This is the basis for using the transforms to solve differential equations.

We will also use the fact that, if \( f(t) \) is a real function, we can write

\[
\hat{f}(s) = \overline{\hat{f}(s)} \quad (2.6)
\]

where the bar indicates the complex conjugate. This property follows from the definition of \( \hat{f}(s) \).

2.1.2 A Filtering Example

The capability of the Laplace transform to filter high frequencies was first considered by Lynch (1985a, 1985b). This is best illustrated by taking the simplest case of a function consisting of a slow and a fast oscillation. We define

\[
f(t) = ae^{i\nu_R t} + Ae^{i\nu_G t}
\]

with \(|\nu_R| \ll |\nu_G|\). The LT of this is given by

\[
\hat{f}(s) = \frac{a}{s - i\nu_R} + \frac{A}{s - i\nu_G}.
\]

The function \( \hat{f} \) has two simple poles on the imaginary axis, at \( s = i\nu_R \) and \( s = i\nu_G \). To invert this to \( f(t) \) we would normally use the inversion integral (2.2) along the straight line \( C \) shown in Figure 2.1.

If, on the other hand, we wish to remove the high frequency behaviour, we first choose a positive real number \( \gamma \) such that \(|\nu_R| < \gamma < |\nu_G|\). Next we define a closed contour \( C^* \) as the circle centred at the origin with radius \( \gamma \), as depicted in Figure 2.1.
We replace $C$ by $C^*$ in the integral in (2.2), yielding the modified inversion

$$f^*(t) \equiv \mathcal{L}^*\{\hat{f}\} = \frac{1}{2\pi i} \oint_{C^*} e^{st} \hat{f}(s) ds$$  \hspace{1cm} (2.7)

The function $f^*(t)$ will only contain contributions from the poles lying inside $C^*$, that is, those with frequencies less than $\gamma$. From Cauchy’s Integral Formula we readily find that

$$f^*(t) = ae^{i\nu_R t}$$

Thus the modified inversion integral (2.7) acts to filter high frequency behaviour, as required.

2.1.3 Laplace Transform Initialisation

This method of modifying the LT inversion in order to filter high frequencies was originally used as an initialisation technique. Lynch (1985a) applied it to a simple

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Figure 2.1: The contour $C^*$ replaces $C$ for the modified LT inversion. (From Lynch (1991), ©Amer. Met. Soc.)
one-dimensional atmospheric model and subsequently it was tested with a limited area barotropic model in Lynch (1985b). The method was formulated for a general system whose state at any time, $X(t)$, is governed by the equation

$$\frac{dX}{dt} + L X + N(X) = 0 \quad (2.8)$$

where $L$ is a linear operator and $N$ is a nonlinear vector function. Using linearity and (2.3), the LT of this equation can be written as

$$s\hat{X} - X_0 + L\hat{X} + \hat{N} = 0 \quad (2.9)$$

Here $X_0$ is the state of the system at time $t = 0$. Direct evaluation of $\hat{N}$ is normally impossible. However, if we can assume that the nonlinear terms are slowly varying, we may approximate $N(X)$ by the constant vector $N_0 = N(X_0)$, evaluated at the initial time. Now using (2.4) we can rearrange (2.9) as

$$\hat{X} = (sI + L)^{-1}\left[ X_0 - N_0/s \right]$$

where $I$ is the identity matrix. If we now apply the modified inversion (2.7) with $t = 0$, we get a filtered initial state $X^*(0)$. As already mentioned, Lynch used this as the basis for an initialisation scheme. An iterative approach was taken in which the nonlinear terms were ignored at the first step. That is,

$$\hat{X}^{(0)} = (sI + L)^{-1}X_0$$

$$X_0^{*(0)} = \mathcal{L}^* \left\{ \hat{X}^{(0)} \right\}_t = 0$$

$$\hat{X}^{(k+1)} = (sI + L)^{-1} \left[ X_0^{*(k)} - \frac{N}{s} \left[ X_0^{*(k)} \right] \right]$$

$$X_0^{*(k+1)} = \mathcal{L}^* \left\{ \hat{X}^{(k+1)} \right\}_t = 0 \quad (2.10)$$

For a barotropic limited area model, Lynch (1985b) found that the first linear step followed by one subsequent step was sufficient to control the noise during a forecast. In fact the linear step can be omitted without loss of accuracy.

This initialisation technique will be used in Chapter 5 when we use 500hPa data as initial conditions for shallow water simulations. Next we examine how the operator $\mathcal{L}^*$ can be practically applied.
2.1.4 Applying the LT Method

In the initialisation scheme outlined above, the transformed quantities \( \hat{X} \) are computed from known values. The inversion to \( X^* \) using \( L^* \) requires the complex integration in (2.7), around the circle \( C^* \). To apply the filter in practice, we replace \( C^* \) by the \( N \) sided polygon \( C^*_N \) to reduce the integration to a summation. The length of each edge is \( \Delta s_n \) and the midpoints are labelled \( s_n \) for \( n = 1, 2, \ldots, N \). Figure 2.2 below shows the case with \( N = 8 \).

![Figure 2.2: For numerical integration, the circle \( C^* \) is replaced by \( C^*_N \). (From Lynch (1991), ©Amer. Met. Soc.)](image)

We can now define the numerical operator used for the modified inversion as

\[
L^*_N \{ \hat{f} \} = \frac{1}{2\pi i} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) \Delta s_n
\]
The vertices, edge length and midpoints are defined respectively as follows

\[ s'_n = \frac{\gamma}{\cos \frac{\pi}{N}} e^{2\pi i n/N} \]

\[ \Delta s_n = s'_n - s'_{n-1} = 2i s'_n e^{-i\pi/N} \sin \frac{\pi}{N} \]

\[ s_n = \gamma e^{i\pi (2n-1)/N} = s'_n e^{-i\pi/N} \cos \frac{\pi}{N} \]  \hspace{1cm} (2.11)

As pointed out in Lynch (1985b), if we divide the summation by

\[ \kappa = \tan \frac{\pi}{N} \]

then the inversion is exact for a constant function. Using this correction factor \( \kappa \) and the definitions in (2.11), we can write

\[ \frac{1}{\kappa} = \frac{2\pi i}{N} \frac{s_n}{\Delta s_n} \]

The inversion integral now becomes

\[ L^*_N \{ \hat{f} \} = \frac{1}{N} \sum_{n=1}^{N} e^{s_n t} \hat{f}(s_n) s_n \]

It was also suggested in Van Isacker and Struylaert (1985) that the exponential term in the expression for the numerical inversion should be replaced by a Taylor series truncated to \( N \) terms. Note that terms in the Taylor series with \( n \geq N \) are aliased onto earlier terms in the series. We write

\[ e_N^z = \sum_{j=0}^{N-1} \frac{z^j}{j!} \]  \hspace{1cm} (2.12)

The reasons for this will be examined in Section 2.3. The final form of the numerical filter to be used is now defined as

\[ L^*_N \{ \hat{f} \} = \frac{1}{N} \sum_{n=1}^{N} e_N^{s_n t} \hat{f}(s_n) s_n \]  \hspace{1cm} (2.13)

### 2.2 LT Integration

For the initialisation scheme, the inversion of the LT was carried out with \( t = 0 \). This can be extended to construct a numerical integration method for differential equations. The resulting method was studied in Van Isacker and Struylaert (1985,
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1986) and Lynch (1986, 1991). The basic idea is to consider the LT over a discrete interval of time $\Delta t$. The transforms can be computed analytically and the modified inversion operator (2.13) is then applied to find a filtered value of the function at the end of the interval.

Again we take the transform of the general equation

$$\frac{dX}{dt} + LX + N(X) = 0$$

and rearrange to get

$$\hat{X} = (sI + L)^{-1}[X_0 - N_0/s]$$

This time we apply the inversion operator with $t = \Delta t$ to get the filtered state at this time

$$X(\Delta t) = \mathcal{L}^*\{\hat{X}\} \bigg|_{t=\Delta t}$$

Having the solution at $t = \Delta t$ we may continue stepwise to extend the forecast. In general we consider the time interval $[\tau \Delta t, (\tau + 1)\Delta t]$. As above, the filtered solution at time $(\tau + 1)\Delta t$ is found by applying the modified inversion to the LT of the equation. Over this general interval, the 'initial condition' as given in (2.3) will be taken at the beginning of the interval, that is, $X^\tau \equiv X(\tau \Delta t)$. The nonlinear terms are also evaluated at this time. Thus the solution at time $(\tau + 1)\Delta t$ is

$$X^{\tau+1} = \mathcal{L}^*\{(sI + L)^{-1}[X^\tau - N^\tau/s]\} |_{t=\Delta t}$$

Alternatively a centred approach may be taken, where we consider the interval $[(\tau - 1)\Delta t, (\tau + 1)\Delta t]$ and the nonlinear terms are evaluated at the centre $\tau \Delta t$. Combining this with the numerical inversion (2.13) we now have the general forecasting procedure as follows:

$$\hat{X}(s) = (sI + L)^{-1} [X^{\tau-1} - N^{\tau}/s]$$

$$X^{\tau+1} = \frac{1}{N} \sum_{n=1}^{N} \hat{X}(s_n) e_n s_n^2 \Delta t$$

(2.14)

Care must be taken, of course, to ensure that $(sI + L)^{-1}$ exists. The matrix $sI + L$ is singular when we have $s = -\lambda$, for $\lambda$ an eigenvalue of $L$. But $|s| = \gamma$, the radius of the contour $\mathcal{C}^*$. The problem can thus be avoided by a suitable choice of $\gamma$, the cutoff frequency.
2.3 Truncated Exponential

We investigate the effect of the truncated exponential in the inversion operator by taking the transform of a constant function and inverting it with $\mathcal{L}_N^*$. We define the function $f(t) \equiv 1$. The LT of this is given in (2.4) as $\hat{f}(s) = \frac{1}{s}$. We attempt to invert this using the operator (2.13) with the full exponential, that is,

$$\mathcal{L}_N^* \left\{ \frac{1}{s} \right\} = \frac{1}{N} \sum_{n=1}^{N} e^{s_n \Delta t}$$

$$= \frac{1}{N} \sum_{n=1}^{N} \sum_{k=0}^{\infty} \frac{(s_n \Delta t)^k}{k!}$$

$$= \frac{1}{N} \sum_{k=0}^{\infty} \sum_{n=1}^{N} \frac{(\gamma \Delta t)^k}{k!} e^{k \pi i (2n-1)/N} \text{ using } (2.11)$$

If $k = mN$ for $m \in \mathbb{N}$ then

$$e^{k \pi i (2n-1)/N} = e^{\pi i m (2n-1)} = (-1)^m$$

For this reason, we write $k = j + mN$ with $j = 0, 1, \ldots, N - 1$. We can then continue the inversion

$$\mathcal{L}_N^* \left\{ \frac{1}{s} \right\} = \frac{1}{N} \sum_{m=0}^{\infty} \sum_{j=0}^{N-1} \sum_{n=1}^{N} \frac{(\gamma \Delta t)^{j+mN}}{(j + mN)!} e^{i \pi (j+mN)(2n-1)/N}$$

$$= \frac{1}{N} \sum_{m=0}^{\infty} \sum_{j=0}^{N-1} \frac{(\gamma \Delta t)^{j+mN}}{(j + mN)!} e^{-i \pi j/N} \sum_{n=1}^{N} e^{i \pi 2n j/N} e^{i \pi 2m n}$$

$$= \frac{1}{N} \sum_{m=0}^{\infty} (-1)^m \sum_{j=0}^{N-1} \frac{(\gamma \Delta t)^{j+mN}}{(j + mN)!} e^{-i \pi j/N} \sum_{n=1}^{N} e^{i \pi 2n j/N}$$

The final term above is a geometric series and sums to zero when $j \neq 0$:

$$\sum_{n=1}^{N} \left( e^{2j \pi i/N} \right)^n = e^{2j \pi i/N} \sum_{n=0}^{N-1} \left( e^{2j \pi i/N} \right)^n$$

$$= e^{2j \pi i/N} \left( \frac{1 - \left( e^{2j \pi i/N} \right)^N}{1 - e^{2j \pi i/N}} \right) = e^{2j \pi i/N} \left( \frac{1 - e^{2j \pi i}}{1 - e^{2j \pi i/N}} \right)$$

$$= 0$$

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For \( j = 0 \) we get

\[
\sum_{n=1}^{N} e^{0} = N
\]

Hence we are left with

\[
\mathcal{L}\left\{\frac{1}{s}\right\} = \frac{1}{N} \sum_{m=0}^{\infty} (-1)^{m} \left\{ \frac{(\gamma \Delta t)^{mN}}{(mN)!} N \right\}
\]

\[
= 1 - \frac{(\gamma \Delta t)^{N}}{N!} + \frac{(\gamma \Delta t)^{2N}}{(2N)!} - \ldots
\]

If the truncated exponential \( e_{N} \) defined in (2.12) is used in the inversion operator, we have only the \( m = 0 \) contribution and the inversion is exact, i.e.

\[
\mathcal{L}\left\{\frac{1}{s}\right\} = 1
\]

Using the full exponential introduces an \( O(\gamma \Delta t)^{N} \) error. We illustrate this by using the LT method to solve for the constant function \( x(t) = 1 \), that is, solve the ordinary differential equation

\[
\frac{dx}{dt} = 0 \quad x(0) = 1
\]

The graph on the left in Figure 2.3 shows the numerical solution along with the exact solution, while the right-hand panel shows a negligible difference between the two. The errors when the full exponential is used are far larger, as seen in Figure 2.4.

![Inverting 1/s](image1)

![Error when inverting 1/s](image2)

Figure 2.3: Comparing the LT solution with the exact for a constant function, using a truncated exponential. The error is \( O(10^{-15}) \).
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Figure 2.4: Comparing the LT solution with the exact for a constant function, using a full exponential. The error is $O(10^{-2})$

It can further be shown (see Appendix A) that, with the truncated exponential, the inversion is exact for all powers of $t$ up to, and including, $N - 1$. In particular we will later use this fact for $f(t) = t$. This is demonstrated in Figure 2.5 where we solve

$$\frac{dx}{dt} = 1$$
$$x(0) = 0$$

Figure 2.5: Comparing the LT solution with the exact for a linear function, using a truncated exponential. The error is $O(10^{-14})$.

It is clearly beneficial to use the truncated exponential in the inversion operator (2.13). This will be used in all subsequent work in the thesis. We will use the fact that

$$e_N^x = e_N^{-ix} \quad (2.15)$$
where $\overline{\cdot}$ denotes the complex conjugate. This can be readily deduced from the definition (2.12). In addition, we define the truncated trigonometric functions

\[
\cos_N x = \text{Re} \left[ e^{ix} \right] \tag{2.16}
\]

\[
\sin_N x = \text{Im} \left[ e^{ix} \right] \tag{2.17}
\]

### 2.4 Filter Response and Stability

With the inversion operator $\mathcal{L}_N^*$ defined by (2.13), we next examine the effect of the filtering operator $\mathcal{L}_N^* \mathcal{L}$ on a single wave component $f(t) = e^{i\omega t}$. This was analysed by Van Isacker and Struylaert (1985) and Lynch (1986), who showed that

\[
\mathcal{L}_N^* \mathcal{L} \{ e^{i\omega t} \} = H_N(\omega) e^{i\omega t} \tag{2.18}
\]

where

\[
H_N(\omega) = \frac{1}{1 + \left( \frac{i\omega}{\gamma} \right)^N} \tag{2.19}
\]

Further details and derivations of this are given in Appendix A.

If we choose a value for $N$ which is a multiple of 4, we ensure that $H_N(\omega)$ is real and $|H_N(\omega)| \leq 1$. Thus its effect is to damp the input, without a phase shift. In addition, the operator $\mathcal{L}_N^* \mathcal{L}$ truncates the original $e^{i\omega t}$ to $N$ terms.

We plot in Figure 2.6 the response function $H_N$ in terms of a nondimensional parameter $x = \frac{\omega}{\gamma}$ for various values of $N$. It has the desired low-pass filter effect of removing those components with frequencies larger than the cutoff ($x > 1$), while the low frequencies are largely unaffected. As $N$ increases we get closer to an ideal step-function filter. We note that $H_N$ is the square of the response function of a Butterworth lowpass filter (Oppenheim and Schafer, 1989).
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Figure 2.6: Filter response $H_N$ as a function of $x = \frac{\omega}{\gamma}$.

Lynch (1986) shows how, when the centred LT method given by (2.14) is used, the response above yields the sufficient stability criterion

$$\Delta t \leq \frac{(N!)^{1/N}}{2\gamma}$$  \hspace{1cm} (2.20)

This is a very lenient condition. With typically used values of $N = 8$ and a cut-off frequency defined by a period $\tau_c = 6$ hours, we get a maximum timestep of around 1.8 hours.

Before testing the LT method in an atmospheric model, a number of simple tests were carried out with ordinary differential equations. This work is presented in Appendix B.

2.5 Symmetry

In this chapter we have developed and analysed the Laplace transform method. Now we look to improve its efficiency by exploiting a symmetry in the inversion contour. As noted in Lynch (1991), if we have a real function $f(t)$ with LT $\hat{f}(s)$, then

$$\frac{1}{2\pi i} \oint_{C^-} e^{st} \hat{f}(s) \, ds = \frac{1}{\pi} \int_{C^+} \text{Im} \left[ e^{st} \hat{f}(s) \, ds \right]$$  \hspace{1cm} (2.21)
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where $C^*$ is the circle in Figure 2.1 and the semicircle $C^+$ is the upper half of $C^*$. This potentially halves the cost of the LT method, as only half of the points $s_n$ on the polygon $C^*_N$ depicted in Figure 2.2 need to be considered.

The points on the inversion contour, as defined in (2.11), satisfy $s_{N+1-n} = \overline{s_n}$ for $n = 1, \ldots, N/2$. This allows us to write the inversion operator (2.13) as

$$L^*_N\{\hat{f}\} = \frac{1}{N} \sum_{n=1}^{N/2} \left\{ s_n \hat{f}(s_n) e^{s_n t} + \overline{s_n} \hat{f}(\overline{s_n}) e^{\overline{s_n} t} \right\}$$

$$= \frac{1}{N} \sum_{n=1}^{N/2} \left\{ s_n \hat{f}(s_n) e^{s_n t} + s_n \hat{f}(s_n) e^{s_n t} \right\}$$

The second line follows by using the property (2.6), which holds if $f$ is a real function. We can now write the above as

$$L^*_N\{\hat{f}\} = 2 \frac{N}{N} \sum_{n=1}^{N/2} \Re \left\{ s_n \hat{f}(s_n) e^{s_n t} \right\}$$

(2.22)

So if we are dealing with real functions, we may halve the number of points required for the numerical inversion while keeping the level of accuracy and stability.

The operator in (2.22) may also be derived by discretising the analytic integral on the right-hand side of (2.21). This is presented in Appendix A, where we also verify that the inversion properties from Sections 2.3 and 2.4 hold for the symmetric inversion operator.
Chapter 3

STSWM and Kelvin Waves

3.1 The Shallow Water System and the Spectral Transform Method

In the previous chapter we outlined the basic theory behind the Laplace transform integration method. We used it to solve some simple ordinary differential equations. The filtering capabilities of the scheme were further investigated in Appendix B. We now seek to test it as a viable scheme for numerical weather prediction (NWP).

Given the complexity of a full atmospheric model, including both dynamics and physical parameterisations, it is more appropriate to test a novel numerical method in a simpler system. The shallow water equations are the traditional choice for this. While considerably more tractable than the full primitive equations, they still exhibit much of the complex behaviour of the real atmosphere; in particular they possess both slow, rotational modes and faster gravity-inertia waves. Thus the system provides an ideal framework for the initial testing of the numerics for a forecasting model. Potential problems with a scheme can be isolated and analysed, which otherwise may have been obscured by the various interacting components of a full model.

As mentioned, the LT method was used in the previous chapter to integrate differential equations in time. We now need to combine this with a spatial discretisation scheme. A number of options exist. In this work we have chosen the spectral transform method. The motivation for this will be given in the next section, based on analogies between the LT and semi-implicit methods.
3.1.1 Semi-Implicit Schemes

The theoretical analysis in Section 2.4 suggested that a key benefit of the LT method is stability, with its potential to allow long timesteps to be used. This has been a long-standing issue in the field of NWP. Early attempts with explicit numerical methods were subject to restrictive Courant-Friedrichs-Lewy (CFL) stability criteria, determined by the speed of the fastest wave in the system. While fully implicit methods offered stability, they were less practical due to the complexity of the resulting discretised equations. Further details of explicit and implicit methods may be found in Kalnay (2006).

The semi-implicit method, introduced by Robert (1969), offered a solution to this problem. By averaging only the terms which lead to gravity waves, Robert was able to gain considerably increased efficiency. Comparable accuracy for timesteps up to six times longer than required by explicit methods was achieved.

When the shallow water equations are discretised with a semi-implicit scheme, one obtains a Helmholtz equation, which needs to be solved at every timestep. Clearly an efficient solver is desirable. This becomes an issue when choosing a spatial scheme for the LT method. We will see that, when the method is applied, we end up with an analogous Helmholtz equation. Whereas the semi-implicit method requires the solution of the equation once every timestep, for the LT scheme we must solve it at each of the $N$ midpoints on the $N$-sided inversion contour in Figure 2.2. Although the symmetry property of Section 2.5 may be used in some models to halve this number, we would still have to solve, for example, four Helmholtz equations per timestep when inverting around a typically-used octagon.

Therefore if we wish to exploit the stability advantages of the LT scheme and compare it with the semi-implicit, it is vital that these potential benefits are not negated by the extra computational overhead. This motivates the coupling of the LT with the spectral transform method, for which the solution of a Helmholtz equation is much simpler and more efficient.

3.1.2 The Spectral Transform Method

Spectral methods were first used by Silberman (1954) to solve the barotropic vorticity equation. Rather than solve for physical fields on a grid, the prognostic
variable is expanded as a truncated series of basis functions and a set of equations for the time-dependent coefficients can then be solved. For spherical geometry the appropriate basis functions are spherical harmonics. Despite having a number of advantages over finite difference methods, such as the lack of a pole problem, this approach proved to be computationally impractical due to the interactions of coefficients arising from nonlinear terms. Over the next few years much research was carried out on the method, culminating in the development of the spectral transform method; Machenhauer (1974) provides a review of this early work.

A purely spectral method considers only the spectral form of the model fields. In the spectral transform method, on the other hand, nonlinear terms are computed on a physical grid and the product is then transformed to spectral space at each timestep. Although this requires transformations back and forth between physical and spectral space, it still proves to be far more cost-effective than the interaction coefficient method. For a series of spherical harmonics truncated to degree \( N \), the spectral transform reduces the number of operations from \( O(N^5) \) to \( O(N^3) \); (Orszag, 1970).

The spectral transform method uses spherical harmonics as basis functions for expansion of the model fields. On a sphere of radius \( a \) with longitude \( \lambda \) and latitude \( \phi \), the Laplacian operator is given by

\[
\nabla^2 \psi = \frac{1}{a^2} \left[ \frac{1}{\cos^2 \phi} \frac{\partial^2 \psi}{\partial \lambda^2} + \frac{1}{\cos \phi} \frac{\partial}{\partial \phi} \left( \cos \phi \frac{\partial \psi}{\partial \phi} \right) \right]
\]

Spherical harmonics are the eigenfunctions of Laplace’s equation and satisfy

\[
\nabla^2 Y_{\ell m}^n = -\frac{\ell (\ell + 1)}{a^2} Y_{\ell m}^n
\]

Writing \( \mu = \sin \phi \), they are defined by \( Y_{\ell m}^n(\lambda, \mu) = e^{i m \lambda} P_{\ell m}^n(\mu) \). The \( P_{\ell m}^n \) are the associated Legendre functions. The appendices in Washington and Parkinson (2005) provide a good introduction and further details of spherical harmonics that are necessary for the spectral transform method.

Examining (3.2) we see that computing the Laplacian of a series of spherical harmonics merely requires scalar multiplications. The solution of a Helmholtz equation is therefore computationally trivial. This provides the motivation for using the transform method with LT time integration. We now discuss a spectral transform model used for shallow water simulations.
Chapter 3: STSWM and Kelvin Waves

3.2 STSWM

The Spectral Transform Shallow Water Model (STSWM) is a freely available model developed at the National Center for Atmospheric Research (NCAR) and described in Hack and Jakob (1992). It is designed to solve the shallow water equations using a spectral transform method and specifically to consider the test suite of Williamson et al. (1992). The original code is written in Fortran 77. An updated version in Fortran 90 was developed by the ICON group at the Max Planck Institute for Meteorology (MPI-M) and the Deutscher Wetterdienst (DWD) [http://icon.enes.org/]. In the next section we provide a brief overview of the model’s discretisation. More detail is given in the report of Hack and Jakob and a similar model is discussed in Bourke (1972). Jakob et al. (1993) specifically describe the changes needed to include orography in the model.

3.2.1 Shallow Water Equations

The shallow water equations govern the behaviour of a shallow, rotating layer of fluid which is homogeneous, incompressible and inviscid (Pedlosky, 1987). In vorticity-divergence form they may be written as

\[
\frac{\partial \zeta}{\partial t} = -\nabla \cdot (\zeta + f)v \\
\frac{\partial \delta}{\partial t} = k \nabla \times (\zeta + f)v - \nabla^2 (\Phi + \frac{v \cdot v}{2}) \\
\frac{\partial \Phi^*}{\partial t} = -\nabla \cdot (\Phi^* v)
\] (3.3)

Here \(v = (u, v)\) is the horizontal velocity vector, \(\zeta = k(\nabla \times v)\) is the relative vorticity, \(\delta = \nabla \cdot v\) is the horizontal divergence and \(f = 2\Omega \sin \phi\) is the Coriolis parameter with the Earth’s angular speed given by \(\Omega\). The free surface geopotential is \(\Phi\) and \(\Phi^*\) represents the geopotential depth; that is, \(\Phi^* = \Phi - \Phi_s\) where \(\Phi_s\) is the geopotential of the surface of the Earth.

The geopotential depth is next written as \(\Phi^* = \bar{\Phi}^* + \Phi^\prime\) where \(\bar{\Phi}^*\) is a time-independent spatial mean. The equations can now be written in the following form,
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in terms of the absolute vorticity $\eta = \zeta + f$:

$$\frac{\partial \eta}{\partial t} = -\frac{1}{a(1 - \mu^2)} \frac{\partial}{\partial \lambda} (U\eta) - \frac{1}{a} \frac{\partial}{\partial \mu} (V\eta)$$

$$\frac{\partial \delta}{\partial t} = \frac{1}{a(1 - \mu^2)} \frac{\partial}{\partial \lambda} (V\eta) - \frac{1}{a} \frac{\partial}{\partial \mu} (U\eta) - \nabla^2 \left( \Phi_s + \Phi' + \frac{U^2 + V^2}{2(1 - \mu^2)} \right)$$

$$\frac{\partial \Phi'}{\partial t} = -\frac{1}{a(1 - \mu^2)} \frac{\partial}{\partial \lambda} (U\Phi') - \frac{1}{a} \frac{\partial}{\partial \mu} (V\Phi') - \bar{\Phi'} \delta$$

where $\mu = \sin \phi$ and

$$U = u \cos \phi$$

$$V = v \cos \phi$$

The prognostic variables of the model are $\eta$, $\delta$ and $\Phi'$. The winds may be found diagnostically by first writing them in terms of a stream function and velocity potential:

$$v = k \times \nabla \psi + \nabla \chi$$

From this we have

$$U = \frac{1}{a} \frac{\partial \chi}{\partial \lambda} - \frac{(1 - \mu^2)}{a} \frac{\partial \psi}{\partial \mu}$$

$$V = \frac{1}{a} \frac{\partial \psi}{\partial \lambda} + \frac{(1 - \mu^2)}{a} \frac{\partial \chi}{\partial \mu}$$

The prognostic fields can be used to find $\psi$ and $\chi$ via

$$\eta = \nabla^2 \psi + f$$

$$\delta = \nabla^2 \chi$$

3.2.2 Spectral Formulation

In a spectral model, all of the fields are represented as truncated series of spherical harmonics; for example, with a triangular truncation,

$$\zeta(\lambda, \mu, t) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \zeta_{\ell m}^{(t)} P_{\ell}^m(\mu) e^{im\lambda}$$

Here $\zeta_{\ell m}^{(t)}$ are the unknown time-dependent spectral coefficients. For the spectral transform method, the nonlinear terms on the right-hand side of (3.4) are computed in
physical space and the product is then expanded in a series. Orthogonality of the spherical harmonics can then be used to obtain a series of equations for the spectral coefficients. We are left with a set of ordinary differential equations of the form

\[
\begin{align*}
\frac{d}{dt}\eta^m_\ell &= N^m_\ell \\
\frac{d}{dt}\delta^m_\ell &= D^m_\ell + \ell(\ell + 1) \frac{\Phi^m_\ell}{a^2} \\
\frac{d}{dt}\Phi^m_\ell &= F^m_\ell - \bar{\Phi}^* \delta^m_\ell
\end{align*}
\] (3.6)

Note that the \(\Phi^m_\ell\) are the spectral coefficients of the perturbation geopotential \(\Phi'\). The prime has been dropped for ease of notation. Full details of the derivation of (3.6) and explicit expressions for \(N^m_\ell\), \(D^m_\ell\) and \(F^m_\ell\) are given in Appendix C.

### 3.2.3 Semi-Implicit STSWM

The STSWM model contains both an explicit and a semi-implicit formulation for solving (3.6). A centred scheme is employed and, in both cases, the vorticity equation is discretised as

\[
\frac{\{\eta^m_\ell\}^{\tau+1} - \{\eta^m_\ell\}^{\tau-1}}{2 \Delta t} = \{N^m_\ell\}^{\tau}
\]

Here the superscript \(\tau\) represents the discrete time level \(t = \tau \Delta t\). An initial forward step of size \(\Delta t\) is necessary. For the semi-implicit scheme, the terms leading to gravity waves in the divergence and continuity equations are averaged to maintain stability. They are discretised as

\[
\frac{\{\delta^m_\ell\}^{\tau+1} - \{\delta^m_\ell\}^{\tau-1}}{2 \Delta t} = \{D^m_\ell\}^{\tau} + \ell(\ell + 1) \frac{\{\Phi^m_\ell\}^{\tau+1} + \{\Phi^m_\ell\}^{\tau-1}}{2a^2}
\]

\[
\frac{\{\Phi^m_\ell\}^{\tau+1} - \{\Phi^m_\ell\}^{\tau-1}}{2 \Delta t} = \{F^m_\ell\}^{\tau} - \bar{\Phi}^* \left( \{\delta^m_\ell\}^{\tau+1} + \{\delta^m_\ell\}^{\tau-1} \right) \frac{1}{2}
\]

The system is now implicit, with coupled expressions for \(\delta^m_\ell\) and \(\Phi^m_\ell\) at the new time level \(\tau + 1\). These can be easily solved, as a result of the spectral form of the Laplacian.
operator. The final timestepping procedure can then be written as

\[
\begin{align*}
\{\eta^m_\ell\}^{\tau+1} &= \{\eta^m_\ell\}^{\tau-1} + 2 \Delta t \{N^m_\ell\}^\tau \\
\{\delta^m_\ell\}^{\tau+1} &= \left[ 1 + \Phi^* \frac{\ell(\ell + 1)}{a^2} \Delta t^2 \right]^{-1} \left( \mathcal{R} + Q \frac{\ell(\ell + 1)}{a^2} \Delta t \right) \\
\{\Phi^m_\ell\}^{\tau+1} &= \left[ 1 + \Phi^* \frac{\ell(\ell + 1)}{a^2} \Delta t^2 \right]^{-1} (Q - R \Phi^* \Delta t)
\end{align*}
\] (3.7)

where

\[
\begin{align*}
\mathcal{R} &= \{\delta^m_\ell\}^{\tau-1} + 2 \Delta t \{D^m_\ell\}^\tau + \Delta t \frac{\ell(\ell + 1)}{a^2} \{\Phi^m_\ell\}^{\tau-1} \\
Q &= \{\Phi^m_\ell\}^{\tau-1} + 2 \Delta t \{F^m_\ell\}^\tau + \Delta t \Phi^* \{\delta^m_\ell\}^{\tau-1}
\end{align*}
\]

This semi-implicit STSWM will be used as the reference model when testing the Laplace transform method. Some of the test cases to be considered do not have analytic solutions. In these situations, high resolution simulations with this model will be used as the reference solution to allow for comparisons, both here and again in Chapter 4.

### 3.3 STSWM: Laplace Transform Formulation

We will now adapt the STSWM code to solve the shallow water equations using the LT method. Again we consider the system of equations (3.6) for the time-dependent spectral coefficients:

\[
\begin{align*}
\frac{d}{dt}\eta^m_\ell &= N^m_\ell \\
\frac{d}{dt}\delta^m_\ell &= D^m_\ell + \frac{\ell(\ell + 1)}{a^2} \Phi^m_\ell \\
\frac{d}{dt}\Phi^m_\ell &= F^m_\ell - \Phi^* \delta^m_\ell
\end{align*}
\]

We take the Laplace Transform of each equation, as described in Section 2.2, to get

\[
\begin{align*}
\mathcal{L}\{\eta^m_\ell\} - \{\eta^m_\ell\}^{\tau-1} &= \frac{1}{s} \{N^m_\ell\}^\tau \\
\mathcal{L}\{\delta^m_\ell\} - \{\delta^m_\ell\}^{\tau-1} &= \frac{1}{s} \{D^m_\ell\}^\tau + \frac{\ell(\ell + 1)}{a^2} \Phi^m_\ell \\
\mathcal{L}\{\Phi^m_\ell\} - \{\Phi^m_\ell\}^{\tau-1} &= \frac{1}{s} \{F^m_\ell\}^\tau - \Phi^* \delta^m_\ell
\end{align*}
\]
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As outlined previously, we are taking our ‘initial’ value at the beginning of the interval, i.e. at \( t = (\tau - 1) \Delta t \). The nonlinear terms of \( N^m_\ell \), \( D^m_\ell \) and \( F^m_\ell \) are evaluated at the middle time level \( \tau \). By taking the transforms of the linear right-hand terms in the divergence and continuity equations, we get a coupled system analogous to the semi-implicit discretisation. We can solve to get

\[
\begin{align*}
  s \hat{\eta}^m_\ell &= \left\{ \eta^m_\ell \right\}^{\tau-1} + \frac{1}{s} \left\{ N^m_\ell \right\}^\tau \\
  s \hat{\delta}^m_\ell &= \left[ 1 + \Phi^* \ell(\ell + 1) \frac{1}{a^2} \right]^{-1} \left( \mathcal{R}' + \frac{1}{s} \ell(\ell + 1) \mathcal{Q}' \right) \\
  s \hat{\Phi}^m_\ell &= \left[ 1 + \Phi^* \ell(\ell + 1) \frac{1}{a^2} \right]^{-1} \left( \mathcal{Q}' - \frac{1}{s} \Phi^* \mathcal{R}' \right)
\end{align*}
\]

(3.8)

where

\[
\begin{align*}
  \mathcal{R}' &= \left\{ \delta^m_\ell \right\}^{\tau-1} + \frac{1}{s} \left\{ D^m_\ell \right\}^\tau \\
  \mathcal{Q}' &= \left\{ \Phi^m_\ell \right\}^{\tau-1} + \frac{1}{s} \left\{ F^m_\ell \right\}^\tau
\end{align*}
\]

Examining (3.7) and (3.8) we find close similarities between the two discretisations. Once we have computed the terms in (3.8), we use the inversion operator \( \mathcal{L}^\ast_N \) to recover the spectral coefficients at the new time \((\tau + 1) \Delta t\), which is at a time \( 2 \Delta t \) after the beginning of the time interval:

\[
\left\{ \eta^m_\ell \right\}^{\tau+1} = \mathcal{L}^\ast_N \left\{ \hat{\eta}^m_\ell \right\} = \frac{1}{N} \sum_{n=1}^{N} s_n \hat{\eta}^m_\ell(s_n) e^{2 \Delta t s_n}
\]

In Section 2.5 we discussed a symmetry that allowed us to invert using a summation involving only \( N/2 \) values. However, this assumes that we are inverting the transforms of real functions. We note that, in this discretisation, we are dealing with the transforms of complex-valued spectral coefficients; for example \( \hat{\eta}^m_\ell = \mathcal{L} \left\{ \eta^m_\ell \right\} \). Thus we are not able to exploit the symmetry properties in the STSWM model.

### 3.4 Test Cases

To test a new numerical scheme, initial conditions must be chosen for tests. It is vital to run a comprehensive set of cases. Each should be capable of testing different
aspects of atmospheric flow and thus have the ability to highlight specific deficiencies in a scheme. Ideally we would use initial conditions for which an analytic solution is known for the shallow water equations, allowing error measurements to be computed. It is also desirable that the modelling community would use an agreed standard set of tests in order to facilitate the intercomparison of model performance. To this end a suite of seven test cases was proposed by Williamson et al. (1992) and has been widely used. Additional cases have since been suggested with increased complexity; for example Galewsky et al. (2004), Nair and Jablonowski (2008). Moving from the shallow water framework, Held and Suarez (1994) propose a benchmark for comparing the dynamical cores of full climate models.

For testing the LT method against the reference semi-implicit in this work, we will mainly use cases 1, 2, 5 and 6 of the Williamson et al. suite. We now briefly describe each of these.

3.4.1 Case 1: Advection of a cosine bell

The first case consists of passive advection by constant nondivergent winds. It does not consider the full shallow water system, just the continuity equation. The initial height field takes the shape of a ‘cosine bell’, given by

\[
\Phi(\lambda, \phi) = \begin{cases} 
gh_0 \left( 1 + \cos \left( \frac{\pi r}{R} \right) \right) & \text{if } r < R \\
0 & \text{if } r \geq R 
\end{cases}
\]

where \( h_0 = 1000 \text{m}, R = a/3 \) and \( r = a \arccos \left( \sin \phi_c \sin \phi + \cos \phi_c \cos \phi \cos (\lambda - \lambda_c) \right) \) is the great circle distance from a given point and the initial centre of the bell \((\lambda_c, \phi_c) = (\frac{3\pi}{2}, 0)\).

The nondivergent winds consist of solid body rotation:

\[
\begin{align*}
    u &= u_0 \left( \cos \phi \cos \alpha + \sin \phi \cos \lambda \sin \alpha \right) \\
v &= -u_0 \sin \lambda \sin \alpha
\end{align*}
\]

The advecting wind is taken as \(2\pi a/(12 \times 24 \times 3600)\) so that the initial bell is advected around the globe and back to its original position in 12 days. The parameter \( \alpha \) determines the orientation of the grid. With \( \alpha = 0 \) we get advection along the equator, while setting \( \alpha = \frac{\pi}{2} \) will rotate the grid so that the bell passes over the pole. In this way the case is useful for identifying problems with the poles.
3.4.2 Case 2: Steady zonal flow

Case 2 consists of a nondivergent zonal flow with a geostrophically balanced height field. The winds and corresponding geostrophically balanced height field $h$ are given by

$$ u = u_0 \left( \cos \phi \cos \alpha + \sin \phi \cos \lambda \sin \alpha \right) $$

$$ v = -u_0 \sin \lambda \sin \alpha $$

$$ gh = gh_0 - \left( a \Omega u_0 + \frac{u_0^2}{2} \right) \left( -\cos \lambda \cos \phi \sin \alpha + \sin \phi \cos \alpha \right)^2 $$

Typical parameter values are $u_0 = 2 \pi a / (12 \times 24 \times 3600) \text{ m s}^{-1}$ and $gh_0 = 2.94 \times 10^4 \text{ m}^2 \text{ s}^{-2}$.

Since this is an exact steady state solution to the shallow water equations, the errors can be readily computed at any stage by comparing the solution with the initial conditions.

3.4.3 Case 5: Flow over an isolated mountain

The initial conditions for case 5 are the similar to those for case 2 with $\alpha = 0$, $h_0 = 5960 \text{ m}$ and $u_0 = 20 \text{ m s}^{-1}$. In this case, however, the bottom surface is no longer flat. Instead, the zonal flow impinges on an isolated mountain, centred at the point $(\lambda_c, \phi_c) = \left( \frac{3\pi}{2}, \frac{\pi}{6} \right)$, given by

$$ h_s = h_{s0} \left( 1 - \frac{r}{R} \right) $$

where $h_{s0} = 2000 \text{ m}$, $R = \pi/9$ and $r^2 = \min \left[ R^2, (\lambda - \lambda_c)^2 + (\phi - \phi_c)^2 \right]$.

No analytic solution exists for this test case. In order to calculate errors and compare methods, it is customary to take a high resolution simulation as the ‘true’ solution. For this we use the STSWM model run at a T213 resolution with a 360 second timestep.

3.4.4 Case 6: Rossby-Haurwitz wave

Test case 6 consists of a Rossby-Haurwitz wave of zonal wavenumber 4, which is a solution to the nondivergent barotropic vorticity equation. This was first used by Phillips (1959) as a test case for the shallow water system and has been commonly used since, even though it is not an exact solution. Care must be taken when using
this case, as Thuburn and Li (2000) showed that the wave is actually unstable and can break down during long integrations. Lynch (2009) accounted for this by showing that the wave is a component of a quasi-resonant triad.

The initial wind is nondivergent and is given by the stream function

$$\psi = -a^2 \omega \sin \phi + a^2 K \cos^4 \phi \sin \phi \cos 4 \lambda$$

The parameters chosen are $\omega = K = 7.848 \times 10^{-6}\text{s}^{-1}$. The initial height is found from the balance equation; the derivation is given by Williamson et al. (1992).

3.4.5 Error Measures and Conservation

Together with the test cases, Williamson et al. (1992) suggest a number of normalised error measurements to allow for ease of comparison. Defining the areal average

$$I(h) = \frac{1}{4\pi} \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(\lambda, \phi) \cos \phi \, d\phi \, d\lambda$$

the three error norms are

$$l_1(h) = \frac{I(|h(\lambda, \phi) - h_T(\lambda, \phi)|)}{I(|h_T(\lambda, \phi)|)}$$

$$l_2(h) = \frac{\{I[(h(\lambda, \phi) - h_T(\lambda, \phi))^2]\}^{\frac{1}{2}}}{\{I(h_T(\lambda, \phi)^2)\}^{\frac{1}{2}}}$$

$$l_\infty(h) = \frac{\max |h(\lambda, \phi) - h_T(\lambda, \phi)|}{\max |h_T(\lambda, \phi)|}$$

(3.9)

where $h_T(\lambda, \phi)$ is the true solution. These are normalised dimensionless measures; for example $l_\infty(h)$ represents the maximum height difference between the numerical and true solution, as a fraction of the maximum height of the true solution.

To examine conservation in the system, a number of normalised invariants are also defined, using the quantity

$$\frac{I[\xi(\lambda, \phi, t)] - I[\xi(\lambda, \phi, 0)]}{I[\xi(\lambda, \phi, 0)]}$$

For mass we take

$$\xi = h^*$$
and for total energy we set

$$\xi = \frac{1}{2} h^* \mathbf{v} \cdot \mathbf{v} + \frac{1}{2} g (h^2 - h_s^2)$$

where $h^*$ is the depth of the fluid layer.

### 3.5 Shallow Water Simulations

The LT version of STSWM was tested using the cases outlined above. The reference model was the original semi-implicit version. Unless otherwise stated, tests in this section were carried out at a spectral T42 resolution with a 1200 second timestep. In this section, the cutoff period for the LT method is $\tau_c = 6$ hours. For the inversion we test both $N = 8$ and $N = 16$.

For cases 5 and 6, fourth order diffusion is used in the semi-implicit runs, with the diffusion coefficients recommended by Jakob et al. (1993):

<table>
<thead>
<tr>
<th>Truncation</th>
<th>Diffusion coefficient (m$^4$ s$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>42</td>
<td>$5.0 \times 10^{15}$</td>
</tr>
<tr>
<td>213</td>
<td>$8.0 \times 10^{12}$</td>
</tr>
</tbody>
</table>

**Case 1**

For the advection test of case 1, we have constant nondivergent winds and so from (3.6) we are only solving

$$\frac{\partial}{\partial t} \Phi^m_{\ell} = F^m_{\ell}$$

For this case only, due to its simplicity, an explicit leapfrog method is the reference. The LT discretisation, after dividing by $s$, becomes simply

$$\widehat{\Phi}^m_{\ell} = \frac{1}{s} \{\Phi^m_{\ell}\}^{\tau-1} + \frac{1}{s^2} \{F^m_{\ell}\}^{\tau}$$

On the right-hand side we have the transform of a constant and a linear function. We saw in Chapter 2 that these inversions are exact. Thus inverting over a time interval of length $2 \Delta t$ we would expect

$$\{\Phi^m_{\ell}\}^{\tau+1} = \{\Phi^m_{\ell}\}^{\tau-1} + 2 \Delta t \{F^m_{\ell}\}^{\tau}$$

which is exactly the explicit reference discretisation. Therefore in this case we should have matching behaviour from the two schemes. This is confirmed in the error plots.
in Figure 3.1. The upper panels give \( l_2 \) (left) and \( l_\infty \) error norms for the case of \( \alpha = 0 \), that is, advection along the equator. The lower panels show the results when we rotate by \( \alpha = \pi/2 \) so that the cosine bell travels over the poles. All schemes are equally accurate. The \( l_2 \) and \( l_\infty \) errors are less than \( 5 \times 10^{-2} \) and \( 5 \times 10^{-3} \) respectively.

![Figure 3.1](image)

**Figure 3.1:** Case 1 with \( \alpha = 0 \) (top) and \( \alpha = \pi/2 \) (bottom): left and right are the \( l_2 \) and \( l_\infty \) errors, respectively, for T42 and \( \Delta t = 1200s \). Errors are in dimensionless units.

**Case 2**

In their discussion of the test case suite, Jakob et al. (1993) note that case 2 is a trivial problem for the spectral method, as the wind and height fields are exactly represented by spherical harmonics. The top panels of Figure 3.2 show \( l_2 \) (left) and \( l_\infty \) (right) errors of just \( O(10^{-14}) \) and \( O(10^{-13}) \), respectively, for both the reference and the LT method with \( N = 16 \).

In the lower panels we add the \( N = 8 \) run. The errors are much larger, at \( O(10^{-4}) \).
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However this seems to be the only situation where we will see a notable difference in accuracy when we vary $N$.

Case 5

As already detailed, there is no analytic solution for the mountain test case. Instead we compute errors by taking the ‘true’ solution from a T213 $\Delta t = 360s$ reference run. The errors for the T42 simulations, plotted in the top panels of Figure 3.3, are of comparable magnitude, for the reference and both LT forecasts.

Figure 3.2: Case 2 with $\alpha = 0$ at T42 and $\Delta t = 1200s$: left and right are the $l_2$ and $l_\infty$ errors, respectively. Note the differing scales in the upper and lower panels.
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Figure 3.3: Case 5 at T42 and $\Delta t = 1200s$. Top row: $l_2$ (left) and $l_\infty$ (right) errors. Bottom row: normalised mass (left) and total energy (right)

On the bottom of Figure 3.3 we plot the normalised mass and total energy. The three forecasts show an almost identical decrease in mass, though at a negligible magnitude of $O(10^{-15})$ after 15 days. While the reference semi-implicit scheme shows a decrease in energy, the two LT runs show an increase of roughly twice this by the end of the forecast. However, the deviation from energy conservation is negligible.

Case 6

For the Rossby-Haurwitz wave of case 6, Jakob et al. (1993) recommend using shorter timesteps than for the other cases, due to the strong winds involved. The high-resolution ‘true’ solution is given by a T213 run with $\Delta t = 180$ seconds. The T42 simulations are run with a timestep of 600 seconds. Errors, along with mass and total energy, are plotted in Figure 3.4. In this case we also run two more LT forecasts,
again using \( N = 8 \) and \( N = 16 \) but with a shorter cutoff period of 3 hours. We see the errors in these (top panels) are much closer to the reference than those with the 6 hour cutoff.

The lower graphs of Figure 3.4 show that all runs again are almost identical in terms of mass lost. The LT \( \tau_c = 3 \) runs are better for energy conservation.

Figure 3.4: Case 6 at T42 and \( \Delta t = 600s \). Top row: \( l_2 \) (left) and \( l_\infty \) (right) errors. Bottom row: normalised mass (left) and total energy (right)

In Figure 3.5 we plot the height of the fluid layer at the point (255.9°E, 40.5°N). The semi-implicit forecast (black line) is given together with the LT \( N = 8 \) (blue) and \( N = 16 \) (red). The upper panel shows the case with \( \tau_c = 6 \) hours while the lower plot is for a 3 hour cutoff. The 3 hour choice is clearly superior, with the \( \tau_c = 6 \) case showing a significant phase difference with the reference.
Figure 3.5: Case 6 at T42 and $\Delta t = 600s$. Height at 255.9$^{\circ}$E, 40.5$^{\circ}$N. The cutoff period for the LT scheme is $\tau_c = 6$ hours (top) and $\tau_c = 3$ hours (bottom).

Summary

In the series of tests, the accuracy and conservation of the LT method compared favourably with that of the reference semi-implicit scheme. We note also that, in
general, the LT accuracy was not significantly improved by increasing $N$, the order of the inversion contour, from 8 to 16.

### 3.6 Phase Error Analysis

The LT method has been shown to perform competitively when compared with the semi-implicit model. As discussed at the beginning of this chapter, semi-implicit methods are popular due to their attractive stability properties. This is achieved at the expense of a slowing of the faster waves present in the system. This is not a serious issue if one is only interested in slower modes, such as meteorologically-significant Rossby waves. There may, however, be cases where we wish to accurately simulate some of the faster waves. In these situations the semi-implicit approach may not be ideal. We will now investigate the effect of semi-implicit averaging on phase speed in the simplest context. After that we compare the effect of the LT discretisation.

We begin with the simple one-dimensional oscillation equation

$$\frac{du}{dt} = i\nu u$$

(3.10)

We follow the approach of Durran (1999) when analysing the two methods. We seek a numerical amplification factor, $A$, such that $u^{\tau+1} = Au^{\tau}$. Writing $A = |A|e^{i\theta}$, a sufficient criterion for stability is given by $|A| \leq 1$. The phase is given by $\theta = \tan^{-1}\left(\frac{\text{Im}(A)}{\text{Re}(A)}\right)$. For the oscillation equation above Durran defines a relative phase change

$$R = \frac{\theta}{\nu \Delta t}$$

(3.11)

A numerical scheme is decelerating if $R < 1$.

**Semi-Implicit**

In order to analyse the semi-implicit method, we evaluate the derivative with a forward finite difference and average the linear term to get

$$\frac{u^{\tau+1} - u^{\tau}}{\Delta t} = i\nu \frac{u^{\tau+1} + u^{\tau}}{2}$$
Rearranging we get $u^{\tau+1} = A_{SI} u^\tau$ with

$$A_{SI} = \frac{1 + \frac{1}{2} i \nu \Delta t}{1 - \frac{1}{2} i \nu \Delta t}$$

We can compute the phase and then the relative phase change using (3.11):

$$R_{SI} = \frac{1}{\nu \Delta t} \tan^{-1} \left( \frac{\nu \Delta t}{1 - \frac{\nu^2 \Delta t^2}{4}} \right)$$  \hspace{1cm} (3.12)

For small values of $\nu \Delta t$, a Taylor series expansion yields

$$R_{SI} \approx 1 - \frac{(\nu \Delta t)^2}{12}$$  \hspace{1cm} (3.13)

and we clearly see that the semi-implicit scheme decelerates waves.

In the next section we will consider a Kelvin wave with zonal wavenumber 5 with a period of about 6.7 hours. For this, a timestep of a 30 minutes in the formula above would give us $R_{SI} \approx 0.98$ while a one hour timestep would yield $R_{SI} \approx 0.92$.

**Laplace Transform**

We next apply the LT method to (3.10), to get

$$s \hat{u} - u^\tau = i \nu \hat{u}$$

Rearranging we get

$$\hat{u} = \frac{u^\tau}{s - i \nu}$$

Inverting analytically with the full $\mathcal{L}^{-1}$ over a $\Delta t$ interval would yield $u^{\tau+1} = u^\tau e^{i \nu \Delta t}$.

Thus we would have an exact representation of the frequency. With the numerical inversion operator $\mathcal{L}_N$, we saw in Chapter 2 that we get

$$u^{\tau+1} = u^\tau H_N(\nu) e^{i \nu \Delta t}$$

So $A_{LT} = H_N(\nu) e^{i \nu \Delta t}$. When we take $N$ to be a multiple of 4, $H_N$ is real and we have

$$R_{LT} = \frac{1}{\nu \Delta t} \tan^{-1} \left( \frac{\sin N(\nu \Delta t)}{\cos N(\nu \Delta t)} \right)$$  \hspace{1cm} (3.14)
where we have used the truncated trigonometric functions defined in (2.16) and (2.17). As in the semi-implicit case, we consider small values of $\nu \Delta t$ and expand (3.14). The details are given in Appendix A, where it is shown that

$$R_{LT} \approx 1 + \frac{N}{(N+1)!} (\nu \Delta t)^N$$  \hspace{1cm} (3.15)

The LT method gives a nearly exact representation of phase speed, with error only due to the discretisation of the inversion operator. This is clearly far more accurate than that for the semi-implicit case in (3.13). If, for example, we use $N = 8$ we get

$$R_{LT} \approx 1 + \left(\frac{\nu \Delta t}{45360}\right)^8$$

The scheme is marginally accelerating, but by a negligible amount. Again we consider the Kelvin wave with period of 6.7 hours; for a 1800 second timestep the formula gives $R_{LT} \approx 1.00000005$.

### 3.7 Hough Modes

We have shown in the previous section that the LT method represents the phase speed of a wave more accurately than the semi-implicit method. This analysis was carried out for the simple oscillation equation. We now look to investigate this numerically in the full shallow water system. Previously we studied simulations of a Rossby-Haurwitz wave. However, as noted, this is a solution of the nondivergent barotropic vorticity equation and not of the shallow water system. Thus it is not ideal as a test case for a careful study of phase speed.

Instead we use the eigenfunctions of the linearised shallow water equations: the Hough modes. Kasahara (1976) provides a description of the Hough modes along with details and code of a numerical method to produce them. This was used to create initial conditions for STSWM. Since these are eigenfunctions for the linearised equations, by taking small enough wave amplitudes we would expect them to propagate almost linearly.

#### 3.7.1 Kelvin Wave Phase Errors

We examine in particular the Kelvin wave. This is an eastward propagating wave characterised by an almost vanishing meridional wind. It is known to play an
important role in a number of atmospheric phenomena. Holton (1975) discusses its role in the dynamics of the Quasi-Biennial Oscillation (QBO) in the stratosphere. A comprehensive review may be found in Baldwin et al. (2001). The Kelvin wave has also been shown to be a crucial factor in the Madden-Julian Oscillation (Zhang, 2005). It is clearly vital, therefore, that the wave can be accurately simulated.

For varying zonal wavenumbers $m$ we can compute the frequency of the corresponding Kelvin wave using the method of Kasahara. We use this to plot the relative phase changes for the semi-implicit and the LT method, given by (3.12) and (3.14) respectively. Figure 3.6 shows the errors plotted against timestep, for two different wavenumbers: $m = 1$ and $m = 5$. For the two cases the LT method (heavy black solid and dashed lines) are indistinguishable and appear to be almost exact. The deceleration is evident in the semi-implicit method (thin solid and dashed lines). As seen from (3.12), the slowing effect increases with larger timesteps and for the higher frequency of the $m = 5$ wave.

![Kelvin Wave Relative Phase Error](image)

Figure 3.6: Relative phase errors for the semi-implicit (SI) and LT methods, for Kelvin waves of zonal wavenumbers $m = 1$ and $m = 5$
3.7.2 Simulations with STSWM

We have shown that we would expect the Kelvin wave to be slowed by the semi-implicit method, whereas the LT method should simulate it with a more accurate speed. We now use STSWM to test this. As mentioned already, we use the method of Kasahara (1976) to create an initial Kelvin wave. The initial height field for zonal wavenumber $m = 5$ is shown in Figure 3.7. The wave is symmetric about the equator and decays with increasing latitude. A mean height of 10km was used with a wave perturbation amplitude of 100m.

![Initial Height, Kelvin wave $m = 5$](image)

**Figure 3.7:** Initial height field (in metres) of the $m = 5$ Kelvin wave

For this value of the mean height, this Kelvin wave has a period of approximately 6.7 hours. We run each forecast for 67 hours and would thus expect the final field to be roughly in phase with the original. This allows us to easily detect a phase error. All runs were carried out at a T63 spectral resolution.

We first consider the semi-implicit solutions. On the top panel of Figure 3.8 we see again the initial height, but for clarity we only plot from $-45^\circ$ to $45^\circ$ latitude and $0^\circ$
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to 90° longitude. The middle and lower graphs below this show 67 hour semi-implicit forecasts with timesteps of 900 and 1800 seconds respectively. As expected from the earlier analysis, we see a slowing down of the wave, with the longer timestep showing a greater deceleration.

In Figure 3.9 we present the 67 hour forecasts with the LT method. Again we have the initial height (top) and two forecasts, with Δt = 900s (middle) and Δt = 1800s (bottom). Here we have used N = 8 and a cutoff period of τc = 3 hours. With the wave period of 6.7 hours, the previously used value of τc = 6 hours would lead to excessive damping of the wave. We can see that the phase speed is simulated accurately at both timesteps; the 67 hour height forecasts are almost perfectly in phase with the initial fields. With N = 16 we plot similar forecasts in Figure 3.10. Again the phase representation is very accurate. As expected, the damping of the wave is reduced by taking the higher value of N.

Figure 3.11 shows the hourly height at a single point close to the equator, (0.0°E, 0.9°N), over the first 10 hours of the forecasts at Δt = 1800s. Here the phase speed differences are easily seen. The solid line marked ‘Exact’ is a sinusoidal wave with a 6.7 hour period, representing the analytic solution. Both LT forecasts, N = 8 (dashed line) and N = 16 (dot-dashed), have nearly identical speeds closely matching the analytic solution. The semi-implicit (solid with circles) is visibly slowed.
Figure 3.8: Semi-implicit forecasts for the Kelvin wave $m = 5$, with a $90^\circ \times 90^\circ$ plotting region: Initial height (top); 67 hour forecast at $\Delta t = 900s$ (middle); 67 hour forecast at $\Delta t = 1800s$ (bottom)
Figure 3.9: LT forecasts for the Kelvin wave \( m = 5 \), with \( N = 8 \) and \( \tau_c = 3 \) hours: Initial height (top); 67 hour forecast at \( \Delta t = 900s \) (middle); 67 hour forecast at \( \Delta t = 1800s \) (bottom)
Figure 3.10: LT forecasts for the Kelvin wave $m = 5$, with $N = 16$ and $\tau_c = 3$ hours: Initial height (top); 67 hour forecast at $\Delta t = 900s$ (middle); 67 hour forecast at $\Delta t = 1800s$ (bottom)
Figure 3.11: Hourly height at 0.0°E, 0.9°N over 10 hours with $\tau_c = 3$ hours
Chapter 4

SWEmodel and Lagrangian LT

4.1 Introduction

In the previous chapter we formulated a LT discretisation of the shallow water equations based on the Eulerian STSWM model. As mentioned in the introduction in Chapter 1, the use of semi-Lagrangian schemes by Robert (1981) led to huge improvements in efficiency over Eulerian methods. These advantages have been exploited by many forecasting centres. Temperton et al. (2001) discuss the advances since the ECMWF’s Eulerian model was replaced by a semi-Lagrangian version in 1991. They report an efficiency gain of the order of a factor of 50 due to a range of developments of the numerical formulation. It is therefore natural to next consider the LT method combined with a semi-Lagrangian treatment of advection.

In an Eulerian model we consider a fixed grid and the changes in model fields at these gridpoints. A Lagrangian framework, on the other hand, follows the trajectories of fluid parcels. Semi-Lagrangian methods take a trajectory approach while retaining a fixed grid by considering the set of parcels arriving at these gridpoints at each timestep. For a given fluid parcel arriving at a particular gridpoint, we may compute a back trajectory to find its departure point at a previous time level. In general this will lie between gridpoints and we will need to interpolate to find field values at this point. In this way, as we consider values at the beginning and end of the trajectory, the numerical domain of dependence is designed to always include the physical domain of dependence, thus ensuring stability. A detailed review of semi-Lagrangian methods is given by Staniforth and Côté (1991).
Trajectory Laplace Transform

The LT STSWM model used a discretisation of the Eulerian form of the shallow water equations. A typical evolution equation has the form

\[ \frac{\partial X}{\partial t} + L = N \]

where \( L \) and \( N \) represent the linear and nonlinear terms respectively. With the Eulerian derivative we are spatially fixed and so only considering changes in time. So when we took the Laplace transform of these equations we used the derivative property (2.3) to get

\[ s \hat{X} - X_{G}^{n-1} + \hat{L} = \frac{1}{s} N^{n}_{G} \]

The subscript \( G \) represents a particular fixed spatial gridpoint.

Lynch (1991) considered a stable semi-Lagrangian treatment of advection when using a LT discretisation although this still explicitly treated the Eulerian derivative term in the same manner as outlined above.

Here we consider a full trajectory semi-Lagrangian Laplace transform approach and consider an evolution equation written in Lagrangian form

\[ \frac{dX}{dt} + L = N \]  \hspace{1cm} (4.1)

The total derivative \( \frac{dX}{dt} \) represents the change along the trajectory of the fluid parcel. We can take the Laplace transform along the time-dependent trajectory of a parcel arriving at a gridpoint \( A \) at time level \((n + 1)\). Formally we are integrating along the trajectory contour \( T \) so that

\[ \hat{X} \equiv \int_{T} e^{-s \tau} X \, d\tau \]

With a three time level approach, this trajectory starts at time \((n - 1)\) at some departure point \( D \), not necessarily coinciding with a gridpoint. This is the ‘initial value’ when taking the transform of a Lagrangian derivative. The transform of the prognostic equation (4.1) can then be written

\[ s \hat{X} - X_{D}^{n-1} + \hat{L} = \frac{1}{s} N^{n}_{M} \]  \hspace{1cm} (4.2)

Here the nonlinear terms have been evaluated at the midpoint \( M \) of the trajectory at time level \( n \).
Chapter Outline

In this chapter a semi-Lagrangian Laplace transform (SLLT) shallow water model will be developed. This is adapted from an existing model called SWEmodel, which will be detailed in the next section. This Eulerian model will be used extensively as a reference.

The trajectory approach discussed above will be used to formulate the SLLT discretisation. This will be compared with a semi-Lagrangian semi-implicit (SLSI) model. The calculation of the departure points will first be examined in a simple advection case before moving on to the full shallow water system. Although the trajectory LT method was introduced in a three time level scheme in (4.2), we will use a two time level discretisation for the shallow water models. As pointed out in Temperton et al. (2001), two time level semi-Lagrangian schemes offer a doubling of efficiency over three time level versions.

The SLSI and SLLT models will be compared with the test cases previously used in Chapter 3. A number of possibilities exist for the SLLT discretisation and these will all be discussed. The stability properties will also be examined.

In Chapter 5 these semi-Lagrangian models will be employed to run forecasts using real atmospheric data as initial conditions. To control noise an initialisation scheme based on the LT method will be incorporated into the models here and its effect will be examined.

4.2 Model

A spectral transform shallow water model called SWEmodel, written in Matlab, was provided by John Drake at Oak Ridge National Laboratory. The spherical harmonic transform routines are described in Drake et al. (2008). The original code solved an Eulerian form of the shallow water equations using a centred time scheme, similar to that of the STSWM model from Chapter 3. In addition, the non-divergent barotropic vorticity equation (BVE) was solved using a three time level semi-Lagrangian method as outlined in Drake and Guo (2001), with all the routines necessary for trajectory calculations included.

A number of changes needed to be implemented and these will be detailed in the
4.2.1 Departure Point Calculation and Interpolations

A number of different methods have been used to calculate the departure points for semi-Lagrangian schemes. Care is needed in particular in spherical geometry near the poles. Ritchie (1987, 1988) considers semi-Lagrangian advection on a Gaussian grid for spectral models and switches to a Cartesian coordinate system for the trajectory calculations. This was the method used in the original SWEmodel code for solving the BVE.

In Ritchie and Beaudoin (1994) a modified method using spherical coordinates applied to a three time level scheme is proposed. For a two time level version with \((\lambda_A, \phi_A)\) denoting an arrival gridpoint at time level \(n + 1\), we write the departure point at the previous time \(n\) as \((\lambda_D, \phi_D)\) and the corresponding trajectory midpoint at time \(n + \frac{1}{2}\) as \((\lambda_M, \phi_M)\). The method of Ritchie and Beaudoin applied to this two time level scheme is then

\[
\begin{align*}
\lambda_M &= \lambda_A - \frac{u_M}{a \cos \phi_A} \frac{\Delta t}{2} \left\{ 1 + \left( \frac{\Delta t}{2} \right)^2 \frac{1}{6a^2} \left( \frac{u_M^2}{\cos^2 \phi_A} - u_M^2 - v_M^2 \right) \right\} \\
\phi_M &= \phi_A - \frac{v_M \Delta t}{a} + \frac{\tan \phi_A}{2} \left( \frac{u_M \Delta t}{a} \right)^2 \\
\lambda_D &= \lambda_A - \frac{u_M \Delta t}{a \cos \phi_A} \left\{ 1 - \tan \phi_A \left( \frac{v_M \Delta t}{a} \right) \right\} \\
\phi_D &= \phi_A - \frac{v_M \Delta t}{a} + \left( \sec^2 \phi_A - \frac{2}{3} \right) \left( \frac{u_M \Delta t}{a} \right)^2 \left( \frac{v_M \Delta t}{a} \right) \tag{4.3}
\end{align*}
\]

Since the formulae for the midpoint \((\lambda_M, \phi_M)\) contain the winds at this point, an iterative solution is necessary:

\[
\begin{align*}
\lambda_M^{(k+1)} &= \lambda_A - \frac{u_M^{(k)}}{a \cos \phi_A} \frac{\Delta t}{2} \left\{ 1 + \left( \frac{\Delta t}{2} \right)^2 \frac{1}{6a^2} \left( \frac{(u_M^{(k)})^2}{\cos^2 \phi_A} - (u_M^{(k)})^2 - (v_M^{(k)})^2 \right) \right\} \\
\phi_M^{(k+1)} &= \phi_A - \frac{v_M^{(k)}}{a} \frac{\Delta t}{2} + \frac{\tan \phi_A}{2} \left( \frac{u_M^{(k)} \Delta t}{a} \right)^2 \\
\end{align*}
\]

An initial guess of the midpoint wind \(v_M^{n+\frac{1}{2}}\) is obtained with the simple two-term extrapolation

\[
\left( v_M^{n+\frac{1}{2}} \right)^{(0)} = \frac{3}{2} v_A^n - \frac{1}{2} v_A^{n-1}
\]
Other possibilities are discussed in Temperton and Staniforth (1987). Three iterations were used for this and at each step new values for $v_M^{n+\frac{1}{2}}$ were computed with bilinear interpolation. The departure points $(\lambda_D, \phi_D)$ can be evaluated after the iterations are complete.

When interpolating model fields to departure or midpoints, bicubic interpolation was used. A discussion of various interpolation options is given in the review paper of Staniforth and Côté (1991). Cubic interpolation is generally considered to give a good balance of accuracy and efficiency.

Although SWEmodel uses $\mu = \sin \phi$ for the spectral transform method, the latitudinal interpolations must be carried out using $\phi$. When interpolating near the poles, neighbouring gridpoints are found by extending the meridians along a great circle.

### 4.2.2 Advection Tests

The first case in the shallow water test suite of Williamson et al. (1992) was described in Section 3.4. It consists of passive advection by constant winds and provides a good initial test for the new trajectory and interpolation routines. With a T63 Gaussian grid we have 192 longitude points. A timestep of 1.5 hours will take 192 steps to produce a 12 day forecast. So at this resolution with $\alpha = 0$ we have a Courant number of 1 and the bell moves a distance of exactly one gridpoint. There should therefore be no interpolation errors. This provides a useful check on the trajectory calculations.

In Figure 4.1 below we see the errors at the end of a 12 day forecast, that is, the final height field minus the initial. The maximum difference is about 0.06 metres. The maximum of the height field was 1000 metres and these tiny values suggest the trajectory calculations are working sufficiently accurately.
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Next we compare the errors for both the Eulerian and the semi-Lagrangian forecasts with a 900 second timestep. Each run shows a comparable accuracy, with maximum errors of the order of 10 metres (Figure 4.2)

To test the accuracy at the poles, we rotate the grid by changing $\alpha$. In their discussion of this test, Williamson et al. (1992) suggest trying a slightly shifted value of $\alpha = \frac{\pi}{2} - 0.05$ to avoid any symmetries. With this value, the bell passes close to each pole during the forecast. The errors for each method are plotted in Figure 4.3. Again, the accuracy is comparable and gives confidence in the semi-Lagrangian trajectory calculations and interpolations.
Ritchie and Beaudoin (1994) use the trajectory approximations given in (4.3) away from the poles and test them for values $|\phi_A| < 80.2^\circ$. Here they have been used throughout for all trajectory calculations, with no problems encountered.

### 4.3 Shallow Water Equations

With confidence in the trajectory calculations and their ability to handle passive advection, we now move on to the full shallow water system. In Drake and Guo (2001), the shallow water equations are written in Lagrangian form as follows:

\[
\begin{align*}
\frac{d\eta}{dt} &= -\eta \delta \\
\frac{d\delta}{dt} + \nabla^2 \Phi &= \eta(\eta - f) - k \cdot v \times \nabla f - \nabla^2 (\Phi_s + \frac{v \cdot v}{2}) \\
&\quad+ \frac{u}{\cos \phi} \left( \nabla^2 (u \cos \phi) - \frac{2 \sin \phi}{a} \zeta \right) \\
&\quad+ \frac{v}{\cos \phi} \left( \nabla^2 (v \cos \phi) + \frac{2 \sin \phi}{a} \delta \right) \\
\frac{d\Phi}{dt} + \Phi^* \delta &= -\Phi \delta 
\end{align*}
\]

Here $\eta = \zeta + f$ is the absolute vorticity, $\zeta$ is the relative vorticity, $\delta$ is the horizontal divergence and $\Phi_s$ is the surface geopotential. The total geopotential height of the fluid’s surface is given by $\Phi_T = \Phi + \Phi^* + \Phi_s$, so that $\Phi^*$ represents a mean geopotential depth and $\Phi$ is the departure from the mean.

Attempting to use a two time level semi-Lagrangian discretisation on this system leads to instability. To overcome this, all linear Coriolis terms must be averaged in
time, not just the terms associated with gravity waves (Temperton and Staniforth, 1987; Côté and Staniforth, 1988). The relative vorticity \( \zeta \) is then used as a prognostic variable instead of \( \eta \). The shallow water system to be solved becomes

\[
\begin{align*}
\frac{d\zeta}{dt} + f\delta + \beta v &= N_\zeta \\
\frac{d\delta}{dt} - f\zeta + \beta u + \nabla^2 \Phi &= N_\delta \\
\frac{d\Phi}{dt} - \frac{d\Phi_s}{dt} + \bar{\Phi}\delta &= N_\Phi
\end{align*}
\]

with

\[
\begin{align*}
N_\zeta &\equiv -\zeta\delta \\
N_\delta &\equiv \zeta^2 - \nabla^2 \left( \frac{v \cdot v}{2} \right) \\
&+ \frac{u}{\cos \phi} \left( \nabla^2 (u \cos \phi) - \frac{2 \sin \phi}{a} \zeta \right) + \frac{v}{\cos \phi} \left( \nabla^2 (v \cos \phi) + \frac{2 \sin \phi}{a} \delta \right) \\
N_\Phi &\equiv -(\Phi - \Phi_s)\delta
\end{align*}
\]

and \( \beta = \frac{1}{a} \frac{\partial f}{\partial \phi} \). In contrast to Drake and Guo, who consider the fluid’s depth, we have written the geopotential in terms of a constant mean height and a deviation from this mean: \( \Phi_T = \bar{\Phi} + \Phi \). With this formulation the orography appears explicitly in the continuity equation and not in the equation for divergence.

We now formulate the two Lagrangian models to be used. The first is the reference semi-Lagrangian semi-implicit (SLSI) model. The second uses the semi-Lagrangian Laplace transform (SLLT) discretisation, based on the trajectory idea of (4.2)

### 4.4 Semi-Lagrangian Semi-Implicit: SLSI

For the discretisation, \( \{ \}_A^{n+1} \) refers to an arrival value at a regular gridpoint at time \( (n + 1)\Delta t \) with the corresponding departure value \( \{ \}_D^n \) at time \( n\Delta t \). The right-hand side terms are evaluated at the midpoint of the trajectory \( \{ \}_M^{n+\frac{1}{2}} \). The
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semi-Lagrangian semi-implicit discretisation of (4.5) is given by

\[
\frac{\zeta^{n+1}_A - \zeta^n_D}{\Delta t} + \frac{\{f\delta\}^{n+1}_A + \{f\delta\}^n_D}{2} + \frac{\{\beta v\}^{n+1}_A + \{\beta v\}^n_D}{2} = \{N\zeta\}^{n+\frac{1}{2}}_M
\]

\[
\frac{\delta^{n+1}_A - \delta^n_D}{\Delta t} - \frac{\{f\zeta\}^{n+1}_A + \{f\zeta\}^n_D}{2} + \frac{\{\beta u\}^{n+1}_A + \{\beta u\}^n_D}{2} + \frac{\nabla^2\Phi^{n+1}_A + \nabla^2\Phi^n_D}{2} = \{N\delta\}^{n+\frac{1}{2}}_M
\]

\[
\frac{\Phi^{n+1}_A - \Phi^n_D}{\Delta t} - \frac{(\Phi_s)^{n+1}_A - (\Phi_s)^n_D}{\Delta t} + \frac{\Phi^{n+1}_A + \Phi^n_D}{2} = \{N\Phi\}^{n+\frac{1}{2}}_M
\]

As mentioned previously, interpolations of fields are carried out using bicubic interpolation. The nonlinear terms on the right are extrapolated in time using

\[
N_A^{n+\frac{1}{2}} = \frac{3}{2} N_A^n - \frac{1}{2} N_A^{n-1}
\]

before being interpolated to the midpoint values \(N_M^{n+\frac{1}{2}}\). Keeping terms at the new time level on the left gives

\[
\zeta^{n+1}_A + \frac{\Delta t}{2} f\delta^{n+1}_A + \frac{\Delta t}{2} \beta v^{n+1}_A = R_\zeta
\]

\[
\delta^{n+1}_A - \frac{\Delta t}{2} f\zeta^{n+1}_A + \frac{\Delta t}{2} \beta u^{n+1}_A + \frac{\Delta t}{2} (\nabla^2\Phi)^{n+1}_A = R_\delta
\]

\[
\Phi^{n+1}_A + \frac{\Delta t}{2} \Phi^{n+1}_A = R_\Phi
\] (4.6)

with

\[
R_\zeta = \left\{ \zeta - \frac{\Delta t}{2} f\delta - \frac{\Delta t}{2} \beta v \right\}^n_D + \Delta t\{N\zeta\}^{n+\frac{1}{2}}_M
\]

\[
R_\delta = \left\{ \delta - \frac{\Delta t}{2} f\zeta - \frac{\Delta t}{2} \beta u - \frac{\Delta t}{2} (\nabla^2\Phi) \right\}^n_D + \Delta t\{N\delta\}^{n+\frac{1}{2}}_M
\]

\[
R_\Phi = \left\{ \Phi - \Phi_s - \frac{\Delta t}{2} \Phi\delta \right\}^n_D + (\Phi_s)_A + \{N\Phi\}^{n+\frac{1}{2}}_M
\]

The horizontal wind terms in (4.6) above can be dealt with by introducing a stream function and velocity potential where

\[
v = k \times \nabla \psi + \nabla \chi
\]
We can write

\[
\zeta = \nabla^2 \psi \\
\delta = \nabla^2 \chi \\
u = \frac{1}{a \cos \phi} \frac{\partial \chi}{\partial \lambda} - \frac{1}{a} \frac{\partial \psi}{\partial \phi} \\
v = \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} + \frac{1}{a} \frac{\partial \chi}{\partial \phi}
\]

**Spectral Solution**

The system is solved in spectral space by expanding each field in terms of spherical harmonics, for example

\[
\zeta = \sum_{q=0}^{N} \sum_{p=-q}^{q} \zeta_{p}^{Q} Y_{q}^{p}(\lambda, \mu)
\]

where \(Y_{q}^{p}(\lambda, \mu) = P_{q}^{p}(\lambda, \mu) e^{i p \lambda}\) and \(\mu = \sin \phi\). We write the Coriolis parameter as \(f = 2\Omega \mu\) and the latitudinal variation \(\beta = \frac{1}{a} \frac{\partial f}{\partial \phi} = \frac{2\Omega \sqrt{1-\mu^2}}{a}\) (note that this analysis must be modified for rotated grids, since the Coriolis terms will have a longitudinal dependence). The expanded vorticity equation then becomes

\[
\sum_{q=0}^{N} \sum_{p=-q}^{q} \left[ \zeta_{p}^{Q} + \Omega \Delta t \mu \delta_{q}^{B} + \Omega \Delta t \sqrt{1-\mu^2} v_{q}^{P} \right] Y_{q}^{p}(\lambda, \mu) = \sum_{q=0}^{N} \sum_{p=-q}^{q} \left[ R_{\zeta}^{Q} \right]_{q}^{p} Y_{q}^{p}(\lambda, \mu)
\]

Orthogonality of the spherical harmonics is used to isolate individual spectral coefficients, by multiplying by the conjugate \(\overline{Y_{q}^{p}}\) and integrating over the range of \(\lambda\) and \(\mu\). The Coriolis terms require some attention and details are given in the Appendix C.

With \(\alpha = \Omega \Delta t\) and \(\epsilon_{n}^{q} = \sqrt{\frac{\ell - m^2}{4\ell + 1}}\), (4.6) transform to

\[
\left[ 1 - \frac{im\alpha}{\ell(\ell + 1)} \right] \zeta_{\ell}^{m} + \frac{\alpha(\ell + 1)}{\ell} \epsilon_{\ell}^{m} \epsilon_{\ell-1}^{m} + \frac{\alpha\ell}{\ell + 1} \epsilon_{\ell+1}^{m} \epsilon_{\ell+1}^{m} = [R_{\zeta}]_{\ell}^{m}
\]

\[
\left[ 1 - \frac{im\alpha}{\ell(\ell + 1)} \right] \delta_{\ell}^{m} - \frac{\alpha(\ell + 1)}{\ell} \epsilon_{\ell}^{m} \zeta_{\ell-1}^{m} - \frac{\alpha\ell}{\ell + 1} \epsilon_{\ell+1}^{m} \zeta_{\ell+1}^{m} = [R_{\delta}]_{\ell}^{m}
\]

\[
\Phi_{\ell}^{m} + \frac{\Delta t}{2} \Phi_{\ell}^{m} = [R_{\Phi}]_{\ell}^{m}
\]

(4.7)
As detailed in Côté and Staniforth (1988), this system decouples into the following

\[
L_m^m \delta_{\ell-2}^m + M_m^m \delta_{\ell}^m + U_m^m \delta_{\ell+2}^m = [R_\delta]_\ell^m + \frac{\Delta t \ell (\ell + 1)}{a^2} [R_\Phi]_\ell^m \\
+ \frac{\alpha (\ell^2 - 1) \varepsilon_{\ell}^m}{\ell (\ell - 1) - i m \alpha} [R_\zeta]_{\ell-1}^m + \frac{\alpha \ell (\ell + 2) \varepsilon_{\ell+1}^m}{(\ell + 1)(\ell + 2) - i m \alpha} [R_\zeta]_{\ell+1}^m
\]

\[
\Phi_{\ell}^m = [R_\Phi]_{\ell}^m - \frac{\Delta t}{2} \Phi \delta_{\ell}^m
\]

\[
\zeta_{\ell}^m = \frac{\ell (\ell + 1) [R_\zeta]_{\ell}^m - \alpha (\ell + 1)^2 \varepsilon_{\ell}^m \delta_{\ell-1}^m - \alpha \ell^2 \varepsilon_{\ell+1}^m \delta_{\ell+1}^m}{\ell (\ell + 1) - i m \alpha}
\]

where

\[
L_m^m = \frac{\alpha^2 (\ell^2 + \ell) \varepsilon_{\ell-1}^m \varepsilon_{\ell}^m}{\ell^2 - \ell - i m \alpha}
\]

\[
M_m^m = 1 - \frac{i m \alpha}{\ell (\ell + 1)} + \left(\frac{\Delta t}{2}\right)^2 \frac{\Phi \ell (\ell + 1)}{a^2}
\]

\[
+ \frac{(\alpha \varepsilon_{\ell}^m)^2 (\ell - 1)(\ell + 1)}{\ell^2 (\ell - 1) - i m \ell \alpha} + \frac{(\alpha \varepsilon_{\ell+1}^m)^2 (\ell + 2)^2}{(\ell + 1)^2 (\ell + 2) - i m \alpha (\ell + 1)}
\]

\[
U_m^m = \frac{\alpha^2 (\ell^2 + \ell) \varepsilon_{\ell+1}^m \varepsilon_{\ell+2}^m}{(\ell + 1)(\ell + 2) - i m \alpha}
\]

If odd and even values of \( \ell \) are considered separately for every \( m \), the above system yields two tridiagonal matrix systems for the coefficients \( \delta_{\ell}^m \) which can be efficiently solved (Durran, 1999). Note that as \( m \) increases, the size of the matrix system decreases since \( \ell = m, m + 1, \ldots, N \). Once \( \delta_{\ell}^m \) is found we can compute the \( \zeta_{\ell}^m \) and \( \Phi_{\ell}^m \) spectral coefficients and then synthesise the physical fields.

### 4.5 Semi-Lagrangian Laplace Transform: SLLT

We begin again with the system (4.5):

\[
\frac{d\zeta}{dt} + f\delta + \beta v = N_\zeta
\]

\[
\frac{d\delta}{dt} - f\zeta + \beta u + \nabla^2 \Phi = N_\delta
\]

\[
\frac{d\Phi}{dt} - \frac{d\Phi_s}{dt} + \Phi\delta = N_\Phi
\]
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Following (4.2), the Laplace transform of the system over a trajectory starting at a departure point denoted \( D \) at time \( n \Delta t \) is given by

\[
\begin{align*}
    s \hat{\zeta} - \zeta_D^n + f \hat{\delta} + \beta v &= \frac{1}{s} \{N\zeta\}^{n+\frac{1}{2}}_M \\
    s \hat{\delta} - \delta_D^n - f \zeta + \beta u + \nabla^2 \Phi &= \frac{1}{s} \{N\delta\}^{n+\frac{1}{2}}_M \\
    s \hat{\Phi} - \Phi_D^n - \left( s \hat{\Phi}_s - (\Phi_s)_D^n \right) + \Phi \hat{\delta} &= \frac{1}{s} \{N\Phi\}^{n+\frac{1}{2}}_M
\end{align*}
\]

Nonlinear terms on the right-hand side are evaluated at the midpoint of the trajectory.

4.5.1 Coriolis Terms

The Coriolis terms on the left need to be considered. Though constant in time, both \( f \) and \( \beta \) vary along a trajectory. This means that transforms of products such as \( \hat{f} \hat{\delta} \) cannot be easily separated and make the system very difficult to decouple.

One approach to overcome this is to assume that the change in \( f \) along a trajectory is negligible, which allows us to then write

\[
\hat{f} \hat{\delta} = f_A \hat{\delta}
\]

We seek justification of this assumption by examining the magnitude of the change of \( f \):

\[
\begin{align*}
    f_D &= 2\Omega \sin \phi_D \\
    &= 2\Omega \sin \left( \phi - \frac{v}{a} \Delta t \right)_A \\
    &= 2\Omega \left( \sin \phi - \left( \frac{v}{a} \Delta t \right) \cos \phi \right)_A + O \left( \frac{v}{a} \Delta t \right)^2 \\
    &\approx (f - \beta v \Delta t)_A
\end{align*}
\]

Typical scales of \( \beta \sim 10^{-11} \text{ m}^{-1} \text{ s}^{-1} \), \( \Delta t \sim 10^3 \text{ s} \) and \( v \sim 10 \text{ m s}^{-1} \) would give \( \beta_A v \Delta t \sim 10^{-7} \text{ s}^{-1} \) while \( f_A \sim 10^{-4} \text{ s}^{-1} \). Similarly

\[
\begin{align*}
    \beta_D &= \frac{2\Omega}{a} \cos \left( \phi - \frac{v}{a} \Delta t \right)_A \\
    &= \frac{2\Omega}{a} \left( \cos \phi + \left( \frac{v}{a} \Delta t \right) \sin \phi \right)_A + O \left( \frac{v}{a} \Delta t \right)^2 \\
    &\approx \left( \beta + \frac{f}{a^2} \frac{v}{a} \Delta t \right)_A
\end{align*}
\]
Now, with \( a \approx 6 \times 10^6 \sim 10^7 \) m, we have \( f_A v \Delta t/a^2 \sim 10^{-14} \text{m}^{-1} \text{s}^{-1} \). If we accept this assumption, the relevant transformations in the SLLT discretisation are

\[
\begin{align*}
\hat{f}\delta & \rightarrow f_A \hat{\delta} \\
\hat{f}\zeta & \rightarrow f_A \hat{\zeta} \\
\hat{\beta}u & \rightarrow \beta_A \hat{u} \\
\hat{\beta}v & \rightarrow \beta_A \hat{v}
\end{align*}
\]

These approximations could be avoided by discretising the shallow water equations in absolute vorticity form, given by (4.4). This would be advantageous in that there would be no transforms of products such as \( \hat{f}\delta \) to consider. As already mentioned, Temperton and Staniforth (1987) report that this formulation is unstable for a two time level SLSI scheme. An LT model based on this form of the equations was tested, but also proved to be unstable.

### 4.5.2 Orography

The transform of the orography \( \hat{\Phi}_s \) also requires care. Like the Coriolis terms discussed above, this is time-independent but varies along a trajectory. If again we assume the changes to be small and treat it as ‘constant’ at its arrival value, the orographic derivative term transforms to

\[
\begin{align*}
\frac{d\Phi_s}{dt} & = s \Phi_s - (\Phi_s)_D^n \\
& = s \left( \frac{(\Phi_s)_A^{n+1}}{s} \right) - (\Phi_s)_D^n \\
& = (\Phi_s)_A^{n+1} - (\Phi_s)_D^n
\end{align*}
\]

An alternative would be to first discretise the derivative as

\[
\begin{align*}
\frac{d\Phi_s}{dt} & = \frac{(\Phi_s)_A^{n+1} - (\Phi_s)_D^n}{\Delta t}
\end{align*}
\]

The terms on the right are now all constants and so we can easily take the LT to get

\[
\begin{align*}
\frac{d\Phi_s}{dt} & = \frac{(\Phi_s)_A^{n+1} - (\Phi_s)_D^n}{s \Delta t}
\end{align*}
\]

Both of these methods are formally equivalent in the nondivergent situation, in which the geopotential equation reduces to

\[
\frac{d\Phi}{dt} = \frac{d\Phi_s}{dt}
\]
A semi-Lagrangian discretisation would give

\[ \Phi_A^{n+1} = \Phi_D^n + (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \]

Taking the LT of (4.9) and considering both treatments of the orography, we have

\[ s \hat{\Phi} - \Phi_D^n = \begin{cases} (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \\ (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \end{cases} \]

Solving for the transform of the geopotential gives

\[ \hat{\Phi} = \frac{1}{s} \Phi_D^n + \begin{cases} \frac{1}{s} [(\Phi_s)_A^{n+1} - (\Phi_s)_D^n] \\ \frac{1}{s^2} \left[ (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \right] \Delta t \end{cases} \]

The inversions involving either a constant or a linear function are exact (see Chapter 2) and so we get the solution at the new time level

\[ \Phi_A^{n+1} = \Phi_D^n + \begin{cases} (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \\ \left[ (\Phi_s)_A^{n+1} - (\Phi_s)_D^n \right] \Delta t \end{cases} \]

In this simple case, therefore, the two methods are consistent in producing the same solution as the semi-Lagrangian method. Both methods were tested in the full shallow water models. The second was found to give superior results in simulations and so was implemented in the main SLLT code to be used in further work.

The orography could also be discretised in Eulerian form. Since it is time-independent, instead of considering \( \frac{d \Phi_s}{dt} \) we can write

\[ \frac{d \Phi_s}{dt} = \mathbf{v} \cdot \nabla \Phi_s \]

The advection term can then be treated with the other nonlinear terms in the continuity equation. This treatment was also tested in the SLLT model and was found to give almost identical results to the Lagrangian treatment.
4.5.3 Spectral Solution

Taking the approximations outlined above, the system of transformed variables to be solved is now

\[
\begin{align*}
\hat{s} \zeta + f \hat{\delta} + \beta \hat{v} &= R_\zeta \\
\hat{s} \delta - f \hat{\zeta} + \beta \hat{u} + \nabla^2 \hat{\Phi} &= R_\delta \\
\hat{s} \Phi + \hat{\Phi} \hat{\delta} &= R_\Phi
\end{align*}
\]  

(4.10)

with

\[
\begin{align*}
R_\zeta &= \zeta^n_D + \frac{1}{s} \{ N_\zeta \}_M^{n+\frac{1}{2}} \\
R_\delta &= \delta^n_D + \frac{1}{s} \{ N_\delta \}_M^{n+\frac{1}{2}} \\
R_\Phi &= \{ \Phi - \Phi_s \}_D^n + (\Phi_s)_A + \frac{1}{s} \{ N_\Phi \}_M^{n+\frac{1}{2}}
\end{align*}
\]

Each transformed variable is a function of space and the complex variable \(s\) and so they can be expanded in terms of spherical harmonics, for example

\[
\hat{\zeta}(s, \lambda, \mu) = \sum_m \sum_\ell \hat{\zeta}_m^\ell(s) Y_\ell^m(\lambda, \mu)
\]

Note that the spectral coefficients in this case are functions of \(s\). The system (4.10) can thus be solved spectrally, in a similar manner to the SLSI scheme, for a given value of \(s\). Using orthogonality as before, we get the following:

\[
\begin{align*}
&\left[ 1 - \frac{im}{\ell(\ell + 1)} \frac{2\Omega}{s} \right] (s \hat{\zeta}_m^\ell) + \frac{(\ell + 1) 2\Omega}{\ell} \frac{s}{\ell} \epsilon_\ell^m (s \hat{\delta}_{\ell-1}^m) \\
&+ \frac{\ell}{\ell + 1} \frac{2\Omega}{s} \epsilon_{\ell+1}^m (s \hat{\delta}_{\ell+1}^m) = [R_\zeta]^m_\ell
\end{align*}
\]

\[
\begin{align*}
&\left[ 1 - \frac{im}{\ell(\ell + 1)} \frac{2\Omega}{s} \right] (s \hat{\delta}_m^\ell) - \frac{(\ell + 1) 2\Omega}{\ell} \frac{s}{\ell} \epsilon_\ell^m (s \hat{\zeta}_{\ell-1}^m) \\
&- \frac{\ell}{\ell + 1} \frac{2\Omega}{s} \epsilon_{\ell+1}^m (s \hat{\zeta}_{\ell+1}^m) - \frac{1}{s} \frac{\ell(\ell + 1)}{a^2} (s \hat{\Phi}_\ell^m) = [R_\delta]^m_\ell
\end{align*}
\]

\[
\begin{align*}
(s \hat{\Phi}_\ell^m) + \frac{1}{s} \hat{\Phi} (s \hat{\delta}_\ell^m) = [R_\Phi]^m_\ell
\end{align*}
\]  

(4.11)
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The above matches (4.7) with the substitution
\[
\frac{\Delta t}{2} \rightarrow \frac{1}{s}
\]
Again we solve two tridiagonal matrix systems for each \( m \). Once all of the spectral coefficients of a field, e.g. \( \hat{\zeta}_m^\ell(s) \), are known we can synthesise the transformed field \( \hat{\zeta}(s) \). For SLLT this must be done for each value of \( s \) on the inversion contour. We can then invert to the physical field with the usual summation
\[
\zeta = \frac{1}{N} \sum_{n=1}^{N} s_n \hat{\zeta}(s_n) e^{s_n \Delta t}
\]

4.5.4 Symmetries

Generally for spectral models we are dealing with the spectral coefficients of a real function, which have the symmetry property
\[
\zeta_{-m} = (-1)^m \overline{\zeta_m}
\]
Then we need only solve for values of \( m \geq 0 \). In the SLLT model, however, we spectrally analyse the complex Laplace transforms, for example, \( \hat{\zeta}_m^\ell \). The above symmetry no longer holds. Therefore we must solve the tridiagonal systems detailed in the previous section for all values of \( m \), both positive and negative. While this adds extra computational cost of the SLLT discretisation, it is possible to make some significant savings.

In Section 2.5 we discussed how a symmetry in the inversion operator could be exploited to potentially halve the number of points in the inversion. This symmetry holds only when we are considering the transforms of real functions. For this reason we could not use it in the STSWM model from Chapter 3. In SWEmodel, however, we take the transform of the real, physical fields and the symmetry will apply. We may invert using the symmetric operator (2.22). This greatly improves the efficiency of the SLLT scheme. This will be analysed in more detail in Section 4.8.

4.6 Stability

Côté and Staniforth (1988) analyse the two time level SLSI scheme outlined above and show that choosing a mean geopotential greater than the maximum value is a
necessary condition for stability. The implications of this will be discussed later when we look at the Rossby-Haurwitz wave test case in Section (4.7.3).

This stability analysis is detailed in Durran (1999) and we now follow this method to examine the stability of the SLLT scheme. We omit the Coriolis force and orography and linearise about a constant angular flow \( \bar{\omega} \) with corresponding zonal wind \( \bar{u} = a \bar{\omega} \cos \phi \). Previously the total geopotential height was written as the sum of a mean value and the perturbation from the mean, \( \Phi = \bar{\Phi} + \Phi' \). In order to study the effect of the extrapolations in the scheme, the perturbation geopotential is here further split into a constant part \( \Phi_e \) and the difference from this, i.e. \( \Phi' = \Phi_e + \Phi'' \). The resulting linear equation set to be considered is thus

\[
\begin{align*}
\frac{d \zeta'}{dt} &= 0 \\
\frac{d \delta'}{dt} + \nabla^2 \Phi'' &= 0 \\
\frac{d \Phi''}{dt} + \bar{\Phi} \delta' &= -\Phi_e \delta' 
\end{align*}
\]

(4.12)

As before, we take the Laplace transform of each equation along a trajectory to get

\[
\begin{align*}
s \hat{\zeta} - \zeta_D^n &= 0 \\
s \hat{\delta} - \delta_D^n + \nabla^2 \hat{\Phi} &= 0 \\
s \hat{\Phi} - \Phi^n_D + \bar{\Phi} \delta &= \frac{1}{s} \{-\Phi_e \delta\}^{n+1/2}_M 
\end{align*}
\]

Here the primes have been dropped from the prognostic variables for convenience of notation. We look for spherical harmonic solutions of the form

\[
\begin{pmatrix}
\zeta^n \\
\delta^n \\
\Phi^n
\end{pmatrix}
= \begin{pmatrix}
\zeta^0 \\
\delta^0 \\
\Phi^0
\end{pmatrix} \Lambda^n e^{im\lambda} P^m_\ell (\mu)
\]

from which we have

\[
\begin{pmatrix}
\zeta^{n+1} \\
\delta^{n+1} \\
\Phi^{n+1}
\end{pmatrix} = \Lambda \begin{pmatrix}
\zeta^n \\
\delta^n \\
\Phi^n
\end{pmatrix}
\]

(4.13)

We are interested in the modulus of the factor \( \Lambda \), which will determine whether the scheme is stable or grows in time. With the constant angular velocity \( \bar{\omega} \) we write the trajectory of a fluid parcel as

\[ \lambda_D = \lambda_A - \bar{\omega} \Delta t \]
and so the departure value of a field is given by

\[ \zeta^n_D = \zeta^n_A e^{-i\theta} \]

where \( \theta = m \bar{\omega} \Delta t \). We use this idea to expand the right-hand side of the continuity equation. The midpoint values are replaced by a two term extrapolation in time at the arrival point:

\[
\frac{1}{s} \{ -\Phi_e \delta \}^{n+1/2}_M \rightarrow -\frac{\Phi_e}{s} \left( \frac{3}{2} \delta^n_A e^{-i\theta} - \frac{1}{2} \delta^n_{-1} e^{-2i\theta} \right)
\]

\[
= -\frac{\Phi_e}{s} \left( \frac{3}{2} \delta^n_A e^{-i\theta} - \frac{1}{2} \Lambda e^{-2i\theta} \right)
\]

We can now write the transformed system as

\[
\begin{pmatrix}
  s & 0 & 0 \\
  0 & s & -\ell(\ell+1) \\
  0 & 0 & s
\end{pmatrix}
\begin{pmatrix}
  \frac{\hat{\zeta}}{s} \\
  \frac{\hat{\delta}}{s} \\
  \hat{\Phi}
\end{pmatrix}
= e^{-i\theta}
\begin{pmatrix}
  0 & 0 & 0 \\
  0 & -\frac{\Phi_e}{s} \left( \frac{3}{2} e^{-i\theta} - \frac{1}{2} \Lambda e^{-2i\theta} \right) & e^{-i\theta}
\end{pmatrix}
\begin{pmatrix}
  \zeta^n_A \\
  \delta^n_A \\
  \Phi^n_A
\end{pmatrix}
\]

This can be solved for the transformed prognostic fields to give

\[
\begin{pmatrix}
  \frac{\hat{\zeta}}{s} \\
  \frac{\hat{\delta}}{s} \\
  \hat{\Phi}
\end{pmatrix}
= e^{-i\theta}
\begin{pmatrix}
  \frac{s^2 + G_\ell^2}{s} \\
  0 & s - \frac{\ell(\ell+1)}{a^2} \Phi_e \left( \frac{3}{2} - \frac{1}{2} \Lambda e^{i\theta} \right) & 0 \\
  0 & -\frac{\Phi_e}{s} \left( \frac{3}{2} - \frac{1}{2} \Lambda e^{i\theta} \right) & s
\end{pmatrix}
\begin{pmatrix}
  \zeta^n_A \\
  \delta^n_A \\
  \Phi^n_A
\end{pmatrix}
\]

where \( G_\ell^2 = \frac{\ell(\ell+1)}{a^2} \bar{\Phi} \). We invert with the numerical operator \( \mathcal{L}_N^* \) to recover the physical fields at time level \( n + 1 \). Writing \( \hat{\ell} = \frac{\ell(\ell+1)}{a^2} \) and \( \hat{\Phi}_e = \Phi_e \left( \frac{3}{2} - \frac{1}{2} \Lambda e^{i\theta} \right) \) we have

\[
\begin{pmatrix}
  \zeta^{n+1}_A \\
  \delta^{n+1}_A \\
  \Phi^{n+1}_A
\end{pmatrix}
= e^{-i\theta}
\begin{pmatrix}
  \mathcal{L}_N^* \left\{ \frac{1}{s} \right\} \\
  0 \\
  0
\end{pmatrix}
\begin{pmatrix}
  \zeta^n_A \\
  \delta^n_A \\
  \Phi^n_A
\end{pmatrix}
= e^{-i\theta}
\begin{pmatrix}
  \frac{s}{s^2 + G_\ell^2} \\
  \frac{\hat{\ell}}{s} \frac{\Phi_e}{s (s^2 + G_\ell^2)} \\
  \frac{s}{s^2 + G_\ell^2}
\end{pmatrix}
\begin{pmatrix}
  \zeta^n_A \\
  \delta^n_A \\
  \Phi^n_A
\end{pmatrix}
\]

We now use (4.13) to write the left-hand side in terms of the fields at \( \left\{ \frac{1}{s} \right\}_A \) and multiply both sides by \( e^{i\theta} \). Defining \( \hat{\Lambda} \equiv \Lambda e^{i\theta} \), we have \( |\hat{\Lambda}| = |\Lambda| \) and the system can finally be written as

\[
M \begin{pmatrix}
  \zeta^n_A \\
  \delta^n_A \\
  \Phi^n_A
\end{pmatrix} = 0
\]
where the matrix $\mathbf{M}$ is given by

$$
\begin{pmatrix}
\tilde{\Lambda} - \mathbf{L}_N^* \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s^2 + G^2} \end{bmatrix} & 0 \\
0 & \tilde{\Lambda} - \mathbf{L}_N^* \begin{bmatrix} \frac{s \ell \Phi_e}{s^2 + G^2} - \frac{\ell \Phi_e}{s (s^2 + G^2)} \\ \frac{s}{s^2 + G^2} \end{bmatrix} \\
0 & \mathbf{L}_N^* \begin{bmatrix} \Phi_e \Phi_e \frac{1}{s^2 + G^2} \\ \frac{1}{s (s^2 + G^2)} \end{bmatrix} - \mathbf{L}_N^* \begin{bmatrix} \frac{s}{s^2 + G^2} \end{bmatrix}
\end{pmatrix}
$$

Using partial fractions and the properties of the operator $\mathbf{L}_N^*$ given in Chapter 2, we can evaluate the inversions in $\mathbf{M}$ using

$$
\begin{align*}
\mathbf{L}_N^* \begin{bmatrix} \frac{1}{s} \\ \frac{s}{s^2 + G^2} \end{bmatrix} &= 1 \\
\mathbf{L}_N^* \begin{bmatrix} \frac{s}{s^2 + G^2} \end{bmatrix} &= \frac{1}{2} H_N(G_\ell) \left( e^{i G_\ell \Delta t} + e^{-i G_\ell \Delta t} \right) \\
\mathbf{L}_N^* \begin{bmatrix} \frac{1}{s^2 + G^2} \end{bmatrix} &= \frac{1}{2i G_\ell} H_N(G_\ell) \left( e^{i G_\ell \Delta t} - e^{-i G_\ell \Delta t} \right) \\
\mathbf{L}_N^* \begin{bmatrix} \frac{1}{s (s^2 + G^2)} \end{bmatrix} &= \frac{1}{G_\ell^2} - \frac{1}{2 G_\ell^2} H_N(G_\ell) \left( e^{i G_\ell \Delta t} + e^{-i G_\ell \Delta t} \right)
\end{align*}
$$

The determinant of $\mathbf{M}$ must be zero for nontrivial solutions. Due to the $\frac{1}{\Lambda}$ term in $\Phi_e$, this yields a quartic equation in $\tilde{\Lambda}$. One solution is $\tilde{\Lambda} = 1$, the stable, stationary Rossby mode mentioned in Côté and Staniforth (1988). The other solutions satisfy the cubic

$$
\tilde{\Lambda}^3 + \left[ \frac{3\varrho}{2} - \left( 2 + \frac{3\varrho}{2} \right) H_N(G_\ell) \cos N(G_\ell \Delta t) \right] \tilde{\Lambda}^2 \\
+ \left[ H_N(G_\ell)^2 |e^{i G_\ell \Delta t}|^2 \left( 1 + \frac{3\varrho}{2} \right) - \varrho H_N(G_\ell) \cos N(G_\ell \Delta t) - \frac{\varrho}{2} \right] \tilde{\Lambda} \\
+ \frac{\varrho}{2} H_N(G_\ell) \cos N(G_\ell \Delta t) - \frac{\varrho}{2} H_N(G_\ell)^2 |e^{i G_\ell \Delta t}|^2 = 0
$$

(4.14)

where we have written $\varrho = \Phi_e / \tilde{\Phi}$ and $\cos N$ is defined in (2.16). This equation must have one real solution corresponding to a computational mode. The other two will represent the eastward and westward travelling gravity waves.

If $\varrho = 0$ the computational mode vanishes and (4.14) reduces to a quadratic. The solutions of this are

$$
\Lambda_{\varrho}^{\pm} = H_N(G_\ell) e^{\pm i G_\ell \Delta t}
$$

(4.15)
and this mode was shown to have the lenient stability criterion (2.20). We note also that $\Lambda_0^+ = \overline{\Lambda_0^-}$. Again following Durran (1999), we expand $\tilde{\Lambda}$ in powers of $\varrho$ which is small by the assumption $|\Phi_e| \ll \Phi$:

$$\tilde{\Lambda} = \Lambda_0 + \Lambda_1 \varrho + O(\varrho^2)$$  \hspace{1cm} (4.16)

Substituting this into (4.14) we get

$$\Lambda_1 = \frac{A \Lambda_0^2 + B \Lambda_0 + C}{3 \Lambda_0^2 - 4 \Lambda_0 H_N \cos_N (G_\ell \Delta t) + H_N(G_\ell)^2 |e_N^{i G_\ell \Delta t}|^2}$$  \hspace{1cm} (4.17)

where

$$A = \frac{3}{2} H_N(G_\ell) \cos_N (G_\ell \Delta t) - \frac{3}{2}$$

$$B = H_N(G_\ell) \cos_N (G_\ell \Delta t) - \frac{3}{2} H_N(G_\ell)^2 |e_N^{i G_\ell \Delta t}|^2 + \frac{1}{2}$$

$$C = \frac{1}{2} H_N(G_\ell)^2 |e_N^{i G_\ell \Delta t}|^2 - \frac{1}{2} H_N(G_\ell) \cos_N (G_\ell \Delta t)$$

With $\Lambda_0$ defined by (4.15) the expression for $\Lambda_1$ in (4.17) reduces to

$$\Lambda_1 = -1 + \frac{1}{4 \Lambda_0} + \frac{3 \Lambda_0}{4}$$  \hspace{1cm} (4.18)

We can use (4.16) to write

$$|\tilde{\Lambda}|^2 = |\Lambda_0|^2 + 2 \varrho \text{Re}[\Lambda_0 \overline{\Lambda_1}] + O(\varrho^2)$$

After substituting $\Lambda_1$ from (4.18) we get a first order approximation

$$|\tilde{\Lambda}|^2 = |\Lambda_0|^2 - 2 \varrho \text{Re}[\Lambda_0] + \frac{\varrho}{2 |\Lambda_0|^2} \text{Re}[\Lambda_0^2] + \frac{3 \varrho}{2} |\Lambda_0|^2$$  \hspace{1cm} (4.19)

With $\Lambda_0 = H_N(G_\ell) e_N^{ \pm i G_\ell \Delta t}$ and using $\sin_N$ from (2.17), we can write

$$|\tilde{\Lambda}|^2 = H_N(G_\ell)^2 |e_N^{i G_\ell \Delta t}|^2 \left(1 + \frac{3 \varrho}{2}\right) - 2 \varrho H_N(G_\ell) \cos_N (G_\ell \Delta t)$$

$$+ \frac{\varrho}{2} \left(\cos_N^2 (G_\ell \Delta t) - \sin_N^2 (G_\ell \Delta t)\right)$$  \hspace{1cm} (4.20)

where we have used the fact from Chapter 2 that once $N$ is chosen to be a multiple of 4, $H_N$ is real with $0 \leq H_N \leq 1$. 

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For scales with large $\ell$, the frequency $G_\ell$ will exceed the cutoff frequency $\omega_c$ of the LT scheme and so $H_N(G_\ell) \to 0$. In this case we have just

$$|\tilde{\Lambda}|^2 = \frac{\varrho}{2} (\cos^2_N(G_\ell \Delta t) - \sin^2_N(G_\ell \Delta t)) \ll 1$$

since $\varrho \ll 1$.

For the larger scales where $G_\ell < \omega_c$ we can use the fact that $H_N \leq 1$ and write

$$|\tilde{\Lambda}|^2 = H_N(G_\ell) |e_N^{i G_\ell \Delta t}|^2 \left(1 + \frac{3\varrho}{2}\right) - \varrho \cos_N(G_\ell \Delta t)$$

$$\leq H_N(G_\ell) |e_N^{i G_\ell \Delta t}|^2 \left(1 + \frac{3\varrho}{2}\right) - \varrho \cos_N(G_\ell \Delta t)$$

$$+ \frac{\varrho}{2} (\cos^2_N(G_\ell \Delta t) - \sin^2_N(G_\ell \Delta t))$$

$$\leq |e_N^{i G_\ell \Delta t}|^2 \left(1 + \frac{3\varrho}{2}\right) - \varrho \cos_N(G_\ell \Delta t) + \frac{\varrho}{2} (\cos^2_N(G_\ell \Delta t) - \sin^2_N(G_\ell \Delta t))$$

provided that

$$H_N(G_\ell) |e_N^{i G_\ell \Delta t}|^2 \left(1 + \frac{3\varrho}{2}\right) \geq 2 \varrho \cos_N(G_\ell \Delta t)$$

i.e. $H_N(G_\ell) |e_N^{i G_\ell \Delta t}|^2 \geq \varrho \left(2 \cos_N(G_\ell \Delta t) - \frac{3}{2} H_N(G_\ell) |e_N^{i G_\ell \Delta t}|^2\right)$

The left-hand side is $O(1)$ while the right is $O(\varrho)$ and so the condition is satisfied.

If the value of $G_\ell \Delta t$ is sufficiently small and $N$ is large, we can assume

$$|e_N^{i G_\ell \Delta t}| \to |e^{i G_\ell \Delta t}| = 1$$

$$\cos_N(G_\ell \Delta t) \to 1$$

$$\cos^2_N(G_\ell \Delta t) - \sin^2_N(G_\ell \Delta t) \to \cos 2G_\ell \Delta t \to 1$$

With these we have

$$|\tilde{\Lambda}|^2 \leq 1 + \frac{3\varrho}{2} - 2\varrho + \frac{\varrho}{2} = 1$$

Without the assumptions we may have a value of $|\tilde{\Lambda}|^2$ slightly larger than 1. However no problems were encountered when running the SLLT model. The model remained stable and didn’t require $\bar{\Phi} \geq \Phi_{\text{max}}$ like the SLSI model.
4.7 Testing SWEmodel

The SLSI and SLLT schemes were tested with cases 2, 5 and 6 from Williamson et al. (1992), as described in Section 3.4. Results were compared with those from the reference Eulerian SWEmodel. The error measures defined in (3.9) were computed using Gaussian quadrature.

4.7.1 Steady state zonal flow

Case 2 consists of a nondivergent zonal flow with a geostrophically balanced height field. As pointed out in Jakob et al. (1993) and Drake and Guo (2001), the steady solution consists of a low order spherical harmonic and so is a trivial problem for a spectral transform model. We would thus expect minimal errors from the Eulerian reference model, while interpolation errors will result from the use of semi-Lagrangian schemes. This is evident in the graph on the left of Figure 4.4, where we have plotted the $l_\infty$ error measure for the three schemes at a T119 resolution and a 600 second timestep. Next to this are the errors for the SLSI and SLLT schemes at the same spatial resolution, but with higher timesteps. Even at a 1 hour timestep, the errors are still very small.

![Figure 4.4: $l_\infty$ errors for case 2](image)

4.7.2 Flow over a mountain

For the mountain test case, no analytic solution exists. As was done in Chapter 3, we use the STSWM model run at a T213 resolution with a 360 second timestep as a
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reference. Solutions are compared every 24 hours and errors are computed based on this. A fourth order diffusion was used in all Eulerian runs i.e. the T213 STSWM and the Eulerian SWEmodel.

Figure 4.5 shows the maximum errors for this case at T119. In the left plot we see comparable accuracy for the three methods at a 600s timestep. The right panel shows SLSI and SLLT runs with longer timesteps. The SLLT errors (blue) show a good consistency in the solution, whereas the SLSI (red) accuracy is deteriorated by increasing timestep.

Figure 4.5: $l_\infty$ errors for case 5

4.7.3 Rossby-Haurwitz waves

For the Rossby-Haurwitz wave test case, a reference T213 STSWM run was again used to compute errors every 24 hours. In this case a smaller timestep of 180 seconds is used, as recommended in Jakob et al. (1993).

Côté and Staniforth (1988) use a logarithmic form of the continuity equation in (4.5) for their two time level scheme. Their linear analysis verifies that this is stable. As was mentioned in Section 4.6, they find that, for the non-logarithmic form, a necessary condition for stability is that the reference geopotential height $\bar{\Phi}$ must exceed the geopotential of the free surface of the fluid. The recommended initial conditions for cases 2 and 5 above satisfy this. Note that $\bar{\Phi}$ is a reference value and not a mean geopotential. For the Rossby-Haurwitz wave, however, the value for $\bar{\Phi}$ specified in Williamson et al. (1992), $\bar{\Phi} = gh_0$ with $h_0 = 8$ km, has to be changed to meet this criterion. For the forecasts using SLSI, a value of $\bar{\Phi} = 1.1 \times 10^5$ m$^2$s$^{-2}$ was used (the
forecast becomes unstable after a few days otherwise). A “Log SLSI” model was also tested, following Côté and Staniforth. This gave almost identical results using the traditional value of $\Phi$.

The analysis in Section 4.6 suggested that SLLT didn’t need the higher value for the mean geopotential. The model was run at the traditional value without any difficulty, confirming the analytical result.

On the left of Figure 4.6 we see $l_\infty$ errors for the various schemes, again at T119 and a timestep of 600 seconds. The SLSI and SLLT models show comparable accuracy. The plot to the right shows the errors with longer timesteps. In this case, the SLSI model would not run stably with a 1 hour timestep. The errors throughout are considerably higher than for case 5. Both the SLSI and SLLT models significantly damp the wave when run with long timesteps; this is more severe for SLSI.

![Figure 4.6: $l_\infty$ errors for case 6](image)

**4.7.4 Cutoff period**

For the Eulerian STSWM model of Chapter 3, it was noted that the LT method gave better results for the Rossby-Haurwitz wave with a cutoff period $\tau_c = 3$ rather than $\tau_c = 6$ hours. For the SLLT model, this does not appear to be an issue. In Figure 4.7 we compare $l_\infty$ errors for cases 5 (left) and 6 (right) using two timesteps and the two cutoff periods. At each timestep, 600 and 2700 seconds, we note little difference between the simulations.
4.8 Efficiency

While the SLLT model shows good performance in terms of accuracy and stability, it is important that this does not come at the price of excessive computational cost. In formulating the model, we saw in Section 4.5.4 that we needed to solve tridiagonal systems for all values of zonal wavenumber $m$, whereas SLSI only required this for $m \geq 0$. However we also found that a symmetry in the inversion contour could be used to halve the number of points necessary for the inversion of the LT.

To quantify the computational cost, 24 hour forecasts were run with a timestep of 600 seconds. Six different spectral resolutions were tested: T21, T42, T63, T79, T106 and T119. The initial conditions were the 500hPa data from 1200 UTC on the 12th of February 1979 (this test case will be used extensively in the next chapter). For each model and resolution, 4 forecasts were carried out and the times were averaged. In Figure 4.8 we plot the relative overhead of the SLLT scheme; that is, the average run-time for the SLLT model at a given resolution divided by the run-time for the SLSI model.
At the lowest resolution, T21, we find that the SLLT model takes nearly twice as long to run. However, this decreases and at T119 the relative cost is only 25-30%. This trend suggests that, while the LT method requires additional work, it is not prohibitive and, crucially, the relative cost decreases with increasing resolution. This is encouraging, particularly when considering the use of the LT method at operational resolutions such as, for example, T1279 at the ECMWF.

4.9 Initialisation

So far in this chapter we have only considered idealised test cases. In the next chapter we will run forecasts with SWEmodel using real atmospheric data. Such data will be out of balance and will require initialisation; an introduction to data assimilation and initialisation may be found in Kalnay (2006).

As discussed in Chapter 2, the filtering inversion of the Laplace transform was originally used as an initialisation technique. We can easily use the SLLT code to this end. The method used by Lynch (1985b) is given in (2.10). The first linear step
may be omitted and then the initialisation is simply an iterated application of the LT integration scheme with $\Delta t = 0$.

In order to test the efficacy of any initialisation scheme, some measure of noise is required. Lynch and Huang (1992), for example, use the mean absolute tendency of surface pressure given by

$$N_1 = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \frac{\partial p_s}{\partial t} \right|_{ij}$$

where $i = 1, \ldots, I$ and $j = 1, \ldots, J$ are gridpoint indices. For this work we use the shallow water analogue

$$N_1 = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \frac{\partial \Phi}{\partial t} \right|_{ij} = \frac{1}{IJ} \sum_{i=1}^{I} \sum_{j=1}^{J} |\nabla \cdot (\Phi \mathbf{v})|_{ij} \quad (4.21)$$

In Chapter 5, 500hPa data from 1200 UTC on the 12th of February 1979 will be used as initial conditions for shallow water simulations. We test now the effect of the Laplace transform initialisation (LTI). We run a 6 hour forecast at T119 and $\Delta t = 600s$. The $N_1$ measure for both SLSI and SLLT are plotted in Figure 4.9. The solid line in each is for when no initialisation is used. The dashed line is the measure when the LTI method has been applied to the data with a cutoff period of 6 hours.

![Figure 4.9: The $N_1$ noise measure for simulations with SLSI (left) and SLLT (right). The solid line is when no initialisation is used. The dashed line is when LTI has been applied with parameters $N = 8$, $\tau_c = 6$ hours.](image)

For SLSI (left), it is clear that the initialisation reduces the noise in the forecast. In particular the initial tendency of the geopotential has been greatly reduced. For SLLT (right), we see that the noise is removed anyway during the forecast, as we are
filtering with each step of the LT integration. The application of the initialisation scheme can be seen, however, to have reduced the noise from the very beginning.

### 4.10 SETTLS Formulation

When formulating the two time level SLSI and SLLT models, the nonlinear terms were evaluated at the midpoints of the trajectories. This can lead to problems in some models (Mariano Hortal, personal communication). In this section we explore other options.

A typical evolution equation is given by

$$\frac{dX}{dt} + L = N$$

where $X$ is any prognostic variable, and $L$ and $N$ represent the linear and nonlinear terms respectively. The discretisation used in the SLSI model was

$$\frac{X_{A}^{n+1} - X_{D}^{n}}{\Delta t} + \frac{L_{A}^{n+1} + L_{D}^{n}}{2} = \frac{N_{M}^{n+\frac{1}{2}}}{2}$$

A number of alternative treatments of the nonlinear terms have been proposed. Tanguay et al. (1992) recommend a spatial averaging which reduces the number of interpolations necessary and also helps to alleviate spurious noise from orographically-forced waves. This issue is the subject of Chapter 5. Spatially averaging for a two time level scheme results in the following discretisation

$$\frac{X_{A}^{n+1} - X_{D}^{n}}{\Delta t} + \frac{L_{A}^{n+1} + L_{D}^{n}}{2} = \frac{N_{A}^{n+\frac{1}{2}} + N_{D}^{n+\frac{1}{2}}}{2} \quad (4.22)$$

As before, the values at time level $n + \frac{1}{2}$ are obtained using a linear extrapolation in time; e.g. $N_{A}^{n+\frac{1}{2}} = \frac{3}{2} N_{A}^{n} - \frac{1}{2} N_{A}^{n-1}$. Gospodinov et al. (2001) discuss how this has been shown to lead to problems, with non-meteorological noise being observed in a number of forecasting centres. This was solved at the ECMWF with the operational introduction of the ‘Stable Extrapolation Two-Time-Level Scheme’, or SETTLS, in April 1998 (Hortal, 2002). This is given by

$$\frac{X_{A}^{n+1} - X_{D}^{n}}{\Delta t} + \frac{L_{A}^{n+1} + L_{D}^{n}}{2} = \frac{1}{2} \left\{ 2 N_{D}^{n} - N_{D}^{n-1} \right\} + N_{A}^{n} \quad (4.23)$$
Gospodinov et al. examine a general class of second-order accurate two time level schemes, characterised by a parameter \( \alpha \), of the form

\[
\frac{X_A^{n+1} - X_D^n}{\Delta t} + \frac{L_A^{n+1} + L_D^n}{2} = \frac{3}{4} (N_A^n + N_D^n) - \frac{1}{4} (N_A^{n-1} + N_D^{n-1}) \]

\[
+ \alpha (-N_A^n + N_A^{n-1} + N_D^n - N_D^{n-1})
\]

Setting \( \alpha = 0 \) yields (4.22), while SETTLS is recovered by choosing \( \alpha = \frac{1}{4} \). In an analysis of this family of schemes, Durran and Reinecke (2004) show that optimal stability is obtained with the SETTLS scheme.

### 4.10.1 SETTLS in SLSI and SLLT

The shallow water equations given by (4.5) were discretised for the SLSI and SLLT models. We now test the SETTLS treatment of the right-hand side nonlinear terms, given by (4.23). Everything else in the models, including the trajectory calculations, remains unchanged.

We once again use the test cases of Williamson et al. (1992). Similar to Section 4.7, we use a T119 resolution. Presented in Figure 4.10 are the \( l_\infty \) errors for the steady zonal flow of case 2. In the left-hand panel we plot the errors for the Eulerian reference and the new SLSI SETTLS and SLLT SETTLS models. The panel to the right shows the errors for the two SETTLS models with higher timesteps. Comparing these with Figure 4.4 we see greatly increased accuracy with the SETTLS discretisation. At 120 hours, the SLSI error is \( O(10^{-5}) \) whereas for SLSI SETTLS the error is \( O(10^{-12}) \).

![Figure 4.10: \( l_\infty \) errors for case 2](image-url)
In the right-hand panel of Figure 4.10, the errors for both models are greater with \( \Delta t = 1800s \) than with the higher value of 3600s. As mentioned already, the errors in this test case come from the semi-Lagrangian scheme and not the spatial scheme. For T119, the Courant numbers at \( \Delta t = 1800s \) and 3600s are 1.25 and 0.625 respectively.

The wind is constant and so the back trajectory with the 3600s timestep brings us to a departure point closer to a gridpoint than for the 1800s timestep. Thus in this simple case we would expect higher errors with the lower timestep.

The graphs in Figure 4.11 show the errors for the mountain test case 5 at four different timesteps. The results are not dramatically improved, as in case 2, but instead are broadly similar. For the SLSI forecasts the SETTLS version is marginally more accurate, while for the SLLT models the original is slightly better.

![Case 5: T119 dt = 600s](image1)

![Case 5: T119 dt = 1800s](image2)

![Case 5: T119 dt = 2700s](image3)

![Case 5: T119 dt = 3600s](image4)

Figure 4.11: \( l_\infty \) errors for case 5

For the Rossby-Haurwitz wave in case 6, the SLSI SETTLS shows a great improvement over the original SLSI at all timesteps. The \( l_\infty \) errors are plotted in Figure 4.12. It was noted in Section 4.7.3 that the wave was unstable at \( \Delta t = 3600 \) for the
SLSI model. With the SETTLS formulation this does not seem to be a problem. For the SLLT models, the SETTLS version also improves the accuracy, particularly at the higher timesteps. The SLSI SETTLS, however, outperforms the SLLT models in this case.

Figure 4.12: $l_\infty$ errors for case 6
Chapter 5

Orographic Resonance

5.1 Background

The coupling of a semi-Lagrangian treatment of advection with a semi-implicit method for stabilising gravity waves allowed numerical forecasts to be carried out with timesteps considerably longer than required for Eulerian schemes (Robert 1981, 1982). While this technique was successful in permitting stable and efficient forecasts, a problem arose in the case of simulating flow over orography.

A simple analysis of the linearised shallow water equations, presented in Section 5.2.1, shows that stationary waves produce an infinite response to orographic forcing when the mean flow equals the gravity wave speed. This physical phenomenon is unlikely to occur given the high speed of gravity waves and so will not generally pose a problem for numerical simulations. However, Coiffier et al. (1987) showed that a semi-Lagrangian semi-implicit (SLSI) discretisation introduces a spurious resonant response at large Courant numbers. Numerical runs confirmed this analysis. As the main advantage of a SLSI scheme was the ability to run at large timesteps, this problem was a cause for concern.

This orographic resonance has been investigated by authors in various shallow water models; for example, Rivest et al. (1994), Rivest and Staniforth (1995), Ritchie and Tanguay (1996), Li and Bates (1996), Lindberg and Alexeev (2000). It has also been observed in baroclinic models; for example, Ritchie and Tanguay (1996), Héreil and Laprise (1996).

A number of solutions have been proposed. Tanguay et al. (1992) show that by
spatially averaging all nonlinear terms, the distortions near orography are reduced, though not fully alleviated. Rivest et al. (1994) show that this can be improved upon by off-centring the semi-implicit averaging. They examine first-order and second-order averaging and recommend the latter for better accuracy. This approach has been investigated in a number of atmospheric models; Héreil and Laprise (1996), Caya and Laprise (1999).

In Ritchie and Tanguay (1996) the authors show that the more efficient first-order off-centring is sufficient if the orographic term in the continuity equation is treated in an Eulerian rather than a Lagrangian manner. They find, however, a truncation error in both approaches, which is smaller in the case of the Eulerian treatment of orography. The ECMWF’s IFS model, for example, follows the approach of Ritchie and Tanguay together with the spatial averaging of Tanguay et al. (1992). The orographic contribution to the advection in the continuity equation is isolated and treated in a spatially averaged manner (Temperton et al., 2001). We note, however, that with the operational resolutions currently used (T1279 with a timestep of 600 seconds), the model does not suffer from orographic resonance (Nils Wedi, personal communication).

In reviewing this approach, Lindberg and Alexeev (2000) note that it has been largely successful but does not fully remove the spurious response. Li and Bates (1996) also show how off-centring can have a negative impact on large scale Rossby waves; the first-order method causes excessive damping. They find that by using a potential vorticity form of the shallow water equations, a less damaging modified off-centring is possible. In their study of a general class of off-centred schemes, Côté et al. (1995) recommend using the least amount of off-centring possible, consistent with alleviating resonance, to minimize errors.

Lindberg and Alexeev also explicitly point out that the problem of orographic resonance is due to the combination of the semi-implicit averaging and the semi-Lagrangian approach. We have already seen in Chapter 3 that the Laplace transform method has advantages over the semi-implicit scheme, by maintaining stability while preserving the phase speed of waves. In Chapter 4 the semi-Lagrangian Laplace transform (SLLT) model was formulated and shown to compare favourably with the reference SLSI scheme. This provides a motivation to investigate the response of orographically forced waves when a SLLT discretisation is used.
In the next section we explore the problem of orographic resonance in a linear shallow water model. Firstly the analytic response to orography is examined. We then discretise the equations and study the effects of the numerical schemes on the solution. After that we use SWEmodel to investigate the problem in full shallow water simulations, using realistic orography and 500hPa data as initial conditions.

5.2 Analysis

Ritchie and Tanguy (1996) analyse the problem in a three time level model. Here we adapt their analysis for a two time level version. The shallow water equations are given in vorticity-divergence form and are linearised about a mean flow $\bar{u} = \bar{a} \bar{\omega} \cos \phi$, where $\bar{\omega}$ is a constant advecting angular velocity. The Coriolis parameter $f$ is taken to be constant. With $\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{\omega} \frac{\partial}{\partial \lambda}$, the linearised system is

$$
\begin{align*}
\frac{d\zeta}{dt} + f \delta &= 0 \\
\frac{d\delta}{dt} - f\zeta + \nabla^2 \Phi &= 0 \\
\frac{d\Phi}{dt} + \bar{\Phi}\delta &= \frac{d\Phi_s}{dt}
\end{align*}
$$

(5.1)

5.2.1 Analytic response

We consider solutions comprised of a single spherical harmonic as follows

$$
\begin{pmatrix}
\zeta \\
\delta \\
\Phi
\end{pmatrix} =
\begin{pmatrix}
\zeta^m_{\ell} \\
\delta^m_{\ell} \\
\Phi^m_{\ell}
\end{pmatrix} e^{i(m\lambda - \nu t)} P^m_{\ell}(\mu)
$$

(5.2)

with

$$
\Phi_s = (\Phi_s)_\ell e^{im\lambda} P^m_{\ell}(\mu)
$$

Substituting this into (5.1) we get

$$
\begin{pmatrix}
i(m\bar{\omega} - \nu) & f & 0 \\
-f & i(m\bar{\omega} - \nu) & -\frac{\ell(\ell+1)}{a^2} \\
0 & \bar{\Phi} & i(m\bar{\omega} - \nu)
\end{pmatrix}
\begin{pmatrix}
\zeta^m_{\ell} \\
\delta^m_{\ell} \\
\Phi^m_{\ell}
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
igm\bar{\omega}(\Phi_s)_\ell e^{im\nu t}
\end{pmatrix}
$$

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We look for orographically-forced stationary solutions and so take $\nu = 0$. Solving then for $\Phi^m_\ell$ we get

$$\Phi^m_\ell = \left( \frac{f^2 - m^2\omega^2}{f^2 + \bar{\Phi} \frac{\ell(\ell+1)}{a^2} - m^2\omega^2} \right) (\Phi_s)_\ell^m$$

We can express this as

$$\Phi^m_\ell = \frac{(m\omega)^2 - f^2}{(m\omega)^2 - G^2_\ell} (\Phi_s)_\ell^m$$

where

$$G^2_\ell \equiv f^2 + \frac{\ell(\ell+1)}{a^2} \bar{\Phi}$$

is the squared frequency of the gravity-inertia wave. We find a resonance if this frequency equals that of the mean flow.

5.2.2 Numerical response: SLSI

Next we consider a two time level semi-Lagrangian semi-implicit treatment of the linearised system. The orography is treated in a Lagrangian manner and no explicit diffusion is included. Discretising (5.1) we get

$$\frac{\zeta^{n+1}_A - \zeta^n_D}{\Delta t} + \frac{f}{2} \left\{ \delta A^{n+1} + \{ \delta \} D^n \right\} = 0$$

$$\frac{\delta A^{n+1}_D - \delta D^n}{\Delta t} - \frac{f}{2} \left\{ \zeta A^{n+1} + \{ \zeta \} D^n \right\} + \frac{\nabla^2 \Phi_A^{n+1} + \{ \nabla^2 \Phi \}_D^n}{2} = 0$$

$$\frac{\Phi^{n+1}_A - \Phi^n_D}{\Delta t} + \Phi^{n+1}_D + \Phi^{n}_D = \frac{(\Phi_s)_A - (\Phi_s)_D}{\Delta t}$$

We again seek to examine the response of stationary waves to orographic forcing and so look for solutions of the form

$$\begin{pmatrix} \zeta \\ \delta \\ \Phi \\ (\Phi_s) \end{pmatrix} = \begin{pmatrix} \zeta^m_\ell \\ \delta^m_\ell \\ \Phi^m_\ell \\ (\Phi_s)_\ell^m \end{pmatrix} e^{im\lambda} P^m_\ell(\mu)$$

(5.5)
The effect of the discretised time derivative on these modes is
\[
\frac{\zeta_n^{n+1} - \zeta_n^0}{\Delta t} = \frac{\zeta_m e^{im\lambda A} P_m - \zeta_m e^{im(\lambda_A - \bar{\omega} \Delta t)} P_m}{\Delta t} = \zeta_m e^{im\lambda A} P_m \left\{ \frac{1 - e^{-i\bar{\omega} \Delta t}}{\Delta t} \right\}
\]
\[
= \zeta^{n+1} \frac{2i}{\Delta t} e^{-i\theta} \left\{ \frac{e^{i\theta} - e^{-i\theta}}{2i} \right\}
\]
\[
= \zeta^{n+1} \frac{2i}{\Delta t} e^{-i\theta} \sin \theta
\]
where \( \theta \equiv \frac{m\bar{\omega}\Delta t}{2} \). Similarly,
\[
\frac{\zeta_n^{n+1} + \zeta_n^0}{2} = \zeta_A^{n+1} e^{-i\theta} \left\{ \frac{e^{i\theta} + e^{-i\theta}}{2} \right\} = \zeta_A^{n+1} e^{-i\theta} \cos \theta
\]

Applying these results to the discretised system and multiplying each equation by \( \Delta t e^{i\theta} \) yields
\[
\begin{bmatrix}
2i \sin \theta & \bar{\Phi} \Delta t \cos \theta & 0 \\
-f \Delta t \cos \theta & 2i \sin \theta & -\ell(\ell+1) \Delta t \cos \theta \\
0 & 0 & 2i \sin \theta
\end{bmatrix}
\begin{bmatrix}
\zeta^m \\
\delta^m \\
\Phi^m
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
2i \sin \theta (\Phi_s)^m
\end{bmatrix}
\]

Solving the linear system for \( \Phi \):
\[
\Phi^m = \left\{ \frac{f^2 \cos^2 \theta - (\frac{\sin \theta}{\bar{\omega}})^2 (m \bar{\omega})^2}{G^2 \cos^2 \theta - (\frac{\sin \theta}{\bar{\omega}})^2 (m \bar{\omega})^2} \right\} (\Phi_s)^m
\]

(5.6)

Following Rivest et al. (1994) and Ritchie and Tanguay (1996), we consider the ratio of the numerical to the physical response and so divide the above by (5.3) to get
\[
R_{SLSI} = \frac{f^2 - \left(\frac{\tan \theta}{\bar{\omega}}\right)^2 (m \bar{\omega})^2}{G^2 - \left(\frac{\tan \theta}{\bar{\omega}}\right)^2 (m \bar{\omega})^2} \left(\frac{m \bar{\omega}}{G^2 - f^2}\right)
\]

(5.7)

We note first that \( R_{SLSI} \) has a zero denominator when \((m \bar{\omega})^2 - f^2 = 0\). As pointed out by Ritchie and Tanguay, this is not a resonance but is due to a zero value of the numerator of the physical response (5.3).

There is, however, a spurious numerical resonance when
\[
\frac{\tan \theta}{\bar{\omega}} = \pm \frac{G_\ell}{m \bar{\omega}}
\]

(5.8)
Since $G_\ell$ is large, resonance occurs near the singular points of $\tan \theta$, that is,

$$\theta = \frac{2k + 1}{2}\pi, \text{ for } k \in \mathbb{Z} \quad (5.9)$$

We present plots of $R_{\text{SLSI}}$ for a T119 truncation, i.e., $m = 0, 1, \ldots, 119$. Following Ritchie and Tanguay we take $\ell = m$ and use the values $f = 10^{-4} \text{ s}^{-1}$ and $\Phi = 5.6 \times 10^4 \text{ m}^2 \text{ s}^{-2}$. We choose the advecting wind to be $50 \text{ m} \text{ s}^{-1}$ at the equator so that $\bar{\omega} = \frac{50}{a} \text{ s}^{-1}$.

In Figure 5.1 we take four values of $\Delta t$: 600, 1800, 2700 and 3600 seconds. The response is plotted in terms of a scaled wavenumber $m\Delta \lambda/\pi$. For these parameter values, the case of zero physical response mentioned earlier occurs at $m\Delta \lambda/\pi \approx 0.07$ and this is seen by the jumps near this value.

For an exact solution we would have the response equal to 1, indicated by the dashed red line in each plot. For $\Delta t = 600s$ and $1800s$ we get acceptable results. At $\Delta t = 2700s$ the smaller scale waves are being significantly amplified. At the longest timestep of $\Delta t = 3600s$ there is clearly a resonant response when $m\Delta \lambda/\pi$ is close to 0.6. This is the first resonance given by (5.9), when $\theta = \frac{\pi}{2}$.

5.2.3 Numerical response: SLLT

We now turn to the SLLT scheme to examine its effect on orographically forced waves. Again we consider the linearised shallow water equations with constant $f$ given by (5.1). Taking the Laplace transform of each equation along a trajectory, as discussed in Chapter 4, and again looking for steady state spherical harmonic solutions of the form (5.5), we get

$$s\hat{\zeta} - \hat{\zeta}_D^n + f\hat{\delta} = 0$$
$$s\hat{\delta} - \hat{\delta}_D^n - f\hat{\zeta} - \frac{\ell(\ell + 1)}{a^2}\hat{\Phi} = 0$$
$$s\hat{\Phi} - \Phi_D^n + \Phi_\delta = s\hat{\Phi}_s - (\Phi_s)_D^n$$

We consider the trajectory $\lambda = \lambda_D + \bar{\omega}t$ and then with $\Phi_s = (\Phi_s)_\ell e^{im\lambda} P_\ell^m(\mu)$ we
can write

\[ \hat{\Phi}_s \equiv \mathcal{L} \left\{ (\Phi_s)_\ell^m e^{i m (\lambda D + \bar{\omega} t)} P^m_{\ell} (\mu) \right\} \]
\[ = \mathcal{L} \left\{ (\Phi_s)_\ell^m e^{i m \lambda D} P^m_{\ell} (\mu) e^{i m \bar{\omega} t} \right\} \]
\[ = (\Phi_s)_D^n \mathcal{L} \left\{ e^{i m \bar{\omega} t} \right\} \]

We need to take the transform of \( e^{i m \bar{\omega} t} \). This is a function of time only, since \( \bar{\omega} \) is a constant with no spatial variation. Therefore we are taking a traditional Laplace transform and can write \( \mathcal{L} \left\{ e^{i m \bar{\omega} t} \right\} = \frac{1}{s - i m \bar{\omega}} \). Using this we can write the transformed
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system as

\[
\begin{pmatrix}
  s & f & 0 \\
  -f & s & -\ell (\ell + 1) \\
  0 & \Phi & s
\end{pmatrix}
\begin{pmatrix}
  \zeta \\
  \delta \\
  \Phi
\end{pmatrix}
= 
\begin{pmatrix}
  \zeta^n_D \\
  \delta^n_D \\
  \Phi^n_D
\end{pmatrix}
+ 
\begin{pmatrix}
  0 \\
  0 \\
  \frac{i m \bar{\omega}}{s - i m \bar{\omega}} (\Phi_s)^n_D
\end{pmatrix}
\]

The right-hand side has been split so as to isolate the response due to the orographic term. We focus on this response; that is, we consider the system

\[
\begin{pmatrix}
  s & f & 0 \\
  -f & s & -\ell (\ell + 1) \\
  0 & \Phi & s
\end{pmatrix}
\begin{pmatrix}
  \zeta_{\text{orog}} \\
  \delta_{\text{orog}} \\
  \Phi_{\text{orog}}
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0 \\
  \frac{i m \bar{\omega}}{s - i m \bar{\omega}} (\Phi_s)^n_D
\end{pmatrix}
\]

Solving for \((\Phi)_{\text{orog}}\) we get

\[
(\Phi)_{\text{orog}} = \frac{s^2 + f^2}{s \left( s^2 + f^2 + \bar{\Phi} (\ell + 1) \right)} \left\{ \frac{i m \bar{\omega}}{s - i m \bar{\omega}} (\Phi_s)^n_D \right\}
\]

\[
= \frac{s^2 + f^2}{s (s^2 + G^2_l)} \left\{ \frac{i m \bar{\omega}}{s - i m \bar{\omega}} (\Phi_s)^n_D \right\}
\]

\[
= \frac{i m \bar{\omega}}{s} \left(\frac{s}{s^2 + G^2_l} (s - i m \bar{\omega}) + \frac{f^2}{s (s^2 + G^2_l) (s - i m \bar{\omega})} \right)
\]

where \(G_l\) is the gravity wave frequency defined in (5.4). We see immediately that we cannot have \(s = 0\), \(s = i m \bar{\omega}\) or \(s = \pm G_l\). But \(|s| = \gamma\), the radius of the inversion contour and so these situations can be avoided by a careful contour choice.

This expression for \((\Phi)_{\text{orog}}\) can be expanded using partial fractions to give

\[
(\Phi)_{\text{orog}} = \frac{f^2 - (m \bar{\omega})^2}{(G^2_l - (m \bar{\omega})^2) (s - i m \bar{\omega})} (\Phi_s)^n_D
\]

\[+ \frac{m \bar{\omega} \left( 1 - \frac{f^2}{G^2_l} \right)}{2 (G_l - m \bar{\omega}) (s - i G_l)} (\Phi_s)^n_D\]

\[+ \frac{m \bar{\omega} \left( 1 - \frac{f^2}{G^2_l} \right)}{2 (G_l + m \bar{\omega}) (s + i G_l)} (\Phi_s)^n_D\]

\[- \left( \frac{f}{G_l} \right)^2 \frac{1}{s} (\Phi_s)^n_D\]

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The first six terms here have the form \( c/(s - i \nu) \) and the final term \( c/s \), where \( c \) is some term which is independent of \( s \). We saw in Sections (2.4) and (2.3), respectively, that when the inversion operator \( \mathcal{L}_N \) is applied to functions of this form we get

\[
\mathcal{L}_N^* \left\{ \frac{c}{s - i \nu} \right\} = c H_N(\nu) e^{|\nu| \Delta t} \\
\mathcal{L}_N^* \left\{ \frac{c}{s} \right\} = c
\]

where

\[
H_N(\nu) \equiv \frac{1}{1 + \left( \frac{\nu}{\gamma} \right)^N}
\]

These results immediately give us the physical solution obtained when we apply the numerical inversion operator to each term in \((\tilde{\Phi}_s)_{orog}\). After inverting and rearranging we get

\[
(\Phi_{orog})_A = R' (\Phi_s)_D
\]

where

\[
R' = \frac{f^2 - (m \bar{\omega})^2}{G^2} H_N(m \bar{\omega}) e^{i m \bar{\omega} \Delta t} - \left( \frac{f}{G} \right)^2 \\
+ \frac{m \bar{\omega} \left( 1 - \left( \frac{f}{G} \right)^2 \right)}{2 (G - m \bar{\omega})} H_N(G) e^{i G \Delta t} \\
- \frac{m \bar{\omega} \left( 1 - \left( \frac{f}{G} \right)^2 \right)}{2 (G + m \bar{\omega})} e^{-i G \Delta t}
\]

Since

\[
(\Phi_s)_D = (\Phi_s)_t^m e^{i m \lambda_d} P_{\ell}^m(\mu) \\
= (\Phi_s)_t^m e^{i m \lambda_a} e^{-i m \bar{\omega} \Delta t} P_{\ell}^m(\mu) \\
= (\Phi_s)_t^m e^{i m \lambda_a} e^{-i 2 \theta} P_{\ell}^m(\mu)
\]

we can divide (5.10) by \( e^{i m \lambda_a} P_{\ell}^m(\mu) \) and write \((\Phi_{orog})_t^m = e^{-i 2 \theta} R' (\Phi_s)_t^m \). As in the
SLSI case, we divide the numerical response by the analytic response (5.3) and define

\[ R_{\text{SLLT}} = H_N(m\bar{\omega}) e^{i2\theta} e^{-i2\theta} \left( \frac{f}{G_\ell} \right)^2 \left( \frac{G_\ell^2 - (m\bar{\omega})^2}{f^2 - (m\bar{\omega})^2} \right) e^{-i2\theta} + e^{-i2\theta} H_N(G_\ell) \frac{m\bar{\omega}}{f^2 - (m\bar{\omega})^2} \left\{ m\bar{\omega} \cos_N (G_\ell \Delta t) + i G_\ell \sin_N (G_\ell \Delta t) \right\} \]

(5.11)

where we have used the truncated trigonometric functions \( \cos_N \) and \( \sin_N \) as defined in (2.16) and (2.17) respectively. The response in (5.11) will only have a zero denominator for \( f^2 = (m\bar{\omega})^2 \). This is, however, the case of zero physical response as mentioned in the SLSI analysis. Thus we expect no spurious resonant response to orography using a SLLT discretisation.

To illustrate this, we plot \( R_{\text{SLLT}} \) in Figure 5.2 with parameters matching those for SLSI in Figure 5.1. In addition we use the values \( N = 8 \) and \( \tau_c = 6 \) hours. As before, we see where the physical solution is zero at \( m\Delta\lambda/\pi \approx 0.07 \). For SLLT, however, there is no resonant behaviour present.

On comparison with the SLSI resonances in Figure 5.1, it may appear that we are resonance-free simply because the problematic wavenumbers have been removed by the LT filtering. However, it is important to note again that the expression in (5.11) shows no artificial resonance, regardless of wavenumber. To demonstrate this we plot \( R_{\text{SLSI}} \) and \( R_{\text{SLLT}} \) in Figure 5.3, this time for T213 resolution and a 2 hour timestep. For the SLLT discretisation we choose a less severe cutoff of 3 hours and \( N = 16 \). With these parameter values we see resonant behaviour in SLSI around \( m\Delta\lambda/\pi \) between 0.1 and 0.2. The SLLT plot to the right of this show that these scales are being retained, but with no resonance.
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Figure 5.2: SLLT with $N = 8$ and $\tau_c = 6$ hours: the numerical response to orographic forcing divided by the physical response. The dashed red line is $R = 1$, where the numerical solution equals the analytic solution.

Figure 5.3: Responses for SLSI and SLLT, T213 with $\Delta t = 7200s$, $N = 16$ and $\tau_c = 3$ hours: the numerical response to orographic forcing divided by the physical response. The dashed red line is $R = 1$, where the numerical solution equals the analytic solution and the dot-dashed blue line is $R = 0$, where the numerical solution is zero due to filtering.
5.3 Shallow Water Experiments

In the previous section we showed that, in a linearised model, the semi-Lagrangian Laplace transform discretisation does not suffer from spurious resonant responses of stationary waves to orographic forcing. Now we move to numerical tests with the fully nonlinear shallow water system. In the previous chapter we outlined the spectral SWEmodel with both the SLSI and SLLT formulations. This already has the option for a varying surface height and so we can easily incorporate real orographic data into the model.

5.3.1 Initial Conditions

In the simulations using SWEmodel in Chapter 4, idealised initial conditions were used. This time we wish to simulate more realistic flow. A number of previous studies have used the analysis at 12 UTC on the 12th of February 1979 [Rivest et al. (1994), Rivest and Staniforth (1995), Li and Bates (1996), Ritchie and Tanguay (1996)]. There is a strong flow over the Rocky mountains, so this is a very suitable case study for investigating orographic resonance.

The initial conditions are the 500hPa winds and geopotential height, as well as the surface geopotential, taken from the ERA-40 dataset of the ECMWF (Uppala et al., 2005). At the beginning of a model run, the fields are loaded and interpolated to the relevant grid. We then initialise these with the Laplace transform initialisation method, as outlined in the Section 4.9. The initial height is plotted at T119 resolution (about 110km at the equator) in Figure 5.4 with the orography shown in Figure 5.5.

5.3.2 Numerical Forecasts

The simulations to be presented were all run at a T119 resolution with no diffusion added. The SLLT parameters used were $N = 8$ and $\tau_c = 6$ hours. We saw in Section 5.2.2 that with a 600 second timestep we should have no trouble with orographic resonance, since this only becomes an issue at high Courant numbers. Plotted in Figures 5.6 and 5.7 are 24-hour height forecasts for SLSI and SLLT respectively. Both models give similar results.
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Figure 5.4: Initial height at T119 (Contour interval = 60m)

Figure 5.5: Model orography at T119 (Contour interval = 200m)
Chapter 5: Orographic Resonance

Figure 5.6: SLSI 24-hour height forecast (Contour interval = 60m)

Figure 5.7: SLLT 24-hour height forecast (Contour interval = 60m)
As already mentioned, the initial conditions show a strong flow over the Rocky Mountains. We focus our attention on the North American continent and show the 24-hour forecasted height in this area for model timesteps of 600, 1800, 2700 and 3600 seconds. The analysis in Section 5.2 suggests that we may encounter problems with orographic resonance at the two higher timesteps.

In Figures 5.8 and 5.9 we have the 600 and 1800s forecasts respectively. The two models are in broad agreement. The 2700s timestep forecast in Figure 5.10 has some noise over the Rockies for the SLSI run, on the left. This spurious noise is far more pronounced when a 3600s timestep is used, as seen in Figure 5.11. As expected from the linear analysis, the SLLT model does not suffer from this resonance and the forecasts are well matched for each of the four different timesteps.
Figure 5.8: 24-hour height forecasts at $\Delta t = 600s$ for SLSI (top) and SLLT (bottom)
Figure 5.9: 24-hour height forecasts at $\Delta t = 1800\text{s}$ for SLSI (top) and SLLT (bottom)
Figure 5.10: 24-hour height forecasts at $\Delta t = 2700s$ for SLSI (top) and SLLT (bottom)
Figure 5.11: 24-hour height forecasts at $\Delta t = 3600$ s for SLSI (top) and SLLT (bottom)
5.3.3 SETTLS Models

We have used the SLSI and SLLT models to investigate orographic resonance. Their formulations were detailed in Chapter 4. In particular the nonlinear terms were evaluated at the midpoint of the trajectories. As mentioned already, Tanguay et al. (1992) showed that spatially averaging all nonlinear terms helps to reduce the distortion. In Section 4.10, SLSI and SLLT were reformulated using the SETTLS treatment of the nonlinear terms, given by (4.23). This involves an averaging of the nonlinear terms to the midpoint of the trajectory.

So far we have considered a Lagrangian treatment of orography; that is, we have discretised the derivative \( \frac{d\Phi_s}{dt} \) in the continuity equation. An alternative Eulerian treatment was discussed in Section 4.5.2, where we write

\[
\frac{d\Phi_s}{dt} = v \cdot \nabla S
\]

and treat the advection with the other nonlinear terms. Ritchie and Tanguay (1996) showed that spatial averaging was more effective at reducing resonance when the orography was treated in this Eulerian manner.

We now examine what effect these formulations will have on the noise observed in the SLSI forecasts in the previous section. The same initial data and resolution as before are used. For the SLLT models, the use of SETTLS makes little difference. With a 600 second timestep the rms differences between the 24 hour height forecasts using SLLT and SLLT SETTLS is 30cm. For higher timesteps, the differences are 5m (\( \Delta t = 1800s \)), 7.5m (\( \Delta t = 2700s \)) and 13m (\( \Delta t = 3600s \)).

In Figure 5.12 we plot a 24-hour height forecast with a 600 second timestep for SLSI SETTLS (upper panel) and SLSI SETTLS using the Eulerian treatment of orography (lower). These are almost identical to the non-SETTLS forecasts in Figure 5.8. The rms difference between the SLSI and SLSI SETTLS forecasts is about 1m; with the Eulerian orography the difference is about 1.1m.

The plots in Figure 5.13 show the same model forecasts but with a 3600s timestep. Comparing these with the SLSI forecasts in Figure 5.11, we see that the noise over the Rockies is less severe with the use of SETTLS in the top panel. The Eulerian treatment of orography with SETTLS (bottom) yields the best results for a SLSI model. However the noise still does not disappear completely. The SLLT forecast in Figure 5.11 is still the most effective for removing the spurious response.
Figure 5.12: 24-hour height forecast with $\Delta t = 600s$ using SLSI SETTLS (top) and with an Eulerian treatment of orography (bottom)
Figure 5.13: 24-hour height forecast with $\Delta t = 3600\text{s}$ using SLSI SETTLS (top) and with an Eulerian treatment of orography (bottom)
Chapter 6

Conclusion

6.1 Summary and Conclusions

The aim of this research project was to investigate the potential of the Laplace transform filtering method as a viable time integration scheme for atmospheric models. It has been shown, both analytically and numerically, to perform competitively when compared with reference semi-implicit schemes and to have a number of additional benefits.

In Chapters 3 and 4, two shallow water models based on the LT discretisation method were developed and tested against reference semi-implicit versions. These were the Eulerian STSWM model and the semi-Lagrangian SWEmodel. Standard test cases were used throughout to test the models’ performance. In both the Eulerian and Lagrangian frameworks, the LT method showed comparable accuracy with the semi-implicit models. For the STSWM model, the LT version also roughly matched the semi-implicit reference in terms of mass and energy conservation. Like the semi-implicit, the LT method has good stability properties and the LT Eulerian model could be run at timesteps matching the reference.

The semi-Lagrangian LT (SLLT) model allows simulations with longer timesteps than would be possible for Eulerian models. A linear analysis verified the stability of the scheme and showed that stability was independent of the choice of reference geopotential.

In addition to rivalling the semi-implicit’s stability properties, the LT method was shown to have some advantages. In Chapter 3 the computational phase speed
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of propagating waves was examined. While semi-implicit schemes slow down faster-moving waves in order to maintain stability, the LT integration method was shown to represent the speed with much greater accuracy. This property may be of particular importance when investigating phenomena involving dynamically significant fast waves.

In Chapter 5 the semi-Lagrangian LT discretisation was demonstrated not to suffer from the spurious orographic resonance found in a number of semi-implicit semi-Lagrangian models. While various ways to reduce the resonance have been proposed, they do not completely eliminate the problem and can, in the case of off-centring, lead to a deterioration in accuracy. In this regard, the resonance-free LT scheme has clear benefits.

Due to the numerical integration around the inversion contour using a summation, the LT scheme involves some additional computational overhead. The more points taken for the summation, the more work needed. We saw, however, in Chapter 3 that increasing from \( N = 8 \) to \( N = 16 \) points made little difference in terms of accuracy. In Chapter 4 the efficiency of the SLLT model was examined in more detail, using forecasts run with \( N = 8 \). Symmetry in the inversion allowed us to reduce the number of points actually used to \( N/2 \). It was found that the relative cost of the SLLT scheme compared to the reference semi-implicit semi-Lagrangian model decreased with increasing spatial resolution. For typical resolutions in current operational models, the overhead should not pose any problem.

6.2 Possible Future Extensions

Having tested the LT method in the shallow water framework, the obvious next step would be to implement the technique in a full baroclinic atmospheric model. The filtering properties of the scheme could be of particular benefit in the context of nonhydrostatic models. Extremely fast modes, such as vertically propagating acoustic waves, are problematic for the stability of numerical solvers. These could be effectively removed from the system with the LT filter.

In considering the use of the LT method in an operational NWP model, efficiency becomes an important factor. As mentioned in the previous section, the relative cost of the LT scheme decreases with increasing resolution. This trend is encouraging. In
full atmospheric models, the physical parameterisations account for a large proportion of the work at every timestep. As the LT scheme would still only require the computation of these terms once per timestep, the efficiency of the timestepping at operational resolutions should be competitive.

The issue of conservation has not been examined in detail for the LT scheme. In particular the formulation of the semi-Lagrangian LT model in Chapter 4 was not specifically designed to conserve mass. A number of mass-conserving schemes have been proposed, for example Kaas (2008), and so it may be possible to improve upon the SLLT formulation. This would be especially important if the LT scheme was to be considered for use in climate simulation in a general circulation model.

With this SLLT model it was found that the use of the SETTLS treatment of nonlinear terms led to some improvements in accuracy. It would be worth investigating various formulations when developing the model further.

In this work the Laplace transform is applied to a continuous function in the equations. If, on the other hand, the time is first discretised, the appropriate formalism is the Z transform (ZT), the discrete analogue of the LT. Details of the ZT may be found in Doetsch (1971). The ZT approach was used by Lynch (1991) in a regional model. The schemes in this thesis are easily modified to use the ZT formulation. Any advantages of this approach remain to be determined.
Appendix A

Further Proofs for the LT Method

A.1 Inversion Operator

A.1.1 Exact Polynomial Inversion

In the description of the Laplace transform method in Chapter 2, it was shown that if a truncated exponential is used in the inversion integral, then the inversion is exact for any constant function $f(t) = a$; that is, if

$$\mathcal{L}^{-1}_N \left\{ \hat{f}(s) \right\} \equiv \frac{1}{N} \sum_{n=1}^{N} \hat{f}(s_n) e^{s_n \Delta t} s_n$$

then

$$\mathcal{L}^{-1}_N \mathcal{L} \{ a \} = a$$

We now consider the more general case of transforming and inverting powers of $t$. For $k = 1, 2, \ldots$, we have

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^k} \right\} = \frac{t^{k-1}}{(k-1)!}$$
Appendix A: Further Proofs for the LT Method

We apply the operator $\mathbf{L}_N^*$ to $\hat{f}(s) = \frac{1}{s^k}$ to get

$$
\mathbf{L}_N^* \left\{ \frac{1}{s^k} \right\} = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{s^k} e^{\gamma N \Delta t} s_n
$$

$$
= \frac{1}{N} \sum_{n=1}^{N} \frac{1}{(\gamma e^{\pi(2n-1)/N})^{k-1}} \sum_{j=0}^{N-1} \frac{\Delta t^j}{j!} \gamma^{j-k+1} e^{\pi(2n-1)(j-k+1)/N}
$$

We consider the special case of $j-k+1 = 0$. For $j = k-1$ we have $e^{\pi(2n-1)(j-k+1)/N} = 1$ and the relevant term in the summation reduces to

$$
\frac{\Delta t^{k-1}}{(k-1)!}
$$

Since $j \leq N - 1$ this can only occur if $k \leq N$. The sum over $n$ for every other $j$ can be shown to equal zero:

$$
\mathbf{L}_N^* \left\{ \frac{1}{s^k} \right\} = \frac{\Delta t^{k-1}}{(k-1)!} + \frac{1}{N} \sum_{n=1}^{N} \sum_{j=0, j \neq k-1}^{N-1} \frac{\Delta t^j}{j!} \gamma^{j-k+1} e^{\pi(2n-1)(j-k+1)/N}
$$

The final sum is now

$$
\sum_{n=1}^{N} \left( e^{2\pi i (j-k+1)/N} \right)^n = e^{2\pi i (j-k+1)/N} \sum_{n=0}^{N-1} \left( e^{2\pi i (j-k+1)/N} \right)^n
$$

$$
= e^{2\pi i (j-k+1)/N} \left\{ \frac{1 - \left( e^{2\pi i (j-k+1)/N} \right)^N}{1 - e^{2\pi i (j-k+1)/N}} \right\}
$$

$$
= 0
$$

Thus, for $1 \leq k \leq N$ we have shown that

$$
\mathbf{L}_N^* \left\{ \frac{1}{s^k} \right\} = \frac{\Delta t^{k-1}}{(k-1)!}
$$

and so we have exact inversion for polynomials in $t$ of degree $N - 1$.

Considering again the inversion of $\hat{f}(s) = \frac{1}{s^k}$, we saw that

$$
\mathbf{L}_N^* \left\{ \frac{1}{s^k} \right\} = \frac{1}{N} \sum_{n=1}^{N} \sum_{j=0}^{N-1} \frac{\Delta t^j}{j!} \gamma^{j-k+1} e^{\pi(2n-1)(j-k+1)/N}
$$
Appendix A: Further Proofs for the LT Method

We focused on the case of \( j = k - 1 \) which required \( 1 \leq k \leq N \). We now seek to
generalise this. We are looking for a specific value of \( j \) such that \( j - k + 1 = mN \) for
\( m \in \mathbb{Z} \) (previously we had \( m = 0 \)). Higher values of \( k \) will give a nonzero value for
\( m \). In this case the exponential term in the sum becomes

\[
e^{\pi i (2n-1)m} = (-1)^{m}
\]

Clearly if \( m \) is odd we may not get a correct inversion.

A.2 Filter Response

Van Isacker and Struylaert (1985) analyse the solution of the following general
differential equation with the LT method:

\[
\frac{\partial y}{\partial t} = Ay + b(y)
\]  

(A.1)

where \( A \) contains all the linear terms and \( b \) is a nonlinear function. This leads them
to consider integrals of the form

\[
\frac{1}{2\pi i} \oint_{C^*} \frac{s^{m+lN}}{sI - A} ds
\]

with \( 0 \leq m < N \) and \( m \in \mathbb{Z} \).

Numerically integrating as described in Chapter 2 gives

\[
\frac{1}{2\pi i} \oint_{C^*} \frac{s^{m+lN}}{sI - A} ds \simeq \frac{1}{N} \sum_{n=1}^{N} \left( \frac{s_n^{m+lN}}{s_n I - A} \right) s_n
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} \frac{s_n^{m+lN}}{I - \frac{A}{s_n}}
\]

Equation (B.5) of Van Isacker and Struylaert (1985) gives an exact formula for the
numerical integration; here we consider the specific case of \( m = l = 0 \)

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1}{I - \frac{A}{s_n}} = \frac{1}{1 + \left( \frac{A}{s_n} \right)^N}
\]  

(A.2)

This formula is the key to deriving the filter response to a wave-like input given in
Section 2.4, since we are there interested in solving

\[
\frac{dy}{dt} = i\omega y
\]

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The proof given of this formula assumes that \(|\frac{A}{s}| < 1\). In terms of filtering frequencies, this would correspond to the situation where the solution has a frequency less than the cut-off. This can be easily expanded to the case where \(|\frac{A}{s}| > 1\). We will use the general result that if \(|x| > 1\), then

\[
\frac{1}{1-x} = \frac{1}{x} \left(\frac{1}{1-x} \right) = -\frac{1}{x} \sum_{k=0}^{\infty} \left(\frac{1}{x}\right)^k = -\sum_{k=1}^{\infty} \left(\frac{1}{x}\right)^k
\] (A.3)

where we have used the fact that \(|\frac{1}{x}| < 1\). Applying this to the summation when \(|\frac{A}{s_n}| > 1\) we get

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 - \frac{A}{s_n}} = -\frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{\infty} \left(\frac{s_n}{A}\right)^k
\]

\[
= -\frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{\infty} \left(\frac{\gamma e^{\pi i(2n-1)/N}}{A}\right)^k
\]

\[
= -\frac{1}{N} \sum_{k=1}^{\infty} \left(\frac{\gamma}{A}\right)^k e^{-\pi ik/N} \sum_{n=1}^{N} (e^{2\pi ik/N})^n
\]

The last sum will always be zero unless \(k = mN\) for \(m \in \mathbb{Z}\). Since \(k \geq 1\) we must have \(m \geq 1\). Then

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 - \frac{A}{s_n}} = -\frac{1}{N} \sum_{m=1}^{\infty} \left(\frac{\gamma}{A}\right)^{mN} e^{-\pi im} \sum_{n=1}^{N} (e^{2\pi im})^n
\]

\[
= -\sum_{m=1}^{\infty} \left(\frac{\gamma}{A}\right)^{mN} (-1)^{-m}
\]

\[
= -\sum_{m=1}^{\infty} \left(-\left(\frac{A}{\gamma}\right)^N\right)^{-m}
\]

With \(|s_n| = \gamma\) we have \(|\frac{A}{\gamma}| > 1\) and so, using the earlier result (A.3) we can conclude that

\[
\frac{1}{N} \sum_{n=1}^{N} \frac{1}{1 - \frac{A}{s_n}} = \frac{1}{1 - \left(-\left(\frac{A}{\gamma}\right)^N\right)} = \frac{1}{1 + \left(\frac{A}{\gamma}\right)^N}
\]
Appendix A: Further Proofs for the LT Method

A.3 Symmetric Inversion Operator

In Section 2.5 we saw how the number of points required in the inversion operator could be halved when dealing with transforms of real functions. We had, from (2.22):

\[ \mathcal{L}_N^* \{ \hat{f} \} = \frac{2}{N} \sum_{n=1}^{N/2} \text{Re} \left\{ s_n \hat{f}(s_n) e_{N}^{*n} \right\} \]

We now derive this from a discretisation of the integral symmetry given in (2.21), as noted in Lynch (1991):

\[ \frac{1}{2\pi i} \oint_{C^*} e^{st} \hat{f}(s) \, ds = \frac{1}{\pi} \int_{C^+} \text{Im} \left[ e^{st} \hat{f}(s) \right] \, ds \]

Following the approach in Section 2.1.4 we define

\[ \mathcal{L}_N^+ \{ \hat{f}(s) \} = \frac{1}{\kappa'} \frac{1}{\pi} \sum_{n=1}^{N/2} \text{Im} \left[ e_{N}^{*n} \hat{f}(s_n) \Delta s_n \right] \]

Here \( \kappa' \) is the correction factor which may be needed to ensure exact inversion of a constant and will be determined below. From the definitions in (2.11) we can write \( \Delta s_n = 2i \tan (\pi/N) s_n \) and so

\[ \mathcal{L}_N^+ \{ \hat{f}(s) \} = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \sum_{n=1}^{N/2} \text{Im} \left[ i e_{N}^{*n} \hat{f}(s_n) s_n \right] \]

\[ = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \sum_{n=1}^{N/2} \text{Re} \left[ e_{N}^{*n} \hat{f}(s_n) s_n \right] \]

A.3.1 Inverting A Constant

As was done in Section 2.3 for the operator \( \mathcal{L}_N^* \) we examine the numerical inversion of the function \( \hat{f}(s) = \frac{1}{s} \) which inverts analytically to \( f(t) = 1 \):

\[ \mathcal{L}_N^+ \left\{ \frac{1}{s} \right\} = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \sum_{n=1}^{N/2} \text{Re} \left[ e_{N}^{*n} t \right] \]

\[ = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \sum_{n=1}^{N/2} \text{Re} \left[ \sum_{j=0}^{N-1} \frac{(\gamma t)^j}{j!} e^{i \pi (2n-1) j/N} \right] \]

\[ = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \text{Re} \left[ \sum_{j=0}^{N-1} \frac{(\gamma t)^j}{j!} e^{-i \pi j/N} \sum_{n=1}^{N/2} \left( e^{i 2 \pi j/N} \right)^n \right] \]

\[ = \frac{1}{\kappa'} \frac{1}{\pi} \tan (\pi/N) \text{Re} \left[ \frac{N}{2} + \sum_{j=1}^{N-1} \frac{(\gamma t)^j}{j!} e^{-i \pi j/N} \sum_{n=1}^{N/2} \left( e^{i 2 \pi j/N} \right)^n \right] \]
Appendix A: Further Proofs for the LT Method

As before we consider the final geometric series:

\[
\sum_{n=1}^{N/2} \left( e^{i2\pi j/N} \right)^n = e^{i2\pi j/N} \sum_{n=0}^{N/2-1} \left( e^{i2\pi j/N} \right)^n
\]

\[
= e^{i2\pi j/N} \frac{1 - \left( e^{i2\pi j/N} \right)^{N/2}}{1 - e^{i2\pi j/N}}
\]

\[
= e^{i2\pi j/N} \frac{1 - e^{i\pi j}}{1 - e^{i2\pi j/N}}
\]

\[
= e^{i2\pi j/N} \frac{1 - (-1)^j}{1 - e^{i2\pi j/N}}
\]

\[
= \frac{1 - (-1)^j}{e^{-i2\pi j/N} - 1}
\]

Previously when we summed \( N \) terms, this series vanished. Now however if we take the correction factor from Section 2.1.4 and set

\[
\kappa' = \frac{\tan \frac{\pi}{N}}{\pi}
\]

we get

\[
\mathcal{L}_N^+ \left\{ \frac{1}{s} \right\} = 1 + \frac{2}{N} \text{Re} \left[ \sum_{j=1}^{N-1} \frac{(\gamma t)^j}{j!} e^{-i\pi j/N} \frac{1 - (-1)^j}{e^{-i2\pi j/N} - 1} \right]
\]

\[
= 1 + \frac{2}{N} \text{Re} \left[ \sum_{j=1}^{N-1} \frac{(\gamma t)^j}{j!} e^{-i\pi j/N} - e^{i\pi j/N} \right]
\]

\[
= 1 + \frac{2}{N} \text{Re} \left[ \sum_{j=1}^{N-1} \frac{(\gamma t)^j}{j!} \frac{1 - (-1)^j}{-2i \sin(\pi j/N)} \right]
\]

\[
= 1
\]

The last line follows since all of the terms in the sum are purely imaginary. The inversion of a constant is therefore exact when we define

\[
\mathcal{L}_N^+ = \frac{2}{N} \sum_{n=1}^{N/2} \text{Re} \left[ e^{s_n^* t} \hat{f}(s_n) s_n \right] \quad (A.4)
\]

This is confirmed by using the scheme to solve for a constant, as in Section 2.3. The results are shown in Figure A.1. We now have \( \mathcal{L}_N^+ \) matching the symmetric \( \mathcal{L}_N^* \) operator from (2.22). We maintain the property of exact inversion of a constant function. We now verify that the filter response is the same as that of the full operator.
Appendix A: Further Proofs for the LT Method

A.3.2 Filter Response

We examine the effect of $\mathcal{L}_N^+$ on a wave component. In (2.18) we saw that

$$\mathcal{L}_N^+ \{e^{i\omega t}\} = H_N(\omega) e^{i\omega t}$$

To use the symmetry property in (2.21), we must have a real valued function. Therefore we consider $f(t) = \cos \omega t$, from which $\hat{f}(s) = \frac{s}{s^2 + \omega^2}$ and so

$$\mathcal{L}_N^+ \{\cos \omega t\} = \mathcal{L}_N^+ \left\{ \frac{s}{s^2 + \omega^2} \right\}$$

$$= \frac{2}{N} \sum_{n=1}^{N/2} Re \left[ e^{sn} \left\{ \frac{s^2}{s_n^2 + \omega^2} \right\} \right]$$

$$= \frac{2}{N} Re \left[ \sum_{n=1}^{N/2} \sum_{j=0}^{N-1} \frac{(s_n t)^j}{j!} \frac{1}{1 + \frac{\omega^2}{s_n^2}} \right]$$

Figure A.1: Comparing the LT solution with the exact for a constant function, with symmetry in the inversion integral
Appendix A: Further Proofs for the LT Method

For now we assume $|\omega s_n| < 1$ and we can then write $\left(1 + \frac{\omega^2}{s_n^2}\right)^{-1}$ as the sum of an infinite geometric series. Thus

\[
\Omega_N^+ \left\{ \frac{s}{s^2 + \omega^2} \right\} = \frac{2}{N} \text{Re} \left\{ \sum_{n=1}^{N/2} \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \frac{(s_n t)^j}{j!} \left( -\frac{\omega^2}{s_n^2} \right)^k \right\}
\]

\[
= \frac{2}{N} \text{Re} \left\{ \sum_{n=0}^{N/2-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} \left( -\omega^2 \right)^k (s_n t)^j \left( -\omega^2 \right)^k \right\}
\]

\[
= \frac{2}{N} \text{Re} \left\{ \sum_{n=0}^{N/2-1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{t^j}{j!} \left( -\omega^2 \right)^k \gamma_{j-2k} e^{i \pi (2n-1) (j-2k)/N} \right\}
\]

\[
= \frac{2}{N} \text{Re} \left\{ \sum_{j=0}^{N-1} \sum_{k=0}^{\infty} \frac{t^j}{j!} \left( -\omega^2 \right)^k \gamma_{j-2k} e^{i \pi (j-2k)/N} \sum_{n=0}^{N/2-1} \left( e^{i 2 \pi (j-2k)/N} \right)^n \right\}
\]

(A.5)

We can show that all the above terms will be zero unless $j - 2k$ is a multiple of $N$. If not, the terms at the end can be summed as a geometric series:

\[
e^{i \pi (j-2k)/N} \sum_{n=0}^{N/2-1} \left( e^{i 2 \pi (j-2k)/N} \right)^n = e^{i \pi (j-2k)/N} \frac{1 - e^{i \pi (j-2k)/N}}{1 - e^{i 2 \pi (j-2k)/N}}
\]

\[
= \frac{1 - (-1)^j}{e^{i \pi (j-2k)/N} - e^{i \pi (j-2k)/N}}
\]

\[
= i \frac{1 - (-1)^j}{2 \sin \left( \pi \frac{(j-2k)}{N} \right)}
\]

This term is purely imaginary. All other terms in (A.5) are real and so taking the real part of the sum gives us zero.

Now suppose we have $j - 2k = m'N$ for an integer $m'$. Since $0 \leq j \leq N - 1$ and $k \geq 0$ we must have $m' \leq 0$. We can then rewrite $k = (j + mN)/2$ where $m \geq 0$. This demands that $j$ be even, since $k$ is an integer and $N$ is even. Using this change
of indices, (A.5) becomes

\[ \mathcal{L}_N^+ \left\{ \frac{s}{s^2 + \omega^2} \right\} = \frac{2}{N} \text{Re} \left[ \sum_{j=0}^{N-1} \sum_{m=0}^{\infty} \frac{t^j}{j!} (-\omega^2)^{(j+mN)/2} \gamma^{-mN} e^{-i \pi m} \sum_{n=0}^{N/2-1} e^{-i 2\pi mn} \right] \]

\[ = \frac{2}{N} \text{Re} \left[ \sum_{j=0}^{N-1} \sum_{m=0}^{\infty} \frac{t^j}{j!} (-\omega^2)^{(j+mN)/2} \gamma^{-mN} (-1)^m \frac{N}{2} \right] \]

\[ = \text{Re} \left[ \sum_{j=0}^{N-1} \frac{(\omega t)^j}{j!} (-1)^{j/2} \sum_{m=0}^{\infty} \left( (-1)^{1+\frac{j}{2}} \left( \frac{\omega}{\gamma} \right)^N \right)^m \right] \]

When we choose \( N \) to be a multiple of 4, \( 1 + (N/2) \) is odd and so the final sum above is

\[ \sum_{m=0}^{\infty} \left( -\left( \frac{\omega}{\gamma} \right)^N \right)^m = \frac{1}{1 + \left( \frac{\omega}{\gamma} \right)^N} \]

since we have already assumed that \( \omega < |s_n| = \gamma \). Then

\[ \mathcal{L}_N^+ \left\{ \frac{s}{s^2 + \omega^2} \right\} = \text{Re} \left[ \frac{1}{1 + \left( \frac{\omega}{\gamma} \right)^N} \sum_{j=0}^{N-1} \frac{(\omega t)^j}{j!} (-1)^{j/2} \right] \]

\[ = \frac{1}{1 + \left( \frac{\omega}{\gamma} \right)^N} \left\{ 1 - \frac{\omega t}{2!} + \frac{(\omega t)^2}{4!} - \cdots - \frac{(\omega t)^{N-2}}{(N-2)!} \right\} \]

The factor multiplying the sum is \( H_N(\omega) \). Although there appears to be an \( i \) missing when compared with its original definition in (2.19), we insist that \( N \) is a multiple of 4 and so this doesn’t matter. The sum is the truncated series corresponding to \( \text{Re} \left[ e^{i \omega t} \right] \equiv \cos_N(\omega t) \). Thus we have

\[ \mathcal{L}_N^+ \mathcal{L} \{ \cos \omega t \} = H_N(\omega) \cos_N \omega t \quad (A.6) \]

The operator \( \mathcal{L}_N^+ \mathcal{L} \) acts just like the full \( \mathcal{L}_N^+ \mathcal{L} \): a sinusoidal wave component is truncated and its amplitude is modified by \( H_N \).
The proof has so far assumed that $|\omega_{sn}| < 1$. We now consider the case $|\omega_{sn}| > 1$.

We have

$$L_N^+ L \{\cos \omega t\} = \frac{2}{N} \sum_{n=1}^{N/2} Re \left[ e^{\frac{i t}{N}} \frac{s_n^2}{s_n^2 + \omega^2} \right]$$

$$= \frac{2}{N} Re \left[ \sum_{n=1}^{N/2} \sum_{j=0}^{N-2} \frac{(s_n t)^j}{j!} \frac{1}{1 + \omega^2 s_n^2} \right]$$

If $|\omega_{sn}| > 1$ we can use (A.3) to write

$$\frac{1}{1 + \frac{\omega^2}{s_n^2}} = -\sum_{k=1}^{\infty} \left( -\left( \frac{s_n}{\omega} \right)^2 \right)^k$$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \left( \frac{s_n}{\omega} \right)^{2k}$$

Thus

$$L_N^+ L \{\cos \omega t\} = \frac{2}{N} Re \left[ \sum_{n=1}^{N/2} \sum_{j=0}^{N-1} \sum_{k=1}^{\infty} \frac{t^j}{j!} (s_n t)^j (-1)^{k+1} \left( \frac{s_n}{\omega} \right)^{2k} \right]$$

$$= \frac{2}{N} Re \left[ \sum_{j=0}^{N-1} \sum_{k=1}^{\infty} \frac{t^j}{j!} (-1)^{k+1} \omega^{-2k} \gamma j^{2k} \sum_{n=1}^{N/2} e^{i \pi (2n-1)(j+2k)/N} \right]$$

We can write the final sum of exponentials as

$$e^{i \pi (j+2k)/N} \sum_{n=0}^{N/2-1} \left( e^{i 2\pi (j+2k)}/N \right)^n$$

Similar to the analysis in Section 2.5, whenever $j+2k$ is not a multiple of $N$, all terms will be purely imaginary and so taking the real part gives us zero.

Now suppose $j+2k = mN$ where $m$ is an integer. Since $k \geq 1$ and $0 \leq j \leq N-1$ we must have $m \geq 1$. This also demands the $j$ be even and we can write $k = (mN - j)/2$.

The summation above becomes

$$e^{i \pi m} \sum_{n=0}^{N/2-1} e^{i 2\pi m n} = (-1)^m \frac{N}{2}$$
Then

\[ \mathcal{L}_N^+ \mathcal{L} \{ \cos \omega t \} = \frac{2}{N} \text{Re} \left[ \sum_{j=0}^{N-1} \sum_{m=1}^{\infty} \frac{t^j}{j!} \left( -1 \right)^{mN-j+2/2} \omega^{-mN+j} \gamma^m N \left( -\frac{1}{2} \right)^m \right] \]

\[ = \text{Re} \left[ \sum_{j=0}^{N-1} \sum_{m=1}^{\infty} \frac{(\omega t)^j}{j!} \left( -1 \right)^{j/2} \left( -1 \right)^{(mN+2)/2} \left( -\frac{\gamma}{\omega} \right)^m \right] \]

\[ = \sum_{j=0}^{N-1} \frac{(\omega t)^j}{j!} \left( -1 \right)^{j/2} \left( -\sum_{m=1}^{\infty} \left( -\left( \frac{\gamma}{\omega} \right)^m \right) \right) \]

\[ = \frac{1}{1 + \left( \frac{\omega}{\gamma} \right)^N} \sum_{j=0}^{N-1} \frac{(\omega t)^j}{j!} \left( -1 \right)^{j/2} \]

In the above we have used the fact that \( N \) is chosen to be a multiple of 4 and so \((-1)^{(mN+2)/2} = -1\). The last line then follows by using (A.3) again. This final result is the same as that for the \(|\omega_s| < 1\) case and so we have again that

\[ \mathcal{L}_N^+ \mathcal{L} \{ \cos \omega t \} = H_N(\omega) \cos N \omega t \]

### A.4 Relative Phase Change

In Chapter 3, the relative phase change for the LT method is given by (3.14). Writing \( x = \nu \Delta t \) for simplicity, this is

\[ R_{LT} = \frac{1}{x} \tan^{-1} \left( \frac{\sin_N(x)}{\cos_N(x)} \right) \]

We now wish to consider the behaviour of \( R_{LT} \) for small \( x \). Since we always choose \( N \) to be a multiple of four, we can write

\[ \sin_N(x) = \sin x - \epsilon_s(x) \]

\[ \cos_N(x) = \cos x - \epsilon_c(x) \]

where

\[ \epsilon_s(x) \equiv \frac{x^{N+1}}{(N+1)!} + O(x^{N+3}) \]

\[ \epsilon_c(x) \equiv \frac{x^N}{N!} + O(x^{N+2}) \]

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Appendix A: Further Proofs for the LT Method

Now
\[
\frac{\sin_N(x)}{\cos_N(x)} = \frac{\sin x}{\cos x - \epsilon_c(x)} - \frac{\epsilon_s(x)}{\cos x - \epsilon_c(x)}
\]
\[
= \frac{\sin x}{\cos x \left(1 - \frac{\epsilon_c(x)}{\cos x}\right)} - \frac{\epsilon_s(x)}{\cos x \left(1 - \frac{\epsilon_c(x)}{\cos x}\right)}
\]
\[
\approx \tan x \left(1 + \frac{\epsilon_c(x)}{\cos x}\right) - \frac{\epsilon_s(x)}{\cos x} \left(1 + \frac{\epsilon_c(x)}{\cos x}\right)
\]  \hspace{1cm} (A.9)

With the definitions in (A.8) we can write
\[
\frac{\epsilon_s(x)}{\cos x} \approx \frac{x^{N+1}}{(N+1)!} \left(1 + \frac{x^2}{2!}\right) = \frac{x^{N+1}}{(N+1)!} + O(x^{N+3})
\]
\[
\frac{\epsilon_c(x)}{\cos x} \approx \frac{x^N}{N!} \left(1 + \frac{x^2}{2!}\right) = \frac{x^N}{N!} + O(x^{N+2})
\]

Using these with (A.9) yields
\[
\frac{\sin_N(x)}{\cos_N(x)} \approx \tan x \left(1 + \frac{x^N}{N!} + O(x^{N+2})\right)
\]
\[
- \left(\frac{x^{N+1}}{(N+1)!} + O(x^{N+3})\right) \left(1 + \frac{x^N}{N!} + O(x^{N+2})\right)
\]
\[
= \tan x + \frac{x^N}{N!} \tan x - \frac{x^{N+1}}{(N+1)!} + O(x^{N+2})
\]
\[
\approx \tan x + \frac{N}{(N+1)!} x^{N+1}
\]  \hspace{1cm} (A.10)

since \(\tan x = x + \frac{1}{3} x^3 + \ldots\), and we have ignored \(O(x^{N+2})\) terms. The addition formula for the inverse tan function gives
\[
\tan^{-1}(\tan x) + \tan^{-1}\left(\frac{N}{(N+1)!} x^{N+1}\right) = \tan^{-1}\left(\frac{\tan x + \frac{N}{(N+1)!} x^{N+1}}{1 - \frac{N}{(N+1)!} x^{N+1} \tan x}\right)
\]
\[
\approx \tan^{-1}\left(\tan x + \frac{N}{(N+1)!} x^{N+1}\right)
\]  \hspace{1cm} (A.11)

where in the final line we have ignored the \(O(x^{N+2})\) contribution from \(\frac{N}{(N+1)!} x^{N+1} \tan x\).
Combining (A.10) and (A.11) we can now return to (A.7) to get

\[
R_{LT} = \frac{1}{x} \tan^{-1} \left( \frac{\sin_N(x)}{\cos_N(x)} \right)
\]

\[
\approx \frac{1}{x} \tan^{-1}(\tan x) + \frac{1}{x} \tan^{-1} \left( \frac{N}{(N+1)!} x^{N+1} \right)
\]

\[
\approx 1 + \frac{N}{(N+1)!} x^N
\]

using the expansion \( \tan^{-1} x = x - \frac{1}{3} x^3 + \ldots \) and again ignoring higher order terms.

This result is given in (3.15).
Appendix B

Simple Applications of the LT Integration Technique

In Chapter 2 the Laplace transform integration technique was introduced and developed. Its properties as a low-pass filter were examined. Here we explore the application of the LT method to a system of ordinary differential equations.

B.1 The Swinging Spring

The elastic pendulum, or swinging spring, consists of a mass attached to one end of a spring. The spring can stretch but not bend. The other end of the spring is fixed. We denote the mass by \( m \), the spring constant by \( k \), the natural spring length by \( l_0 \) and the extended equilibrium length by \( l \). The system exhibits two motions: elastic oscillations (the spring) with frequency \( \omega_z = \sqrt{\frac{k}{m}} \) and a pendulum motion (the swing) with frequency \( \omega_R = \sqrt{\frac{g}{l}} \) where \( g \) is the acceleration due to gravity. We define the dimensionless ratio of frequencies

\[
\epsilon \equiv \frac{\omega_R}{\omega_z}
\]

When the spring is resting in equilibrium we have

\[
m g = k(l - l_0)
\]

This can be used to write

\[
\epsilon^2 = \frac{\omega_R^2}{\omega_z^2} = \frac{mg}{kl} = 1 - \frac{l_0}{l}
\]
Appendix B: Simple Applications of the LT Integration Technique

For a stiff spring with \( l \approx l_0 \) we have \( \epsilon^2 \ll 1 \); that is, \( \omega_R \ll \omega_z \). The system then consists of a combination of slow and fast motions and this makes it ideal for testing the filtering effect of the LT method.

Lynch (2002) explores the dynamics of the swinging spring in detail. In terms of a Cartesian coordinate system with the origin at the equilibrium position of the mass, the governing equations can be written as

\[
\begin{align*}
\frac{d^2 x}{dt^2} + \omega^2_R x + \left[ \omega^2_z \left( \frac{r - l_0}{r} \right) - \omega^2_R \right] x &= 0 \\
\frac{d^2 y}{dt^2} + \omega^2_R y + \left[ \omega^2_z \left( \frac{r - l_0}{r} \right) - \omega^2_R \right] y &= 0 \\
\frac{d^2 z}{dt^2} + \omega^2_z z + \left[ \omega^2_z \left( \frac{r - l_0}{r} \right) (z - l) + g - \omega^2_z z \right] &= 0
\end{align*}
\]  

(B.1)

where \( r = \sqrt{x^2 + y^2 + (z - l)^2} \) is the distance to the point of support.

In Section 2.2 the LT integration technique was formulated for the general vector equation

\[
\frac{dX}{dt} + LX + N(X) = 0
\]  

(B.2)

We define the state vector

\[
X(t) \equiv \left( x, \frac{dx}{dt}, y, \frac{dy}{dt}, z, \frac{dz}{dt} \right)^T
\]

We can then write (B.1) in vector form with

\[
L = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
\omega^2_R & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & \omega^2_R & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & \omega^2_z & 0
\end{pmatrix}
\]

and

\[
N(X) = \begin{pmatrix}
\left[ \omega^2_z \left( \frac{r - l_0}{r} \right) - \omega^2_R \right] x \\
0 \\
\left[ \omega^2_z \left( \frac{r - l_0}{r} \right) - \omega^2_R \right] y \\
0 \\
\omega^2_z \left( \frac{r - l_0}{r} \right) (z - l) + g - \omega^2_z z
\end{pmatrix}
\]
Appendix B: Simple Applications of the LT Integration Technique

B.2 Numerical Simulations

The LT forecasting method was given in (2.14) to solve the general vector system (B.2) with a centred scheme. This was implemented in Matlab to solve (B.1). The solutions were compared with those from Matlab’s ode45 Runge-Kutta method.

The initial conditions for the simulations presented here are

\[ \mathbf{X}(0) = (0.1, 0, 0, 0, 0) \]

The parameter values chosen are as follows: \( m = 1 \text{ kg}, \quad g = \pi^2 \text{ m s}^{-2}, \quad l = 1 \text{ m}, \quad k = 25 \pi^2 \text{ kg s}^{-2} \). With these we have \( \omega_R = \pi \text{ s}^{-1} \) and \( \omega_z = 5 \pi \text{ s}^{-1} \) so that \( \epsilon = 0.2 \).

We run for a time of 5s with a timestep \( \Delta t = 0.01 \text{s} \).

For the LT solution we use \( N = 8 \) for the numerical inversion. Since we are dealing with real quantities, we exploit the symmetry in the inversion and use the operator (2.22).

We plot solutions using the two methods. We first consider the unfiltered LT case; that is, we choose the cutoff frequency \( \gamma \) to be \( 40\text{s}^{-1} \), so as to greatly exceed \( \omega_z = 5 \pi \text{ s}^{-1} \). Thus we are not filtering the higher frequency and the solutions should match. These are plotted in Figure B.1, where we show the trajectories in the x-direction (top left), y-direction (top right) and z-direction (bottom left). On the bottom right we have the horizontal x-y projection.

We now investigate the filtering effect of the LT scheme. We choose \( \gamma = 10 \) so that \( \omega_R < \gamma < \omega_z \). With this we would expect to keep the low frequency behaviour while eliminating the faster oscillation in the vertical direction. This is evident in Figure B.2. We see the filtered behaviour in the z-direction (bottom left). For the x and y trajectories in the plots on the top, the slowly evolving mode is still present.

We note that for this centred LT method, a Robert-Asselin filter (Asselin, 1972) was needed to control the computational mode. It was not, however, required for the shallow water models used in the rest of this work.
Appendix B: Simple Applications of the LT Integration Technique

![Graphs showing trajectories and horizontal projection with Runge-Kutta (red-dashed) and unfiltered LT (blue) solutions.]

Figure B.1: Reference Runge-Kutta (red-dashed) and unfiltered LT (blue) solutions
Figure B.2: Reference Runge-Kutta (red-dashed) and filtered LT (blue) solutions
Appendix C

Details of Spectral Solutions

C.1 Spherical Harmonics

The spectral transform method involves the expansion of a model field as a truncated series of spherical harmonics. We first provide a brief overview of the relevant properties of spherical harmonics, as appropriate for spectral models. Further details may be found in Washington and Parkinson (2005). In the subsequent sections we discuss the details of the spectral discretisation of the two models used in this thesis: STSWM and SWEmodel.

Spherical harmonics are the eigenfunctions of Laplace’s equation on a sphere, here of radius $a$, satisfying

$$\nabla^2 Y_{m}^{\ell} = -\frac{\ell(\ell + 1)}{a^2} Y_{m}^{\ell} \quad \text{(C.1)}$$

They are defined by

$$Y_{m}^{\ell}(\lambda, \mu) = e^{i m \lambda} P_{m}^{\ell}(\mu)$$

where $\mu = \sin \phi$. Here $m$ is the zonal wavenumber and $\ell - |m|$ gives the number of zeros between the poles. The associated Legendre functions $P_{m}^{\ell}$ are the solutions of the second order associated Legendre equation

$$(1 - \mu^2)y'' - 2\mu y' + \left[\ell(\ell + 1) - \frac{m^2}{1 - \mu^2}\right] y = 0$$

They are given by the formula

$$P_{m}^{\ell}(\mu) = (1 - \mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_{\ell}$$
Appendix C: Details of Spectral Solutions

with the Legendre polynomials defined as

\[ P_\ell(\mu) = \frac{1}{2\ell!} \frac{d^\ell}{d\mu^\ell} [(\mu^2 - 1)^\ell] \]

The associated Legendre polynomials are orthogonal and satisfy

\[ \int_{-1}^{1} [P_\ell^m(\mu)]^2 d\mu = \frac{2}{2\ell + 1} \frac{(\ell + m)!}{(\ell - m)!} \]

Thus we can redefine them as follows in order that they normalise to one:

\[ P_\ell^m(\mu) = \left( \frac{2\ell + 1}{2} \frac{(\ell - m)!}{(\ell + m)!} \right)^{\frac{1}{2}} (1 - \mu^2)^{\frac{m}{2}} \frac{d^m}{d\mu^m} P_\ell \]

They have the following symmetry property

\[ P_{-\ell}^m(\mu) = (-1)^m P_\ell^m(\mu) \]

The following recurrence relationships will be used frequently:

\[ \mu P_\ell^m = \varepsilon_\ell^m P_{\ell-1}^m + \varepsilon_\ell^{m+1} P_{\ell+1}^m \quad (C.2) \]

\[ (1 - \mu^2) \frac{d}{d\mu} P_\ell^m = (\ell + 1) \varepsilon_\ell^m P_{\ell-1}^m - \ell \varepsilon_\ell^m P_{\ell+1}^m \quad (C.3) \]

where

\[ \varepsilon_\ell^m \equiv \sqrt{\frac{\ell^2 - m^2}{4\ell^2 - 1}} \]

C.2 STSWM Model

The Eulerian spectral transform STSWM is presented in Chapter 3. From (3.4) we see that the shallow water equations can be written as

\[ \frac{\partial \eta}{\partial t} = \frac{1}{a(1 - \mu^2)} \frac{\partial A}{\partial \lambda} - \frac{1}{a} \frac{\partial B}{\partial \mu} \]

\[ \frac{\partial \delta}{\partial t} = \frac{1}{a(1 - \mu^2)} \frac{\partial B}{\partial \lambda} - \frac{1}{a} \frac{\partial A}{\partial \mu} - \nabla^2 E - \nabla^2 \Phi' \quad (C.4) \]

\[ \frac{\partial \Phi'}{\partial t} = \frac{1}{a(1 - \mu^2)} \frac{\partial C}{\partial \lambda} - \frac{1}{a} \frac{\partial D}{\partial \mu} - \Phi' \delta \]
Appendix C: Details of Spectral Solutions

with

\[
\begin{align*}
A & \equiv U\eta \\
B & \equiv V\eta \\
C & \equiv U\Phi' \\
D & \equiv V\Phi' \\
E & \equiv \Phi_s + \frac{U^2 + V^2}{2(1 - \mu^2)}
\end{align*}
\]

For any particular model field \( \xi \) we expand as follows

\[
\xi(\lambda, \mu, t) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \xi_{\ell}^{m}(t) e^{im\lambda} P_{\ell m}(\mu)
\]

(C.5)

Here a triangular truncation has been used. Orthogonality of the Legendre functions can be used to obtain a particular spectral coefficient. We multiply by \( Y_{\ell m} = e^{-im\lambda} P_{\ell m}(\mu) \) and integrate over the range of \( \lambda \) and \( \mu \):

\[
\xi_{\ell}^{m} = \frac{1}{2\pi} \int_{-1}^{1} \int_{0}^{2\pi} \xi(\lambda, \mu) e^{-im\lambda} P_{\ell m}(\mu) d\lambda d\mu
\]

We examine each integral separately. Firstly,

\[
\xi^{m} = \frac{1}{2\pi} \int_{0}^{2\pi} \xi(\lambda, \mu) e^{-im\lambda} d\lambda
\]

can be evaluated using a Fast Fourier Transform method, with

\[
\xi^{m} = \frac{1}{T} \sum_{k=1}^{I} \xi(\lambda_k, \mu) e^{-im\lambda_k}
\]

(C.6)

where

\[
\lambda_k = \frac{2\pi k}{T}
\]

The second integral is

\[
\xi_{\ell}^{m} = \int_{-1}^{1} \xi^{m}(\mu) P_{\ell m}(\mu) d\mu
\]

To numerically evaluate this Gaussian quadrature is used, so that

\[
\xi_{\ell}^{m} = \sum_{j=1}^{J} \xi^{m}(\mu_j) P_{\ell m}(\mu_j) w_j
\]

(C.7)
Appendix C: Details of Spectral Solutions

Here, the $\mu_j$ are the Gaussian latitudes - the roots of $P_J(\mu)$. The corresponding weights are

$$w_j = \frac{2(1-\mu_j^2)}{(JP_{J-1}(\mu_j))^2},$$
satisfying $\sum_{j=1}^{J} w_j = 2$

To avoid aliasing in both transforms we require, for triangular truncation

$$J \geq \frac{3L+1}{2}$$
$$I \geq 3L + 1$$

In order to transform the right-hand side terms of (C.4) we need to consider the spectral form of the various differentiated terms. Using integration by parts we get the following results:

$$\left\{ \frac{\partial \xi}{\partial \lambda} \right\}_m = i m \xi^m$$
$$\left\{ \frac{1}{a} \frac{\partial \xi}{\partial \mu} \right\}_\ell = - \int_{-1}^{1} \frac{1}{a} \xi^m \frac{dP_m^m(\mu)}{d\mu} \, d\mu$$

and so

$$\left\{ \frac{1}{a(1-\mu^2)} \frac{\partial \xi}{\partial \lambda} \right\}_\ell = \sum_{j=1}^{J} i m \xi^m(\mu_j) \frac{P_m^m(\mu_j)}{a(1-\mu_j^2)} w_j$$
$$\left\{ \frac{1}{a} \frac{\partial \xi}{\partial \mu} \right\}_\ell = - \sum_{j=1}^{J} \xi^m(\mu_j) \frac{H_m^m(\mu_j)}{a(1-\mu_j^2)} w_j$$

where

$$H_m^m(\mu) \equiv (1-\mu^2) \frac{dP_m^m}{d\mu}$$

Using (C.1), we immediately obtain

$$\left\{ \nabla^2 \xi \right\}_\ell = - \frac{\ell(\ell+1)}{a^2} \sum_{j=1}^{J} \xi^m(\mu_j) P_m^m(\mu_j) w_j$$

For the spectral transform method we evaluate the nonlinear terms $A$ to $E$ on the Gaussian grid $(\lambda_k, \mu_j)$ and then transform the result. Using the preceding results, the system (C.4) becomes

$$\frac{\partial}{\partial t} \eta^m = \mathbf{N}^m$$
$$\frac{\partial}{\partial t} \delta^m = \mathbf{D}^m + \frac{\ell(\ell+1)}{a^2} \Phi^m$$
$$\frac{\partial}{\partial t} \Phi^m = \mathbf{F}^m - \bar{\Phi}^* \delta^m$$
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with

\[ N^m_\ell = - \sum_{j=1}^{J} [i m A^m(\mu_j) P^m_\ell(\mu_j) - B^m(\mu_j) H^m_\ell(\mu_j)] \frac{w_j}{a (1 - \mu_j^2)} \]

\[ D^m_\ell = - \sum_{j=1}^{J} [-i m B^m(\mu_j) P^m_\ell(\mu_j) - A^m(\mu_j) H^m_\ell(\mu_j)] \frac{w_j}{a (1 - \mu_j^2)} \]

\[ + \frac{\ell (\ell + 1)}{a^2} \sum_{j=1}^{J} E^m(\mu_j) P^m_\ell(\mu_j) w_j \]

\[ F^m_\ell = - \sum_{j=1}^{J} [i m C^m(\mu_j) P^m_\ell(\mu_j) - D^m(\mu_j) H^m_\ell(\mu_j)] \frac{w_j}{a (1 - \mu_j^2)} \]

This is the form of the equations in the STSWM model used in Chapter 3.

C.3 SLSI SWEmodel

The semi-Lagrangian SWEmodel in Chapter 4 also uses a spectral method, although in a slightly different manner. In the STSWM model outlined above, non-linear products are computed in physical space before every term is transformed to spectral space. We then have a system of ODEs to solve for the spectral coefficients.

For the semi-Lagrangian model, we first discretise the equations in time. Spectral transformations are then carried out and we can solve the resulting algebraic system for the spectral coefficients of the fields at the new time level.

C.3.1 General Spectral Coefficient

When discretised for the SLSI scheme, the vorticity equation takes the form

\[ \zeta + \frac{\Delta t}{2} f \delta + \frac{\Delta t}{2} \beta v = R \zeta \]

where the left-hand terms are at the arrival points at the new time level. As already mentioned in Section 4.4 we expand each term as a series of spherical harmonics

\[ \sum_{q=0}^{N} \sum_{p=-q}^{q} \left[ \zeta^p_{q} + \Omega \Delta t \mu \delta^p_{q} + \frac{\Omega \Delta t}{a} \sqrt{1 - \mu^2} \nu^p_{q} \right] Y^p_{q}(\lambda, \mu) = \sum_{q=0}^{N} \sum_{p=-q}^{q} [R \zeta^p_{q}] Y^p_{q}(\lambda, \mu) \]

An equation for a particular coefficient \( \zeta^m_\ell \) is then found by multiplying by the conjugate \( Y^m_{\ell} \) and integrating over the sphere.

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For the $\Delta t^2 f \delta$ term we get

$$\int_{-1}^{1} \frac{1}{2 \pi} \int_{0}^{2\pi} \left( \sum_{q=0}^{N} \sum_{p=-q}^{q} \frac{\Delta t}{2} (2 \Omega) \mu \delta^p_q Y^p_q \right) \nabla^m \lambda d\mu d\lambda$$

$$= \Omega \Delta t \sum_{q=0}^{N} \delta^m_q \int_{-1}^{1} P^m_q (\mu P^m_\ell) \, d\mu$$

$$= \Omega \Delta t \sum_{q=0}^{N} \delta^m_q \int_{-1}^{1} P^m_m (\varepsilon^m_{\ell+1} P^m_{\ell+1} + \varepsilon^m_{\ell-1} P^m_{\ell-1}) \, d\mu$$

$$= \alpha \varepsilon^m_{\ell+1} \delta^m_{\ell+1} + \alpha \varepsilon^m_{\ell} \delta^m_{\ell-1}$$

where $\alpha \equiv \Omega \Delta t$.

The wind in the $\beta$ term is written in terms of a stream function and velocity
potential, so that

$$\frac{\Delta t}{2} \beta v = \frac{\Delta t}{2} \beta \left( \frac{1}{a \cos \phi} \frac{\partial \psi}{\partial \lambda} + \frac{1}{a} \frac{\partial \chi}{\partial \phi} \right)$$

$$= \frac{\alpha}{a^2} \frac{\partial \psi}{\partial \lambda} + \frac{\alpha}{a^2} (1 - \mu^2) \frac{\partial \chi}{\partial \mu}$$

using $\beta = \frac{1}{a} \frac{\partial f}{\partial \phi} = \frac{2 \Omega}{a} \cos \phi$ and $\mu = \sin \phi$. The first term is expanded as follows

$$\int_{-1}^{1} \frac{1}{2 \pi} \int_{0}^{2\pi} \left( \frac{\alpha}{a^2} \sum_{q} \sum_{p} \left[ \frac{\partial}{\partial \lambda} \psi^p_q \right] Y^p_q \right) \nabla^m \lambda d\mu d\lambda = -\frac{im\alpha}{\ell(\ell+1)} \zeta^m$$

where we have used the fact that $\nabla^2 \psi = \zeta$ and so in terms of spectral coefficients, $\psi^m_\ell = -\frac{a^2}{\ell(\ell+1)} \zeta^m_\ell$. Similarly $\chi^m_\ell = -\frac{a^2}{\ell(\ell+1)} \delta^m_\ell$. 

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The expansion for the second term, $\frac{a^2}{a^2} (1 - \mu^2) \frac{\partial \chi}{\partial \mu}$, is

$$
\int_{-1}^{1} \frac{1}{2\pi} \int_{0}^{2\pi} \left( \frac{\alpha}{a^2} \sum_{q} \sum_{p} (1 - \mu^2) \frac{\partial}{\partial \mu} \left( - \frac{a^2}{q(q+1)} \delta_{q}^{p} \right) Y_{q}^{p} \right) Y_{\ell}^{m} d\lambda d\mu
$$

$$
= -\alpha \sum_{q} \left( \frac{1}{q(q+1)} \right) \delta_{q}^{m} \int_{-1}^{1} (1 - \mu^2) \frac{d}{d\mu} P_{q}^{m} P_{\ell}^{m} d\mu
$$

$$
= -\alpha \sum_{q} \left( \frac{1}{q(q+1)} \right) \delta_{q}^{m} \int_{-1}^{1} P_{\ell}^{m} \left( (q + 1) \delta_{q}^{m} P_{q-1}^{m} - q \delta_{q+1}^{m} P_{q+1}^{m} \right) d\mu
$$

$$
= -\alpha \delta_{\ell+1}^{m} \frac{\ell + 2}{(\ell + 1)(\ell + 2)} \delta_{\ell+1}^{m} + \alpha \delta_{\ell+1}^{m} \frac{\ell - 1}{(\ell - 1)\ell} \delta_{\ell-1}^{m}
$$

$$
= \frac{\alpha}{\ell} \delta_{\ell+1}^{m} \delta_{\ell-1}^{m} - \frac{\alpha}{\ell + 1} \delta_{\ell+1}^{m} \delta_{\ell+1}^{m}
$$

Combining all terms, we finally have an equation for a particular coefficient $\zeta_{\ell}^{m}$

$$
\left[ 1 - \frac{im\alpha}{\ell(\ell + 1)} \right] \zeta_{\ell}^{m} + \left[ \left( 1 + \frac{1}{\ell} \right) \alpha \zeta_{\ell}^{m} \right] \delta_{\ell+1}^{m} + \left[ \left( 1 - \frac{1}{\ell + 1} \right) \alpha \zeta_{\ell+1}^{m} \right] \delta_{\ell+1}^{m} = (R_{\zeta})_{\ell}^{m}
$$

Equations for $\delta_{\ell}^{m}$ and $\Phi_{\ell}^{m}$ can be derived in a similar manner.

C.3.2 Special Cases for Spectral System

The equations for $\zeta_{\ell}^{m}$, $\delta_{\ell}^{m}$ and $\Phi_{\ell}^{m}$ can be decoupled to give the system (4.8). This is solved for each $m$. We now consider special cases of this system which require separate treatment.

$m = N$

When $m = N$ the system reduces to a single equation for $\delta_{N}^{N}$:

$$
\left[ 1 - \frac{im\alpha}{N + 1} + \left( \frac{\Delta t}{2} \right)^2 \frac{N(N+1)}{a^2} + \frac{(\alpha \zeta_{N}^{N})^2 N(N+1)}{(N+1)^2 - \frac{1}{N+1}} \right] \delta_{N}^{N} = [R_{\delta}]_{N}^{N} + \frac{\Delta t N(N+1)}{2 \frac{N(N+1)}{a^2}} [R_{\Phi}]_{N}^{N}
$$

(C.9)
where we have used the fact that $\varepsilon^N_N = 0$ and $[F]_\ell^N = 0$ when $\ell > N$ for any field $F$.

Now we solve for the other two fields with

\[
\Phi^N_N = [R\Phi]^N_N - \frac{\Delta t}{2}\bar{\Phi}\delta^N_N
\]

\[
\zeta^N_N = \frac{[R\zeta]^N_N}{1 - \frac{\alpha}{N+1}}
\]

(C.10)

$m = 0, \ell = 0$

\[
\zeta^0_0 = [R\zeta]^0_0
\]

\[
\delta^0_0 = [R\delta]^0_0
\]

\[
\Phi^0_0 = [R\Phi]^0_0 - \frac{\Delta t}{2}\bar{\Phi}\delta^0_0
\]

$m = 0, \ell = 1$

\[
\left[1 + \left(\frac{\Delta t}{2}\right)^2 \frac{\alpha^2}{a^2}\bar{\Phi} + \frac{3}{4}(\alpha\varepsilon_2^0)^2\right] \delta_1^0 + \frac{1}{3}\alpha^2\varepsilon_2^0\varepsilon_3^0 \delta_3^0 = \right]

\[
[R\delta]^0_1 + \left(\frac{\Delta t}{2}\right)^2 \frac{\alpha^2}{a^2} [R\Phi]^0_1 + 2\alpha\varepsilon_1^0 [R\zeta]^0_0 + \frac{\alpha}{2}\varepsilon_2^0 [R\zeta]^0_2
\]

\[
\zeta_1^0 = [R\zeta]^0_1 - 2\alpha\varepsilon_1^0\delta_0^0 - \frac{\alpha}{2}\varepsilon_2^0\delta_2^0
\]

\[
\Phi_1^0 = [R\Phi]^0_1 - \frac{\Delta t}{2}\bar{\Phi}\delta_1^0
\]
Appendix C: Details of Spectral Solutions

\( m = 0, \ell = 2, 3, \ldots, N \)

\[
\left[ 1 + \left( \frac{\Delta t}{2} \right)^2 \Phi \frac{\ell (\ell + 1)}{a^2} + \frac{(\alpha \varepsilon_{\ell})^2 (\ell - 1)(\ell + 1)}{\ell^2} + \frac{(\alpha \varepsilon_{\ell+1})^2 \ell (\ell + 2)}{(\ell + 1)^2} \right] \delta_{\ell}^0
\]

\[
+ \left[ \frac{\alpha^2 (\ell + 1)\varepsilon_{\ell-1}^0 \varepsilon_{\ell}^0}{\ell - 1} \right] \delta_{\ell-2}^0 + \left[ \frac{\alpha^2 \ell \varepsilon_{\ell+1}^0 \varepsilon_{\ell+2}^0}{\ell + 2} \right] \delta_{\ell+2}^0
\]

\[
= [R_\delta]_\ell^0 + \frac{\Delta t}{2} \frac{\ell (\ell + 1)}{a^2} [R_\Phi]_\ell^0 + \frac{\alpha (\ell + 1)}{\ell} \varepsilon_{\ell}^0 [R_\zeta]_{\ell-1}^0 + \frac{\alpha \ell}{(\ell + 1)} \varepsilon_{\ell+1}^0 [R_\zeta]_{\ell-1}^0
\]

\[\Phi_{\ell}^0 = [R_\Phi]_\ell^0 - \frac{\Delta t}{2} \delta_{\ell}^0\]

\[\zeta_{\ell}^0 = \frac{\ell (\ell + 1)}{\ell (\ell + 1)} [R_\zeta]_{\ell}^0 - \alpha (\ell + 1) \varepsilon_{\ell+1}^0 \delta_{\ell+1}^0 - \alpha \ell^2 \varepsilon_{\ell+1}^0 \delta_{\ell+1}^0\]

\( m = -N \)

We use the fact that \( \varepsilon_{\ell}^{-m} = \varepsilon_{\ell}^m \):

\[
\left[ 1 + \frac{i \alpha}{N + 1} + \left( \frac{\Delta t}{2} \right)^2 \Phi \frac{N (N + 1)}{a^2} + \frac{(\alpha \varepsilon_{N+1})^2 N (N + 2)}{(N + 1)^2 (N + 2) + i \alpha N (N + 1)} \right] \delta_{-N}^N
\]

\[
= [R_\delta]_{-N}^N + \frac{\Delta t}{2} \frac{N (N + 1)}{a^2} [R_\Phi]_{-N}^N
\]

\[\Phi_{-N}^N = [R_\Phi]_{-N}^N - \frac{\Delta t}{2} \bar{\Phi} \delta_{-N}^N\]

\[\zeta_{-N}^N = \frac{[R_\zeta]_{-N}^N}{1 + \frac{\alpha}{N + 1}}\] (C.11)
Appendix D

Submitted Papers

The work reported in this thesis was submitted as a two-part paper to the Quarterly Journal of the Royal Meteorological Society in August 2010.

The first part of this paper, “Laplace transform integration of the shallow water equations. Part 1: Eulerian formulation and Kelvin waves”, describes the basic theory of the Laplace transform method from Chapter 2. The work with the STSWM model and Kelvin waves from Chapter 3 is then presented.

The second part, “Laplace transform integration of the shallow water equations. Part 2: Lagrangian formulation and orographic resonance”, reports on the work with the semi-Lagrangian LT method and orographic resonance, as covered here in Chapters 4 and 5.

The two parts are presented in the form submitted to QJRMS on the 20th of August 2010.
Laplace transform integration of the shallow water equations. Part 1: Eulerian formulation and Kelvin waves

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A filtering integration scheme is developed, using a modification of the contour used to invert the Laplace transform (LT). It is shown to eliminate components with frequencies higher than a specified cut-off value. Thus it is valuable for integrations of the equations governing atmospheric flow. The scheme is implemented in a shallow water model with an Eulerian treatment of advection. It is compared to a reference model using the semi-implicit (SI) scheme. The LT scheme is shown to treat dynamically important Kelvin waves more accurately than the SI scheme. Copyright © 0000 Royal Meteorological Society

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1. Introduction

The purpose of this work is to investigate a filtering integration technique suitable for application to a range of physical problems, in particular to numerical weather prediction. In operational NWP, efficiency is crucial, as we must produce regular and timely forecasts. In integrating the equations of motion, we need to use the longest timestep possible while still retaining acceptable accuracy.

Explicit finite difference schemes are limited by the CFL criterion, as stability is governed by the fastest waves present in the system. Fully implicit methods lead to complicated coupled nonlinear systems, which are impractical to solve in an operational context. The development of the semi-implicit method by Robert (1969) was a major breakthrough. By averaging terms leading to fast-moving gravity waves, Robert was able to achieve acceptable accuracy using a time step considerably larger than that required for explicit methods.

Despite these advances, there remain a number of issues with the semi-implicit method. In particular, the method maintains stability by slowing down the fast-moving waves in the system (see, e.g., Lynch, 2006, pp. 85–87). This may be problematic if we need to simulate a phenomenon that is influenced by such waves. Every discretisation technique invariably has its own strengths and weaknesses and there is still no ‘perfect’ scheme. It is important, therefore, that research into numerical methods for atmospheric models is continued. With this motivation we investigate a numerical scheme that offers some significant advantages over existing schemes.

High frequency noise has been a problem throughout the history of NWP. As outlined in Lynch (2006), this was the main cause of the failure of Richardson’s forecast. Various initialization techniques have been developed to address this problem. One such method, first presented in Lynch (1985a, 1985b), used a modified inversion to the Laplace transform (LT) to remove high frequency components from the initial conditions. The LT initialization scheme was reviewed in Daley (1991). In Van Isacker and Struylaert (1985), Lynch (1986) and Lynch (1991), this method was extended beyond initialization and a filtering time-stepping scheme was developed from the idea. The work presented in this paper further develops the LT discretisation as a viable numerical scheme for NWP. The implementation of the scheme is described and the benefits of the technique over existing schemes are demonstrated by a combination of analytical and numerical approaches.

This study is presented in two parts. In Part 1 we implement an LT scheme in a model using Eulerian advection and demonstrate its advantages for simulating atmospheric waves, in particular Kelvin waves. In Part 2 (Clancy and Lynch, 2010) we combine the LT scheme with a semi-Lagrangian advection scheme. We show that it is accurate and that it is free from the problem of orographic resonance that is found with semi-implicit schemes.
In §2, the background theory and mathematical formulation of the LT integration scheme is presented. After these preliminaries, we test the LT method’s efficacy as a numerical solver for the partial differential equations governing the atmosphere. In §3, a spectral model using the LT method for its temporal discretisation is developed, using an existing shallow water model (STSWM) as a basis and reference. The scheme is evaluated using various standard test cases. Along with a linear analysis in a simple oscillation equation, we perform shallow water simulations of Kelvin waves to investigate the effect of the LT scheme on phase speeds and, in §4, demonstrate its benefits over the semi-implicit scheme. Finally, a summary of the main results and conclusions is given in §5.

2. The Laplace transform integration method

2.1. Basic definitions

Given a function \( f(t) \) with \( t \geq 0 \), the Laplace transform (LT) is defined as

\[
\hat{f}(s) \equiv \mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t) \, dt
\]

where the contour \( C \) is a line parallel to the imaginary axis in the s-plane, to the right of all the singularities of \( \hat{f} \). Further theory and applications of the Laplace transform may be found in Doetsch (1971).

The ability of the Laplace transform to filter high frequencies is illustrated by the simple example of a function consisting of a slow and a fast oscillation. We define \( f(t) = a e^{\nu_R t} + A e^{i \nu_G t} \) with \( |\nu_R| \ll |\nu_G| \). The LT of this function is given by

\[
\hat{f}(s) = \frac{a}{s - i \nu_R} + \frac{A}{s - i \nu_G}. \tag{2}
\]

The function \( \hat{f} \) has two simple poles on the imaginary axis, at \( s = i \nu_R \) and \( s = i \nu_G \). To invert this to \( f(t) \) we would normally use the inversion integral (2) along the straight line \( C \) shown in Figure 1 (left panel).

To remove the high frequency component, we choose a positive real number \( \gamma \) such that \( |\nu_R| < \gamma < |\nu_G| \). Then we define a closed contour \( C^* \) as the circle centred at the origin, with radius \( \gamma \), as depicted on the left in Figure 1. We replace \( C \) by \( C^* \) in the integral in (2), yielding the modified inversion

\[
f^*(t) \equiv \mathcal{L}^*\{\hat{f}\} = \frac{1}{2\pi i} \oint_{C^*} e^{st} \hat{f}(s) \, ds \tag{3}
\]

The function \( f^*(t) \) contains only contributions from the poles lying within \( C^* \), that is, those with frequencies less than \( \gamma \). From Cauchy’s Integral Formula we readily find that

\[
f^*(t) = a e^{\nu_R t}
\]

Thus, the modified inversion integral (3) acts to filter high frequency behaviour, as required.

2.2. Laplace transform integration

The LT method was originally used as an initialisation technique (Lynch, 1985a,b). The extension to time integration was studied in Van Isacker and Struyfelaert (1985, 1986) and Lynch (1986, 1991). The basic idea is to consider the LT over a discrete interval of time \( \Delta t \). The transforms can be computed analytically and the modified inversion operator (3) is applied to find a filtered value at the end of the interval. We consider the transform of the general equation

\[
\frac{dX}{dt} + LX + N(X) = 0
\]

where \( L \) is a linear operator and \( N \) a nonlinear vector function, and rearrange to get

\[
\hat{X} = (sI + L)^{-1} [X^0 - N^0/s] \tag{4}
\]

The initial value is \( X^0 \) and we have held the nonlinear term at its initial value \( N^0 \). We apply the inversion operator at time \( t = \Delta t \) to get the filtered state at this time

\[
X(\Delta t) = \mathcal{L}^*\left\{\hat{X}\right\}_{t=\Delta t}
\]

Having the solution at \( t = \Delta t \) we continue stepwise to extend the forecast. In general we consider the time interval \( [\tau\Delta t, (\tau + 1)\Delta t] \). The filtered solution at time \( (\tau + 1)\Delta t \) is found by applying the modified inversion to the LT of the equation. Over this general interval, the ‘initial condition’ as given in (4) will be taken at the beginning of the interval, that is, \( X^\tau \equiv X(\tau\Delta t) \). The nonlinear terms are also evaluated at this time. Thus the solution at time \( (\tau + 1)\Delta t \) is

\[
X^{\tau+1} = \mathcal{L}^*\{ (sI + L)^{-1} [X^\tau - N^\tau/s]\}_{t=\Delta t}
\]

Alternatively, a centred approach may be taken, where we consider the interval \( [\tau\Delta t, (\tau + 1)\Delta t] \) and the nonlinear terms are evaluated at the centre \( \tau\Delta t \). The general forecasting procedure is thus as follows:

\[
\hat{X}(s) = (sI + L)^{-1} [X^{\tau-1} - N^{\tau-1}/s]
\]

\[
X^{\tau+1} = \mathcal{L}^*\left\{\hat{X}\right\}_{t=2\Delta t} \tag{5}
\]

Care must be taken to ensure that \( (sI + L)^{-1} \) exists. The matrix \( sI + L \) is singular when we have \( s = -\lambda \), for \( \lambda \) an eigenvalue of \( L \). But \( |s| = \gamma \), the radius of the contour \( C^* \). The problem can thus be avoided by a suitable choice of \( \gamma \), the cutoff frequency.

2.3. Evaluating the contour integral

The inversion using \( \mathcal{L}^* \) requires the complex integration in (3), around the circle \( C^* \). To apply the filter in practice, we replace \( C^* \) by the \( N \)-sided polygon \( C_N^* \) to reduce the integration to a summation. The length of each edge is \( \Delta s_n \) and the midpoints are labelled \( s_n \) for \( n = 1, 2, \ldots, N \). The right panel of Figure 1 shows the case with \( N = 8 \).

We can now define the numerical operator used for the modified inversion as

\[
\mathcal{L}^*_N\{\hat{f}\} \equiv \frac{1}{2\pi i} \sum_{n=1}^N e^{s_n t} \hat{f}(s_n) \Delta s_n
\]
It was suggested in Van Isacker and Struylaert (1985) that the exponential term in the expression for the numerical inversion should be replaced by a Taylor series truncated to \( N \) terms. We write

\[
e_Z^N = \sum_{j=0}^{N-1} e^{j \omega t} \tag{6}
\]

If we divide the summation in the numerical inversion by

\[
\kappa = \frac{N}{\pi} \tan \frac{\pi}{N}
\]

then the inversion is exact for a constant function, and for any power of \( t \) up to degree \( N - 1 \) (Clancy, 2010).

As noted in Lynch (1991), it can be shown that

\[
1/\kappa = (2\pi i/N) (s_n/\Delta s_n)
\]

so the final form of the numerical filtering inversion integral to be used is

\[
\mathcal{L}_N \{ f \} \equiv \frac{1}{N} \sum_{n=1}^{N} e^{s_n t} f(s_n) s_n
\]

(7)

2.4. Filter response and stability

With the inversion operator \( \mathcal{L}_N \) defined by (7), we consider the effect of the filtering operator \( \mathcal{L}_N \mathcal{L} \) on a single wave component \( f(t) = e^{i\omega t} \). This was analysed by Van Isacker and Struylaert (1985) and Lynch (1986), who showed that

\[
\mathcal{L}_N \mathcal{L} \{ e^{i\omega t} \} = H_N(\omega) e^{i\omega t} \tag{8}
\]

where

\[
H_N(\omega) = \frac{1}{1 + \left( \frac{\omega}{\gamma} \right)^2} \tag{9}
\]

If we always choose a value for \( N \) that is a multiple of 4, we ensure that \( H_N(\omega) \) is real and \( |H_N(\omega)| \leq 1 \). Thus its effect is to damp the input, without phase shift. In addition, the operator \( \mathcal{L}_N \mathcal{L} \) truncates the original \( e^{i\omega t} \) to \( N \) terms. We note the \( H_N \) is the square of the response function of a Butterworth lowpass filter (Oppenheim and Schafer, 1989).

Lynch (1986) showed how, when the centred LT method given by (5) is used, the response above yields the sufficient stability criterion

\[
\Delta t \leq \left( \frac{N!}{2\gamma} \right)^{1/N} \tag{10}
\]

This is a very lenient condition. With typical value \( N = 8 \) and a cut-off frequency defined by a period \( \tau_c = 6 \) hours, we get a maximum timestep of around 1.8 hours, longer than would normally be used in practice.

3. The spectral transform shallow water model

We now test the performance of the LT integration scheme in a shallow water model. A key benefit of the LT method is stability, with its potential to allow long timesteps to be used. We will compare it with a reference semi-implicit method.

When the shallow water equations are discretised with a semi-implicit scheme, one obtains a Helmholtz equation that needs to be solved at every timestep. Clearly an efficient solver is essential. When the LT method is applied, we encounter an analogous Helmholtz equation. Whereas the semi-implicit method requires the solution of the equation once every timestep, for the LT scheme we must solve it at each of the \( N \) midpoints on an \( N \)-gon. It is vital that the benefits of the LT scheme are not negated by the extra computational overhead. This motivates the coupling of the LT scheme with the spectral transform method, for which the solution of a Helmholtz equation is simple and efficient.

The spectral transform method uses spherical harmonics as basis functions for expansion of the model fields. Spherical harmonics are the eigenfunctions of Laplace’s equation and satisfy

\[
\nabla^2 Y_{\ell m} = -\frac{\ell(\ell + 1)}{a^2} Y_{\ell m} \tag{11}
\]

where \( a \) is the radius of the Earth. Writing \( \mu = \sin \phi \), they are defined by \( Y_{\ell m}(\lambda, \mu) = e^{im\lambda} P_{\ell m}^m(\mu) \). The \( P_{\ell m}^m \) are the associated Legendre functions. Washington and Parkinson (2005) provide the further details of spherical harmonics that are necessary for the spectral transform method.

Examining (11) we see that computing the Laplacian of a series of spherical harmonics merely requires scalar multiplications. The solution of a Helmholtz equation is therefore computationally trivial. This provides the motivation for using the transform method with a LT time integration.

3.1. STSWM

The Spectral Transform Shallow Water Model (STSWM) is a freely available model developed at the National Center for Atmospheric Research (NCAR) and described in Hack and Jakob (1992). It is designed to solve the shallow water equations using a spectral transform method and specifically to consider the test suite of Williamson et al. (1992). The original code is written in Fortran 77. An updated version in Fortran 90 was developed by the ICON group at the Max Planck Institute for Meteorology (MPI-M) and the Deutscher Wetterdienst (DWD) [http://icon.enes.org/].

We now provide a brief overview of the model’s discretisation. Full details are given in the report of Hack and Jakob. Jakob et al. (1993) specifically describe the changes needed to include orography in the model.

The shallow water equations are given in the form

\[
\begin{align*}
\frac{\partial \bar{\eta}}{\partial t} &= -\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} (U \bar{\eta}) - \frac{1}{a} \frac{\partial}{\partial \mu} (V \bar{\eta}) \\
\frac{\partial \bar{s}}{\partial t} &= -\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} \bar{V} - \frac{1}{a} \frac{\partial}{\partial \mu} \bar{U} \\
\nabla^2 \left( \Phi_s + \Phi_f \right) &= \frac{U^2 + V^2}{2(1-\mu^2)} \\
\frac{\partial \Phi_f}{\partial t} &= -\frac{1}{a(1-\mu^2)} \frac{\partial}{\partial \lambda} (U \Phi_f) - \frac{1}{a} \frac{\partial}{\partial \mu} (V \Phi_f) - \Phi^* \delta
\end{align*}
\]

Here \((U, V) = (u \cos \phi, v \cos \phi)\) are the horizontal wind images, \(\mu = \sin \phi, \eta = \zeta + f\) is the absolute vorticity and \(\delta\) is the horizontal divergence. The free surface geopotential
has been written as \( \Phi = \Phi^* + \Phi' - \Phi_s \), where \( \Phi^* \) is a time-independent spatial mean geopotential depth and \( \Phi_s \) is the geopotential of the surface of the Earth.

All of the fields are represented as truncated series of spherical harmonics; for example, with a triangular truncation,

\[
\eta(\lambda, \mu, t) = \sum_{l=0}^{L} \sum_{m=-l}^{l} \eta^{lm}_{\ell}(t) e^{im\lambda} P^m_l(\mu)
\]

Here \( \eta^{lm}_{\ell} \) are the time-dependent spectral coefficients. For the spectral transform method, the nonlinear terms on the right-hand side of (12) are computed in physical space and the product is then expanded in a series. Orthogonality of the spherical harmonics can then be used to obtain a series of equations for the spectral coefficients. We are left with a set of ordinary differential equations of the form

\[
\begin{align*}
\frac{\partial}{\partial t} \eta^{lm}_{\ell} &= N^{lm}_{\ell} \\
\frac{\partial}{\partial t} \delta^{lm}_{\ell} &= D^{lm}_{\ell} + \frac{\ell(\ell+1)}{a^2} \Phi^{lm}_{\ell} \\
\frac{\partial}{\partial t} \Phi^{lm}_{\ell} &= F^{lm}_{\ell} - \Phi^* \Phi^{lm}_{\ell}
\end{align*}
\]

(13)

Note that the \( \Phi^{lm}_{\ell} \) are the spectral coefficients of the perturbation geopotential \( \Phi' \); the prime has been dropped for ease of notation.

### 3.2. STSWM: Semi-implicit scheme

We use the semi-implicit STSWM as the reference model in this work. The discretisation of (13) is given by

\[
\begin{align*}
\frac{\eta^{lm}_{\ell}}{2 \Delta t} + \frac{\eta^{lm}_{\ell}}{2 \Delta t} &= \{N^{lm}_{\ell}\}^T \\
\frac{\delta^{lm}_{\ell}}{2 \Delta t} + \frac{\delta^{lm}_{\ell}}{2 \Delta t} &= \{D^{lm}_{\ell}\}^T + \frac{\ell(\ell+1)}{a^2} \Phi^{lm}_{\ell} + \frac{\ell(\ell+1)}{a^2} \Phi^{lm}_{\ell} \\
\frac{\Phi^{lm}_{\ell}}{2 \Delta t} - \Phi^{lm}_{\ell} &= \{F^{lm}_{\ell}\}^T
\end{align*}
\]

Here the superscript \( \tau \) represents the discrete time level \( \ell = \tau \Delta t \). The decoupling of the expressions for \( \delta^{lm}_{\ell} \) and \( \Phi^{lm}_{\ell} \) is simplified due to the spectral form of the Laplacian operator. The final time-stepping procedure can then be written as

\[
\begin{align*}
\eta^{lm}_{\ell} &= \eta^{lm}_{\ell} + 2 \Delta t \{N^{lm}_{\ell}\}^T \\
\delta^{lm}_{\ell} &= \frac{1}{d} \left( \mathcal{R} + \frac{\ell(\ell+1)}{a^2} \mathcal{D} \right) \\
\Phi^{lm}_{\ell} &= \frac{1}{d} \left( \mathcal{Q} - \mathcal{R} \Phi^* \right)
\end{align*}
\]

(14)

where

\[
\begin{align*}
d &= 1 + \Phi^* \frac{\ell(\ell+1)}{a^2} \Delta t^2 \\
\mathcal{R} &= \{\delta^{lm}_{\ell}\}^T + 2 \Delta t \{D^{lm}_{\ell}\}^T + \Delta t \left( \frac{\ell(\ell+1)}{a^2} \Phi^{lm}_{\ell} \right)^T \\
\mathcal{Q} &= \{\Phi^{lm}_{\ell}\}^T + 2 \Delta t \{F^{lm}_{\ell}\}^T + \Delta t \Phi^* \{\delta^{lm}_{\ell}\}^T
\end{align*}
\]

### 3.3. STSWM: Laplace transform formulation

We now adapt the STSWM code to solve the shallow water equations using the LT method. Again we consider the system of equations (13) for the time-dependent spectral coefficients. We take the Laplace transform of each equation, as described in §2.2:

\[
\begin{align*}
s \eta^{lm}_{\ell} &= \{\eta^{lm}_{\ell}\}^{\tau+1} = \frac{1}{s} \{N^{lm}_{\ell}\}^T \\
s \delta^{lm}_{\ell} &= \{\delta^{lm}_{\ell}\}^{\tau+1} = \frac{1}{s} \{D^{lm}_{\ell}\}^T + \frac{\ell(\ell+1)}{a^2} \Phi^{lm}_{\ell} \\
s \Phi^{lm}_{\ell} &= \{\Phi^{lm}_{\ell}\}^{\tau+1} = \frac{1}{s} \{F^{lm}_{\ell}\}^T - \Phi^* \delta^{lm}_{\ell}
\end{align*}
\]

As outlined previously, we are taking our ‘initial’ value at the beginning of the time step, i.e. at \( \ell = (\tau - 1) \Delta t \). The nonlinear terms \( N^{lm}_{\ell}, D^{lm}_{\ell} \) and \( F^{lm}_{\ell} \) are evaluated at the middle time level \( \tau \). By taking the transforms of the linear right-hand terms in the divergence and continuity equations, we get a coupled system analogous to that for the semi-implicit discretisation. We can solve it to get

\[
\begin{align*}
s \eta^{lm}_{\ell} &= \{\eta^{lm}_{\ell}\}^{\tau+1} + \frac{1}{s} \{N^{lm}_{\ell}\}^T \\
s \delta^{lm}_{\ell} &= \frac{1}{d} \left( \mathcal{R} + \frac{1}{s} \frac{\ell(\ell+1)}{a^2} \mathcal{Q} \right) \\
s \Phi^{lm}_{\ell} &= \frac{1}{d} \left( \mathcal{Q} - \frac{1}{s} \Phi^* \mathcal{R} \right)
\end{align*}
\]

(15)

where

\[
\begin{align*}
d' &= 1 + \Phi^* \frac{\ell(\ell+1)}{a^2} \Delta t \\
\mathcal{R}' &= \{\delta^{lm}_{\ell}\}^{\tau+1} + \frac{1}{s} \{D^{lm}_{\ell}\}^T \\
\mathcal{Q}' &= \{\Phi^{lm}_{\ell}\}^{\tau+1} + \frac{1}{s} \{F^{lm}_{\ell}\}^T
\end{align*}
\]

Comparing (14) and (15), we find close similarities between the two discretisations. Once we have computed the terms in (15), we use the inversion operator \( \mathcal{S}^\tau \) to compute the spectral coefficients at the new time \( (\tau + 1) \Delta t \), which is at a time \( 2 \Delta t \) after the beginning of the time interval; for example

\[
\{\eta^{lm}_{\ell}\}^{\tau+1} = \mathcal{S}^\tau \{\eta^{lm}_{\ell}\}^{\tau+1} = \frac{1}{N} \sum_{n=1}^{N} s_n \eta^{lm}_{\ell}(s_n) e^{2 \Delta t s_n}
\]

### 3.4. Numerical simulations

To compare the LT method with the reference semi-implicit scheme, we use the test case suite of Williamson et al. (1992). In particular we consider Case 1 (advection of a cosine bell), Case 2 (steady zonal flow), Case 5 (flow over a mountain) and Case 6 (Rossby-Haurwitz wave).

A number of normalised error measurements are used for ease of comparison. Williamson et al. (1992) recommend the \( l_1(h) \), \( l_2(h) \) and \( l_\infty(h) \) quantities, where \( l_1(h) \) is the mean absolute difference from the reference, \( l_2(h) \) is the root mean square difference, and \( l_\infty(h) \) is the maximum absolute difference, each normalized by the appropriate measure of the reference solution. Normalised
Laplace transform integration

invariants for mass and for total energy are also used to study the conservation properties of the schemes.

The LT version of STSWM was compared to the original semi-implicit version (the reference model) using the test cases outlined above. Unless otherwise stated, tests were carried out at a spectral T42 resolution with a 1200 second timestep. The default cutoff period for the LT method is \( \tau_c = 6 \) hours. For the inversion we test both \( N = 8 \) and \( N = 16 \). For Cases 5 and 6, fourth order diffusion is used in the semi-implicit runs, with the diffusion coefficients recommended by Jakob et al. (1993): for T42 simulations this is \( 5.0 \times 10^{-13}\text{m}^4\text{s}^{-1} \).

For Cases 1 and 2, all schemes performed with high accuracy, and all were of comparable precision (for details, see Clancy (2010)). For Case 5, flow over an isolated mountain, there is no analytical solution. We compute errors by taking the ‘true’ solution from a T213 \( \Delta t = 360\) s reference run with a diffusion coefficient of \( 8 \times 10^{-12}\text{m}^4\text{s}^{-1} \). The errors for the T42 simulations, plotted in Figure 2 (left panel: \( l_2 \) error; right panel: \( l_\infty \) error), are of comparable magnitude for the reference and for the two LT forecasts (using \( N = 8 \) and \( N = 16 \) points). The three forecasts showed an almost identical decrease in mass, though at a negligible magnitude of \( O(10^{-13}) \) after 15 days. The deviation from energy conservation was also negligible for the three forecasts (Clancy, 2010).

For the Rossby-Haurwitz wave of Case 6, Jakob et al. (1993) recommend using shorter timesteps than for the other cases, due to the strong winds involved. The high-resolution ‘true’ solution is given by a T213 run with \( \Delta t = 180\) s. The T42 simulations are run with a timestep of 600 seconds. Errors are plotted in Figure 3. In this case we also ran two more LT forecasts, again using \( N = 8 \) and \( N = 16 \) but with a shorter cutoff period of 3 hours. We see that these forecasts are much closer to the reference than those with the 6 hour cutoff. All runs are comparable in terms of (negligible) mass loss. The LT \( \tau_c = 3 \) runs are best for energy conservation.

### 4. Simulation of Kelvin waves

Semi-implicit methods are popular due to their attractive stability properties. This is achieved at the expense of a slowing of the faster waves present in the system. This is not a serious issue if one is only interested in slower modes. There may, however, be cases where we wish to accurately simulate some of the faster waves. In these situations the semi-implicit approach may not be ideal. We investigate the effect of semi-implicit averaging on phase speed in the simplest context, and compare it to results using the LT discretisation. We then confirm the analytical results by simulating a Kelvin wave using the LT and semi-implicit schemes.

#### 4.0.1. Phase error analysis

We begin with the one-dimensional oscillation equation

\[
\frac{du}{dt} = i \nu u
\]  

(16)

We follow the approach of Durran (1999) when analysing the two methods. We seek a numerical amplification factor, \( A \), such that \( u^{t+1} = A u^t \). Writing \( A = |A| e^{i \vartheta} \), a sufficient criterion for stability is given by \( |A| \leq 1 \). The phase is given by \( \vartheta = \tan^{-1} (\Im(A)/\Re(A)) \). Following Durran, we define a relative phase change

\[
R = \frac{\vartheta}{\nu \Delta t}
\]

(17)

A numerical scheme is decelerating if \( R < 1 \).

We will compute the relative phase change for the semi-implicit scheme

\[
\frac{u^{\tau+1} - u^\tau}{\Delta t} = i \nu \frac{u^{\tau+1} + u^\tau}{2}
\]

Evaluating the relative phase change \( R_{SI} \) using (17) we get

\[
R_{SI} = \frac{1}{\nu \Delta t} \tan^{-1} \left( \frac{\nu \Delta t}{1 - \nu^2 \Delta t^2 / 4} \right)
\]

\[
\approx 1 - \frac{(\nu \Delta t)^2}{12}
\]

(18)

for small values of \( \nu \Delta t \). Clearly, the semi-implicit scheme decelerates waves.

We next apply the LT method to (16), to get

\[
s \hat{u} - u^\tau = i \nu \hat{u}
\]

Inverting analytically with the full integral \( \Sigma^{-1} \) over a \( \Delta t \) interval yields \( u^{\tau+1} = u^\tau e^{i \nu \Delta t} \). Thus we have an exact representation of the frequency. With the numerical inversion operator \( \Sigma^*_N \), we get

\[
u^{\tau+1} = u^\tau H_N(\nu) e^{i \nu \Delta t}
\]

The relative phase change is given by

\[
R_{LT} = \frac{1}{\nu \Delta t} \tan^{-1} \left( \frac{\sin_N(\nu \Delta t)}{\cos_N(\nu \Delta t)} \right)
\]

(19)

where \( \cos_N(\nu \Delta t) \) and \( \sin_N(\nu \Delta t) \) denote, respectively, the real and imaginary parts of \( e^{i \nu \Delta t} \). Taking a Taylor series yields

\[
R_{LT} \approx 1 + \frac{N}{(N + 1)!} (\nu \Delta t)^N
\]

(20)

The details are given in Clancy (2010). The LT method gives a highly precise representation of phase speed, with an error due only to the discretisation of the inversion operator. This is clearly far more accurate than that for the semi-implicit case in (18). If, for example, we use \( N = 8 \) we get

\[
R_{LT} \approx 1 + \frac{(\nu \Delta t)^8}{45360}
\]

The scheme is marginally accelerating, but by a negligible amount.

In the next subsection we will consider a Kelvin wave with zonal wavenumber 5, with a period of about 6.7 hours. With a timestep of 30 minutes, the error in the semi-implicit scheme is then \( R_{SI} \approx 0.98 \) while a one hour timestep yields \( R_{SI} \approx 0.92 \). For the LT scheme with a 1800 second timestep, the error is \( R_{LT} \approx 1.0000005 \), completely negligible.
4.0.2. Numerical integration for the Kelvin wave

We now investigate the performance of semi-implicit and LT schemes in simulating Kelvin waves. These are eastward propagating waves, characterised by almost vanishing meridional wind. They are symmetric about the equator and decay with increasing latitude. They are known to play an important role in a number of atmospheric phenomena. Holton (1975) discusses their role in the dynamics of the Quasi-Biennial Oscillation (QBO) in the stratosphere. A comprehensive review may be found in Baldwin et al. (2001). Kelvin waves have also been shown to be important for the Madden-Julian Oscillation (Zhang, 2005). It is clearly vital, therefore, that these waves are accurately simulated.

Kasahara (1976) provides a description of the Hough modes along with details and code of a numerical method to produce them. This was used to generate initial conditions for STSWM. Since Hough modes are eigenfunctions for the linearised equations, they propagate almost linearly, without change of form, for small amplitudes.

For varying zonal wavenumbers \( m \) we compute the frequency of the Kelvin wave using the method of Kasahara. We use this to plot the relative phase changes for the semi-implicit and the LT method, given in (18) and (19) respectively. Figure 4 shows the errors plotted against the timestep for two wavenumbers: \( m = 1 \) and \( m = 5 \). For the two cases, the LT method (heavy black solid and dashed lines) are indistinguishable and appear to be almost exact.

The deceleration is evident in the semi-implicit method (thin solid and dashed lines). As seen from (18), the slowing effect increases with larger timesteps and for the higher frequency of the \( m = 5 \) wave.

We compared three numerical simulations of Kelvin wave with zonal wave number \( m = 5 \). A mean height of 10 km was used with a wave perturbation amplitude of 100 m. For this value, the period is approximately 6.7 hours. All runs were carried out at a T63 spectral resolution. Figure 5 shows the hourly height at a single point close to the equator, \((0.0^\circ E, 0.9^\circ N)\), over the first 10 hours of the forecasts at \( \Delta t = 1800 \) s. Here the phase speed differences are easily seen. The solid line marked ‘Exact’ is a sinusoidal wave with a 6.7 hour period, representing the analytical solution. Both LT forecasts, \( N = 8 \) (dashed line) and \( N = 16 \) (dot-dashed), have nearly identical speeds closely matching the analytical solution. The semi-implicit solution (solid with circles) is visibly slower.

5. Conclusion

We have developed a time integration method based on a modified inversion of the Laplace transform (LT). It can be configured to simulate low frequency components of the solution whilst eliminating unwanted high frequency oscillations. The method was compared to the semi-implicit (SI) approach. The SI method stabilizes high frequency gravity waves by averaging them. This has the effect of reducing the phase speed of the waves. The LT scheme has been shown to have smaller phase errors than the SI scheme when simulating a Kelvin wave. Since these waves are dynamically important, this is a significant advantage.

In this study we have combined the LT scheme with an Eulerian treatment of advection. Thus, the time step is limited by the strength of the ambient flow. In Part 2 we will combine the LT scheme with semi-Lagrangian advection, and show that it has additional important advantages over the semi-implicit method.

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References


Holton JR. 1975. The Dynamic Meteorology of the Stratosphere and Mesosphere. Meteorological Monographs, Volume 15, Number 37, American Meteorological Society


Figure 4. Relative phase errors for the semi-implicit (SI) and LT methods, for Kelvin waves of zonal wavenumbers $m = 1$ and $m = 5$

Figure 5. Hourly height at 0.0°E, 0.9°N over 10 hours with $\tau_c = 3$ hours
Figure 1. Left: The contour $C^*$ replaces $C$ for the modified LT inversion. Right: The numerical inversion is performed using $C_N^*$. (From Lynch (1991), © Amer. Met. Soc.)

Figure 2. Case 5 at T42 and $\Delta t = 1200s$. $l_2$ error (left panel) and $l_\infty$ error (right panel).

Figure 3. Case 6 at T42 and $\Delta t = 600s$. $l_2$ error (left panel) and $l_\infty$ error (right panel).
Laplace transform integration of the shallow water equations. 
Part 2: Lagrangian formulation and orographic resonance

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In this paper we combine the Laplace transform (LT) scheme with a semi-Lagrangian advection scheme, and implement it in a shallow water model. It is compared to a reference model using the semi-implicit (SI) scheme, with both Eulerian and Lagrangian advection. We show that it is accurate and computationally competitive with these reference schemes. We also show, both analytically and numerically, that the LT scheme is free from the problem of orographic resonance that is found with semi-implicit schemes. Copyright © 0000 Royal Meteorological Society

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1. Introduction

In this paper a semi-Lagrangian Laplace transform (SLLT) shallow water model is developed. It is based on an existing model, SWEmodel. In Clancy and Lynch (2010), referenced below as Part 1, we introduced the Laplace transform (LT) scheme for time integration and implemented it in a model using Eulerian advection. The scheme was shown to have advantages when compared to a reference semi-implicit (SI) scheme. In particular, it was able to simulate Kelvin waves with greater accuracy.

The size of the timestep used with an Eulerian advection scheme is limited by constraints of stability rather than accuracy. By combining the semi-implicit averaging with a semi-Lagrangian treatment of advection, Robert (1981, 1982) was able to perform stable and accurate integrations with even longer timesteps. Bates and McDonald (1982) showed that there was no CFL restriction with the semi-Lagrangian advection scheme, and were the first to implement it in an operational forecast model. Further details on the development of semi-Lagrangian methods may be found in the review of Staniforth and Côté (1991).

A combined semi-Lagrangian and LT scheme for a shallow water model is formulated in §2. We use a spectral method for the spatial discretisation and test the model against various Eulerian and semi-Lagrangian semi-implicit versions. The stability and accuracy of the scheme are analysed. We use a two time level discretisation for the SLLT scheme. Two time level semi-Lagrangian schemes offer a doubling of efficiency compared to three time level versions (Temperton et al., 2001). In §3 the SLLT scheme will be evaluated using the test cases of Williamson et al. (1992). A number of variations of the SLLT discretisation are discussed. Its stability properties are also examined.

We explore the problem of orographic resonance in §4. This is a spurious noise that results from the coupling of the semi-implicit and semi-Lagrangian schemes at high Courant numbers. We investigate the problem analytically and show that, in a simple linear model, the SLLT scheme is free from this problem. We confirm that this result holds for the fully nonlinear shallow water equations by numerical tests. Finally, §5 contains a summary of the main results.

2. The shallow water model: SWEmodel

Two semi-Lagrangian models are developed; one using a semi-implicit discretisation and the other with the LT method. These are based on a spectral transform shallow water model called SWEmodel, written in MatLab. This model is described in Drake and Guo (2001), with the spherical harmonic transform routines documented in Drake et al. (2008). The original code included an Eulerian shallow water model, which we will also use as a reference.
2.1. Semi-Lagrangian semi-implicit: SLSI

The shallow water equations are written in the form

\[ \frac{d \zeta}{dt} + f \delta + \beta v = N_\zeta \]
\[ \frac{d \delta}{dt} - f \zeta + \beta u + \nabla^2 \Phi = N_\delta \]
\[ \frac{d \Phi}{dt} - \delta a \Phi + \Phi \delta = N_\Phi \]

(1)

where \( \zeta \) is the relative vorticity, \( \delta \) is the horizontal divergence and \( \beta = \frac{1}{2} \frac{\partial f}{\partial y} \). The constant reference geopotential is \( \Phi \), with \( \Phi \) representing the deviation of the geopotential field from this reference. The surface geopotential is \( \Phi_s \).

The nonlinear terms are

\[ N_\zeta = -\zeta \delta \]
\[ N_\delta = \zeta^2 - \nabla^2 \left( \frac{v v}{2} \right) + \frac{u}{\cos \phi} \left( \nabla^2 (u \cos \phi) - \frac{2 \sin \phi}{a} \zeta \right) \]
\[ + \frac{v}{\cos \phi} \left( \nabla^2 (v \cos \phi) + \frac{2 \sin \phi}{a} \delta \right) \]
\[ N_\Phi = -(\Phi - \Phi_s) \delta \]

For the semi-Lagrangian semi-implicit (SLSI) model, a two time level discretisation is used. All linear terms are averaged in time, including the Coriolis terms (Temperton and Staniforth, 1987; Côté and Staniforth, 1988). The departure points are computed using the method outlined in Ritchie and Beaudoin (1994). For this iterative technique, an initial guess of the midpoint wind \( \mathbf{v}_M^{n+\frac{1}{2}} \) is obtained with the simple two-term extrapolation

\[ \left( \mathbf{v}_M^{n+\frac{1}{2}} \right)^{(0)} = \mathbf{v}_A^n + \frac{1}{2} \mathbf{v}_A^{n-1} \]

Other possibilities are discussed in Temperton and Staniforth (1987). Three iterations are used for this and at each step new values for \( \mathbf{v}_M^{n+\frac{1}{2}} \) are computed with bilinear interpolation.

When interpolating model fields to departure or midpoints, bicubic interpolation is used. The nonlinear terms are first extrapolated in time using

\[ N_A^{n+\frac{1}{2}} = \frac{3}{2} N_A^n - \frac{1}{2} N_A^{n-1} \]

before being interpolated to the midpoint values \( N_M^{n+\frac{1}{2}} \). A discussion of various interpolation options is given in the review paper of Staniforth and Côté (1991).

The system is solved using a spectral method. Each field is expanded in spherical harmonics, e.g.

\[ \zeta = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbf{c}_\ell^m Y^m_\ell (\lambda, \mu) \]

where \( Y^m_\ell (\lambda, \mu) = e^{im\lambda} P^m_\ell (\lambda, \mu) \) and \( \mu = \sin \phi \). Orthogonality of the spherical harmonics is used to isolate individual spectral coefficients. Following Côté and Staniforth (1988), and defining \( \alpha = \Omega \Delta t \) and \( \varepsilon_\ell = \sqrt{\frac{\ell^2 - m^2}{\ell^2}} \), we get

\[ A_\ell^m \mathbf{c}_\ell^m + B_\ell^m \delta_{\ell+1} - C_\ell^m \delta_{\ell-1} = |R_\ell^m| \]
\[ A_\ell^m \delta_{\ell+1} - B_\ell^m \mathbf{c}_\ell^m - C_\ell^m \delta_{\ell-1} = |R_\ell^m| \]
\[ A_\ell^m \mathbf{c}_\ell^m + B_\ell^m \delta_{\ell+1} - C_\ell^m \delta_{\ell-1} = |R_\ell^m| \]
\[ \Phi_\ell^m + \frac{\Delta t}{2} \delta \Phi_\ell^m = |R_\ell^m| \]

(3)

with

\[ A_\ell^m = 1 - \frac{i m \alpha}{\ell (\ell + 1)}, \quad B_\ell^m = \frac{\alpha (\ell + 1)}{\ell} \varepsilon_\ell, \]
\[ C_\ell^m = \frac{\alpha \ell}{\ell + 1} \varepsilon_\ell^{m+1}. \]

The spectral system can be shown to decouple to

\[ L_\ell^m \delta_{\ell-2} + M_\ell^m \delta_{\ell-1} + U_\ell^m \delta_{\ell+2} + V_\ell^m \delta_{\ell+1} = [R_\ell^m] \]
\[ + \frac{\alpha (\ell^2 - 1)}{\ell (\ell + 1)} \varepsilon_\ell \delta_{\ell-1} + \frac{\alpha (\ell^2 + 2)}{\ell (\ell + 1)} \varepsilon_{\ell+1} - \frac{i m \alpha}{\ell (\ell + 1)} |R_\ell^m| \]
\[ \Phi_\ell^m = |R_\ell^m| \]
\[ \zeta_\ell^m = \frac{\ell (\ell + 1)}{\ell (\ell + 1) - i m \alpha} |R_\ell^m| \]

(4)

where

\[ L_\ell^m = \frac{\alpha^2 (\ell^2 + \ell)}{\ell^2 - \ell - i m \alpha} \varepsilon_\ell^{m+1} \varepsilon_\ell^m \]
\[ M_\ell^m = 1 - \frac{i m \alpha}{\ell (\ell + 1)} + \frac{\Delta t}{2} \frac{\Phi_\ell (\ell + 1)}{\alpha^2} + \frac{\alpha (\varepsilon_\ell^m)^2 (\ell - 1)^2 + (\ell + 1)^2 (\ell + 2)^2}{\ell^2 (\ell - 1)^2 - i m \ell \alpha} + \frac{(\varepsilon_\ell^{m+1})^2 (\ell + 1)^2 (\ell + 2)^2}{(\ell + 1)^2 (\ell + 2) - i m \alpha (\ell + 1)} \]
\[ U_\ell^m = \frac{\alpha^2 (\ell^2 + \ell)}{\ell (\ell + 1) (\ell + 2) - i m \alpha} \]

If odd and even values of \( \ell \) are considered separately for every \( m \), the system yields two tridiagonal matrix systems for the coefficients \( \delta_\ell^m \) which can be efficiently solved (Durrani, 1999).
2.2. Semi-Lagrangian Laplace transform: SLLT

We consider a general evolution equation in Lagrangian form

\[ \frac{dX}{dt} + LX = N(X) \]  

(5)

The total derivative \( \frac{dX}{dt} \) represents the change along the trajectory of the fluid parcel. We take the Laplace transform along the time-dependent trajectory of a parcel arriving at a gridpoint \( A \) at time level \( n + 1 \). Formally, we are integrating along the trajectory contour \( T \) so that

\[ \hat{X} \equiv \int_T e^{-s\tau} X \mathrm{d}\tau \]  

(6)

With a two time level approach, this trajectory starts at time \( n \) at some departure point \( D \), not necessarily coinciding with a gridpoint. This is the ‘initial value’ when taking the transform of a Lagrangian derivative. The transform of the prognostic equation (5) can then be written

\[ s\hat{X} - X^n_D + L\hat{X} = \frac{1}{s} N^{n+\frac{1}{2}}_M \]

Here the nonlinear terms have been evaluated at the midpoint \( M \) of the trajectory and at time level \( n + \frac{1}{2} \).

We apply this to the system (1), yielding

\[ s\hat{\zeta} - \zeta^n_D + \hat{f}\delta + \beta\hat{v} = \frac{1}{s} \{ N \zeta \}^{n+\frac{1}{2}}_M \]
\[ s\hat{\delta} - \delta^n_D - \hat{f}\zeta + \beta\hat{u} + \nabla^2\hat{\Phi} = \frac{1}{s} \{ N \delta \}^{n+\frac{1}{2}}_M \]
\[ s\hat{\Phi} - \Phi^n_D - \frac{\partial\hat{\Phi}}{\partial t} + \Phi\hat{\delta} = \frac{1}{s} \{ N \Phi \}^{n+\frac{1}{2}}_M \]

The Coriolis terms need special consideration. Though constant in time, both \( f \) and \( \beta \) vary along a trajectory. This means that transforms of products such as \( \hat{f}\delta \) cannot be easily separated and make the system very difficult to decouple.

One approach to overcome this is to assume that the changes in \( f \) and \( \beta \) along a trajectory are negligible, which allows us to then write

\[ \hat{f}\delta \rightarrow f_A \hat{\delta}, \quad \hat{f}\zeta \rightarrow f_A \hat{\zeta} \]
\[ \beta\hat{u} \rightarrow \beta_A \hat{u}, \quad \beta\hat{v} \rightarrow \beta_A \hat{v} \]

These are used for the SLLT model in this work. Alternatives were explored, but these approximations were found to give the best results (Clancy, 2010). The transform of the orography \( \frac{\partial \Phi}{\partial z} \) also requires care. Like the Coriolis terms, this is time-independent but varies along a trajectory.

If we assume the changes to be small and treat it as ‘constant’ at its arrival value, the orographic derivative transforms to

\[ \frac{d\Phi_A}{dt} = s\hat{\Phi}_s = (\Phi_s)^{n+1}_A - (\Phi_s)^n_D \]

An alternative would be first to discretise the derivative as

\[ \frac{d\Phi_A}{dt} = \frac{(\Phi_s)^{n+1}_A - (\Phi_s)^n_D}{\Delta t} \]

The terms on the right are now all constants and so we can easily take the LT to get

\[ \frac{d\Phi}{dt} = \frac{(\Phi_s)^{n+1}_A - (\Phi_s)^n_D}{s \Delta t} \]

Both methods were tested: the second was found to give superior results in simulations and so was implemented in the main SLLT code.

With the above approximations, the system to be solved is now

\[ \hat{\zeta} + f\hat{\delta} + \beta\hat{v} = R_{\zeta} \]  

(7)
\[ \hat{\delta} - f\hat{\zeta} + \beta\hat{u} + \nabla^2\hat{\Phi} = R_{\delta} \]  

(8)
\[ s\hat{\Phi} + \Phi\hat{\delta} = R_{\Phi} \]  

(9)

with

\[
R_{\zeta} = \zeta^n_D + \frac{1}{s} \{ N \zeta \}^{n+\frac{1}{2}}_M \\
R_{\delta} = \delta^n_D + \frac{1}{s} \{ N \delta \}^{n+\frac{1}{2}}_M \\
R_{\Phi} = \Phi^n_D + \frac{(\Phi_s)^{n+1} - (\Phi_s)^n_D}{s \Delta t} + \frac{1}{s} \{ N \Phi \}^{n+\frac{1}{2}}_M 
\]

Each transformed variable is a function of space and the complex variable \( s \) and can be expanded in terms of spherical harmonics; for example

\[
\hat{\zeta}(s, \lambda, \mu) = \sum_m \sum_\ell \hat{\zeta}_m^\ell (s) Y^m_\ell (\lambda, \mu) 
\]

Note that the spectral coefficients are functions of \( s \). The system (7) can thus be solved spectrally, in a manner similar to the SLSI scheme, for a given value of \( s \). Using orthogonality as before, we get the following

\[
\hat{\zeta}_m^\ell = \left[ \frac{-i m}{\ell (\ell + 1)} \frac{2 \Omega}{s} \right] \hat{B}_m^\ell \left( s \hat{\delta}_m^\ell - \hat{f}_m^\ell \right) + \left[ \frac{\ell (\ell + 1)}{2 \Omega} \frac{2 \Omega}{s} \right] \hat{C}_m^\ell \left( s \hat{\zeta}_m^\ell + \frac{1}{s} \hat{\Phi}_m^\ell \right) 
\]

(10)

with

\[
\hat{A}_m^\ell = 1 - \frac{i m}{\ell (\ell + 1)} \frac{2 \Omega}{s}, \quad \hat{B}_m^\ell = \left( \frac{\ell (\ell + 1)}{2 \Omega} \right) \frac{2 \Omega}{s} \hat{C}_m^\ell 
\]

\[
\hat{C}_m^\ell = \frac{\ell (\ell + 1)}{\ell (\ell + 1)} \frac{2 \Omega}{s} \epsilon^{\ell+1}_m 
\]

The above matches (3) with \( 1/s \) replacing \( \Delta t/2 \).

Again we solve two tridiagonal matrix systems for each \( \hat{\delta}_m^\ell \) and can then evaluate all the \( \hat{\Phi}_m^\ell \) and \( \hat{\zeta}_m^\ell \). Once these are known we can synthesize each of the transformed fields \( \hat{\delta}, \hat{\Phi} \) and \( \hat{\zeta} \). For SLLT this must be done for each value of \( s \) on the inversion contour. We can then invert to the physical field at the new time level with the summation outlined in Part 1; for example

\[
\delta^{n+1} = \frac{1}{N} \sum_{n=1}^{N} s_n \hat{\delta}(s) e^{s_n \Delta t} 
\]

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2.3. Symmetry

In the SLSI model, we consider the spectral coefficients of a real function which have the symmetry property

$$\delta^m = (-1)^m \delta^m \bar{f}$$

where the bar denotes the complex conjugate. As a result, we only need to solve the tridiagonal systems in (4) for $m \geq 0$. In the case of SLLT, we are dealing with the spectral coefficient of the complex Laplace transform of the fields, for example, $\delta^m \bar{f}$. The symmetry in $m$ will not now hold and we need to solve for all $m$.

It is possible, however, to reduce the computational overhead of the LT method. Lynch (1991) discusses a symmetry that allows us to halve the number of points in the inversion contour. For a real function $f(t)$ with LT $\hat{f}(s)$, it follows from the definition of the transform that

$$\hat{f}(\bar{s}) = \overline{\hat{f}(s)}$$

On the inversion $N$-gon described in Part 1, the points used in the summation satisfy $s_{N+1-n} = \overline{s_n}$ for $n = 1, \ldots, N/2$. We can then write the inversion summation as

$$L_N(f) = \frac{1}{N} \sum_{n=1}^{N/2} \left\{ s_n \hat{f}(s_n) e^{s_n t} + \overline{s_n \hat{f}(\overline{s_n}) e^{\overline{s_n} t}} \right\}$$

$$= \frac{1}{N} \sum_{n=1}^{N/2} \left\{ s_n \hat{f}(s_n) e^{s_n t} + s_n \overline{\hat{f}(s_n)} e^{\overline{s_n} t} \right\}$$

$$= \frac{2}{N} \sum_{n=1}^{N/2} \Re \left\{ s_n \hat{f}(s_n) e^{s_n t} \right\}$$

Thus we are able to halve the number of inversion points needed for the LT method.

2.4. Stability

Côté and Staniforth (1988) used a logarithmic form of the continuity equation in (1) for their two time level scheme. Their stability analysis showed that, for the non-logarithmic form used in our SLSI model, the mean geopotential must be greater than the maximum geopotential height. A stability analysis of the SLLT scheme, following the approach of Durran (1999), was presented in Clancy (2010). He found that, under mild assumptions the scheme is stable. If these conditions are not satisfied, an amplification factor slightly larger than 1 is possible. However, no problems were encountered when running the SLLT model. The model remained stable and did not require $\Phi \geq \Phi_{\text{max}}$ like the SLSI model.

3. Testing STSWM

The SLSI and SLLT schemes were tested with cases 2, 5 and 6 from Williamson et al. (1992). Results were compared with the reference Eulerian STSWM model. For Case 2, a nondivergent zonal flow with a geostrophically balanced height field, the errors for both the SLSI and SLLT schemes were very small, even with a 1 hour timestep. The errors for SLLT were about half the magnitude of those for SLSI.

For the mountain test case, Case 5, no analytic solution exists. As a reference, we used the STSWM model detailed in Part 1, with a T213 resolution and a 360 second timestep. In the left panel of Figure 1 we plot the normalised $t_\infty$ errors for the Eulerian, SLSI and SLLT models at T119 resolution with a 600 second timestep over a 240 hour simulation. For the SLLT runs we used $N = 8$ and $\tau_c = 6$ hours. The three models show comparable accuracy. In the right panel we present the errors for both semi-Lagrangian models at higher timesteps. Errors remain small, even with a one hour timestep. The SLLT model in particular shows little variation with increasing timestep.

For Case 6, the Rossby-Haurwitz wave, a T213 STSWM run was again used as a reference. The value for $\Phi$ specified in Williamson et al. (1992), $\Phi = gh_0$ with $h_0 = 8$ km, had to be changed for the SLSI forecasts to maintain stability, as discussed above. We used $\Phi = 1.1 \times 10^3 \text{m}^2\text{s}^{-2}$ in order to exceed the maximum height. The SLLT model ran at the traditional value without difficulty.

In the left panel of Figure 2 we see $t_\infty$ errors for the various schemes, again for T119 and a timestep of 600 seconds. The right panel shows errors at higher timesteps. The SLSI and SLLT models show comparable accuracy. Both significantly damp the wave when run with long timesteps; this is more severe for SLSI. The SLSI model would not run stably with a one hour timestep. The SLLT run remained stable, but was severely damped.

For both Cases 5 and 6, the SLLT model was also tested using a cut-off value of $\tau_c = 3$ hours. The accuracy remained comparable with the $\tau_c = 6$ simulations. It was, however, unstable for Case 6 with a one hour timestep. This is probably due to the violation of the stability criterion of the LT scheme described in Part 1. We note that this criterion is a sufficient, but not necessary, condition. Case 5 ran successfully with a one hour timestep.

4. Orographic resonance

The coupling of a semi-Lagrangian treatment of advection with a semi-implicit method for stabilising gravity waves allowed numerical forecasts to be carried out with timesteps considerably longer than required for Eulerian schemes (Robert 1981, 1982). While this technique was successful in permitting stable and efficient forecasts, a problem arose in the case of simulating flow over orography.

A simple analysis of the linearised shallow water equations shows that stationary waves produce an infinite response to orographic forcing when the mean flow equals the gravity wave speed. This physical phenomenon is unlikely to occur given the high speed of gravity waves and does not generally pose a problem for numerical simulations. However, Coiffier et al. (1987) showed that a semi-Lagrangian semi-implicit discretisation introduces a spurious resonant response at large Courant numbers. Numerical runs confirmed this analysis. As the main advantage of a SLSI scheme is the ability to run at large timesteps, this problem was a cause for concern.

A number of solutions have been proposed. Tanguay et al. (1992) show that by spatially averaging all nonlinear terms, the distortions near orography are reduced, though not fully alleviated. Rivest et al. (1994) show that this can be improved upon by off-centring the semi-implicit averaging. They examine first-order and second-order averaging and recommend the latter for better accuracy. This approach has investigated in a number of atmospheric models: Héreil and Laprise (1996), Caya and Laprise (1999).
Ritchie and Tanguay (1996) found that the more efficient first-order off-centring is sufficient if the orographic term in the continuity equation is treated in an Eulerian manner rather than a Lagrangian manner. They noted, however, a truncation error evident in both approaches, which is smaller in the case of the Eulerian treatment of orography. The ECMWF model, for example, follows the approach of Ritchie and Tanguay together with the spatial averaging of Tanguay et al. (1992). The orographic contribution to the advection in the continuity equation is isolated and treated in a spatially averaged manner (Temperton et al., 2001).

In reviewing this approach, Lindberg and Alexeev (2000) note that it has been largely successful but does not fully remove the spurious response. Li and Bates (1996) also show how off-centring can have a negative impact on large scale Rossby waves; the first-order method causes excessive damping. They find that by using a potential vorticity form of the shallow water equations, a less damaging modified off-centring is possible. In their study of a general class of off-centred schemes, Côté et al. (1995) recommend using the least amount of off-centring possible, consistent with alleviating resonance, to minimize errors.

4.1. Linear analysis: physical response

The shallow water equations are given in vorticity-divergence form and are linearised about a mean flow \( \bar{u} = u_0 \cos \phi \), where \( \omega \) is a constant advecting angular velocity. The Coriolis parameter \( f \) is taken to be constant. With \( \frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \omega \), the linearised system is

\[
\begin{align*}
\frac{d\zeta}{dt} + f \delta &= 0 \\
\frac{d\delta}{dt} - f \zeta + \nabla^2 \Phi &= 0 \\
\frac{d\Phi}{dt} + \Phi \delta &= d\Phi_s
\end{align*}
\]  

We consider a single spherical harmonic component

\[
\begin{pmatrix}
\zeta \\
\delta \\
\Phi
\end{pmatrix}
= \begin{pmatrix}
\zeta_m^n \\
\delta_m^n \\
\Phi_m^n
\end{pmatrix}
\]

where \( \Phi_s = (\Phi_s)_m^l e^{im\lambda l} P_l^m(\mu) \). We substitute this into (11) and take \( \nu = 0 \) to consider orographically-forced stationary solutions. Solving for \( \Phi_m^n \) we get

\[
\Phi_m^n = \frac{(m \omega^2 - f^2)(\Phi_s)_m^l}{(m \omega^2 - G_l^2)}
\]  

with \( G_l^2 = f^2 + \frac{\ell(\ell + 1)}{a^2} \Phi \) is the squared frequency of the gravity-inertia wave. We find a physical resonance if this frequency equals that of the mean flow.

4.2. Linear analysis: SLSI

Ritchie and Tanguay (1996) analyse the problem in a three time level model. Here we adapt their analysis for a two time level version. First we consider a two time level semi-Lagrangian semi-implicit (SLSI) treatment of the linearised system. The orography is treated in a Lagrangian manner and no explicit diffusion is included. Discretising (11) we get

\[
\begin{align*}
\frac{\zeta_m^{n+1} - \zeta_m^n}{\Delta t} + f \left[ \frac{\delta_m^{n+1} + \delta_m^n}{2} \right] &= 0 \\
\frac{\delta_m^{n+1} - \delta_m^n}{\Delta t} - f \left[ \frac{\zeta_m^{n+1} + \zeta_m^n}{2} \right] + \left( \nabla^2 \Phi \right)_m^{n+1} + \left( \nabla^2 \Phi \right)_m^n &= 0
\end{align*}
\]  

We examine the response of stationary waves to orographic forcing and seek solutions of the form

\[
\begin{pmatrix}
\zeta_m^n \\
\delta_m^n \\
\Phi_m^n
\end{pmatrix}
= \begin{pmatrix}
m \lambda \\
0 \\
0
\end{pmatrix}
\]

\[ e^{im\lambda l} P_l^m(\mu) \]

The effect of the discretised time derivative is, e.g.,

\[
\frac{\zeta_m^{n+1} - \zeta_m^n}{\Delta t} = \omega \delta_m^n + \frac{1}{2} \frac{\nabla^2 \Phi_m^n}{\Delta t} e^{-im\lambda l} P_l^m(\mu)
\]

We apply these results to the discretised system. Each equation is then multiplied by \( \Delta t e^{i\theta} \) and the linear system can be solved for \( \Phi' \):

\[
\Phi'^n_l = \begin{pmatrix}
G_l^2 \cos^2 \theta - \frac{\sin \theta}{\bar{\theta}} \left( m \omega^2 \right) \left( m \omega^2 - G_l^2 \right) \\
G_l^2 \cos^2 \theta - \frac{\sin \theta}{\bar{\theta}} \left( m \omega^2 \right) \left( m \omega^2 - f^2 \right)
\end{pmatrix}\]

Following Rivest et al. (1994) and Ritchie and Tanguay (1996), we consider the ratio of the numerical to the physical response and divide the above by (13) to get

\[
R_{\text{SLSI}} = \frac{\left( \frac{f^2}{G_l^2} - \frac{\sin \theta}{\bar{\theta}} \left( m \omega^2 \right) \left( m \omega^2 - G_l^2 \right) \right)}{\left( G_l^2 - \frac{\sin \theta}{\bar{\theta}} \left( m \omega^2 \right) \left( m \omega^2 - f^2 \right) \right)}
\]

We note first that \( R_{\text{SLSI}} \) has a zero denominator when \( m \omega^2 - f^2 = 0 \). As pointed out by Ritchie and Tanguay, this is not a resonance but is due to a zero value of the physical response (13). There is, however, a spurious numerical resonance when

\[
\tan \theta = \pm \frac{G_l}{m \omega}
\]

Since \( G_l \) is large, resonance occurs near the singular points of \( \tan \theta \), that is,

\[
\theta = \frac{2k + 1}{2} \pi, \quad k \in \mathbb{Z}
\]

We present plots of \( R_{\text{SLSI}} \) for a T119 truncation, i.e., \( m = 0, 1, \ldots, 119 \). Following Ritchie and Tanguay we take \( \ell = m \) and use the values \( f = 10^{-4} \text{s}^{-1} \) and \( \Phi = 5.6 \times 10^4 \text{m}^2\text{s}^{-2} \). We choose the advecting wind to be \( 50 \text{m s}^{-1} \) at the equator so that \( \omega = (50/a) \text{s}^{-1} \).
In the left panel of Figure 3 we take $\Delta t = 600$ and 3600 seconds. The response is plotted in terms of a scaled wavenumber $m\Delta \lambda / \pi$. For these parameter values, the case of zero physical response mentioned earlier occurs at $m\Delta \lambda / \pi \approx 0.07$ and this is seen by the jumps near this value.

For an exact solution we would have the response equal to 1, indicated by the dotted line in each plot. For $\Delta t = 600$s (solid line) we get acceptable results. At the longer timestep of $\Delta t = 3600$s (dashed) there is clearly a resonant response for $m\Delta \lambda / \pi \approx 0.56$. This is the first resonance given by (19), when $\theta = \frac{\pi}{2}$.

4.3. Linear analysis: SLLT

We now turn to the SLLT scheme to examine its effect on orographically forced waves. Taking the Laplace transform of each equation along a trajectory, as discussed in §2.2, and again looking for steady state spherical harmonic solutions of the form (15), we get

$$ s\hat{\mathbf{C}} - \mathbf{D} \hat{\mathbf{C}}_{\Phi} + \mu f\hat{\mathbf{C}} = \mathbf{0} $$

$$ s\hat{\mathbf{D}} - \mathbf{D} \hat{\mathbf{D}}_{\Phi} + \mu f\hat{\mathbf{D}} = \mathbf{0} $$

Considering the trajectory $\lambda = \lambda_D + \omega t$, we can write

$$ \hat{\mathbf{C}}_{\Phi} = \sum (\lambda_D + \omega t) \mathbf{e}^{im\lambda_D} P_m^m(\mu) $$

$$ \hat{\mathbf{D}}_{\Phi} = \sum (\lambda_D + \omega t) \mathbf{e}^{im\lambda_D} P_m^m(\mu) $$

We must consider the transform of $e^{im\omega t}$. This is a function of time only, since $\omega$ is a constant with no spatial variation. Therefore we can write $\sum \mathbf{e}^{im\omega t} = 1/(s - im\omega)$. Using this we can write the transformed system as

$$ \mathbf{M} \hat{\mathbf{X}} = \mathbf{R}_s $$

where $\mathbf{X} = (\zeta, \delta, \Phi)^T$.

$$ \mathbf{M} = \begin{pmatrix} s f & 0 & 0 \\ -f & s & \frac{l(l+1)}{a^2} \\ 0 & \frac{1}{s} & \hat{\mathbf{D}}_{\Phi} \end{pmatrix} $$

and

$$ \mathbf{R}_s = \begin{pmatrix} 0 \\ 0 \\ i m \omega \hat{\mathbf{D}}_{\Phi} \end{pmatrix} $$

The right-hand side has been split so as to isolate the response due to the orographic term. We focus on this response; that is, we consider the system

$$ \mathbf{M} \hat{\mathbf{X}}_{\text{org}} = \mathbf{R}_s $$

Solving for $\hat{\mathbf{C}}_{\text{org}}$ we get

$$ \hat{\mathbf{C}}_{\text{org}} = \left(\hat{\mathbf{D}}_{\Phi}\right)^{-1} \sum \frac{s^2 + f^2}{s(s + G_{\ell})} (s - im\omega) $$

We see immediately that we cannot have $s = 0$, $s = im\omega$ or $s = \pm iG_{\ell}$. But $|s| = \gamma$, the radius of the inversion contour and so these situations can be avoided by a careful contour choice.

The operator $\mathbf{L}^*_N$ is applied to $\hat{\mathbf{C}}_{\text{org}}$ to recover the physical field. The expression for $\hat{\mathbf{C}}_{\text{org}}$ can first be expanded using partial fractions. We can then apply $\mathbf{L}^*_N$ and use the following results from Part 1 of this work:

$$ \mathbf{L}^*_N \left\{ \frac{1}{s - \mp i\nu} \right\} = H_N(\nu) e^{\pm \mu \Delta t} \quad \text{and} \quad \mathbf{L}^*_N \left\{ \frac{1}{s} \right\} = 1 $$

These results give us the physical solution obtained when we apply the numerical inversion operator to each term in $\hat{\mathbf{C}}_{\text{org}}$. After inverting we get

$$ (\hat{\mathbf{C}}_{\text{org}}) = R' (\hat{\mathbf{C}}_{\Phi}) $$

where

$$ R' = (\mathbf{f}^2 - (m\omega)^2) \frac{G_{\ell}^2}{G_{\ell}^2 - (m\omega)^2} H^+(m\omega) - \frac{(f \mathbf{G}_{\ell})^2}{G_{\ell}^2 - (m\omega)^2} + \frac{m\omega \mathbf{F}}{2(G_{\ell} + m\omega)} H^+(G_{\ell}) $$

$$ + \frac{m\omega \mathbf{F}}{2(G_{\ell} + m\omega)} H^-(G_{\ell}) $$

with

$$ H^\pm(x) = H_N(x) e^{\pm \mu \Delta t} \quad \text{and} \quad \mathbf{F} = 1 - \left(\frac{f \mathbf{G}_{\ell}}{G_{\ell}^2 - (m\omega)^2}\right)^2 $$

Since

$$ (\hat{\mathbf{C}}_{\Phi}) = (\mathbf{f}^2 - (m\omega)^2) \frac{G_{\ell}^2}{G_{\ell}^2 - (m\omega)^2} e^{im\lambda_D} P_m^m(\mu) $$

we can divide (20) by $e^{im\lambda_D} P_m^m(\mu)$ and write $(\hat{\mathbf{C}}_{\text{org}})^m = e^{-i\theta} R'(\phi_{\text{org}})^m$. As in the SLSI case, we divide the numerical response by the analytic response (13) to get

$$ R_{\text{SLLT}} = H^+(m\omega) e^{-i\theta} - \frac{f^2}{G_{\ell}^2 - (m\omega)^2} \left(\frac{G_{\ell}^2 - (m\omega)^2}{f^2 - (m\omega)^2}\right) e^{-i\theta} $$

$$ + e^{-i\theta} H_N(G_{\ell}) \left(\frac{m\omega}{G_{\ell}} \left(1 - \frac{f^2}{G_{\ell}^2 - (m\omega)^2}\right) \cos N(G_{\ell} \Delta t) + i G_{\ell} \sin N(G_{\ell} \Delta t)\right) $$

where $\cos N(x)$ and $\sin N(x)$ denote, respectively, the real and imaginary parts of $e^{i\lambda_D}$. The response in (21) will have a zero denominator only for $f^2 = (m\omega)^2$. This is, however, the case of zero physical response as mentioned in the SLSI analysis. Thus we expect no spurious resonant response to orography using a SLLT discretisation.

To illustrate this, we plot $R_{\text{SLLT}}$ in the right-hand panel of Figure 3 with parameters matching those for SLSI. In addition we use the values $N = 8$ and $\tau_c = 6$ hours. For SLLT, however, there is no resonant behaviour present.

On comparison with the SLSI resonances on the left in Figure 3, it may appear that we are resonance-free simply because the problematic wavenumbers have been removed by the LT filtering. However, it is important to note again that the expression in (21) shows no artificial resonance, regardless of wavenumber. To demonstrate this we plot

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$R_{\text{SLSI}}$ and $R_{\text{SLLT}}$ in Figure 4, this time for T213 resolution and a 2 hour timestep. For the SLLT discretisation we choose a less severe cutoff of 3 hours and $N = 16$. With these parameter values we see resonant behaviour in SLSI around $m\Delta \lambda /\pi$ between 0.1 and 0.2. The SLLT plot to the right of this shows that these scales are being retained, but with no resonance.

4.4. Shallow water experiments

We now describe numerical tests with the fully nonlinear shallow water system, using the SLSI and SLLT models. A number of previous authors have used the analysis at 12 UTC on the 12th of February 1979 as a case study [Rivest et al. (1994), Rivest and Staniforth (1995), Li and Bates (1996), Ritchie and Tanguay (1996)]. There is a strong flow over the Rocky mountains and so this is very suitable for investigating orographic resonance.

The initial conditions are the 500hPa winds and geopotential height, as well as the surface geopotential, taken from the ERA-40 dataset of the ECMWF (Uppala et al., 2005). We initialize with the Laplace transform initialization method, as outlined in Part 1.

The simulations were all run at a T119 resolution with no added diffusion. The SLLT parameters used were $N = 8$ and $\tau_s = 6$ hours. As predicted by the linear analysis, forecasts using a 600 second timestep did not suffer from any spurious orographic resonance, since this only becomes an issue at high Courant numbers. We focus on the North American continent and show the 24-hour forecast height using a one hour timestep. The linear analysis suggests that we may encounter problems with orographic resonance for the higher timestep. In Figure 5 we plot the height at 24 hours using the SLSI (left) and SLLT (right) models. The SLSI model shows very pronounced noise over the mountains. As expected from the linear analysis, the SLLT model does not suffer from this resonance.

4.5. SETTLS Formulation

When discretising the shallow water equations (1) for SLSI and SLLT, we evaluated the nonlinear terms at the midpoint of the trajectory. A number of alternative treatments of the nonlinear terms have been proposed. As mentioned, Tanguay et al. (1992) recommend a spatial averaging which helps to alleviate orographic noise and also reduces the number of interpolations necessary. Gospodinov et al. (2001) discuss how this has been shown to lead to problems, with non-meteorological noise being observed in a number of forecasting centres. This was solved at the ECMWF with the operational introduction of the ‘Stable Extrapolation Two-Time-Level Scheme’, or SETTLS (Hortal, 2002). Durran and Reinecke (2004) show that, out of the class of schemes studied by Gospodinov et al., optimal stability is obtained with SETTLS.

We incorporated the SETTLS treatment of the nonlinear terms into the SLSI. Everything else in the models, including the trajectory calculations, was unchanged. When tested with the cases used in §3, SETTLS version of SLSI showed improved accuracy over the original (see Clancy (2010) for further details).

For the orographic resonance test case, we plot the 24 hour forecasted height using SLSI SETTLS with a one hour timestep in Figure 6. Comparing with Figure 5, we see that the SLSI SETTLS treatment reduces the distortions over the mountains, but nevertheless it is still present. The SLLT discretisation is still the most effective for removing the spurious response.

5. Conclusion

We have developed a semi-Lagrangian shallow water model using the Laplace transform filtering integration method. This permits forecasts with longer timesteps than could be used for an Eulerian model and compares favourably with a reference semi-Lagrangian semi-implicit (SLSI) scheme in terms of accuracy. We investigated the problem of orographic resonance associated with SLSI discretisations. By means of a linear analysis and also by shallow water simulations, the SLLT model was shown to be free from this spurious noise.

The Laplace transform integration method has been tested in a shallow water model. Its advantages should also hold for baroclinic models used in operational NWP. Its capacity to filter high-frequency waves should be of particular benefit in the context of nonhydrostatic models.

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References


**Figure 6.** 24-hour height forecasts at $\Delta t = 3600$ s for SLSI SETTLS
Figure 1. $l_{\infty}$ errors for Case 5

Figure 2. $l_{\infty}$ errors for Case 6

Figure 3. Numerical response to orographic forcing divided by the physical response. The dotted line is $R = 1$, where the numerical solution equals the analytic solution. Left: SLSI. Right: SLLT. Note that the extremes at $m\Delta\lambda/\pi \approx 0.07$ are artefacts, due to the vanishing of the physical solution.
Figure 4. Responses for SLSI and SLLT, T213 with $\Delta t = 7200$ s, $N = 16$ and $\tau_c = 3$ hours: the numerical response to orographic forcing divided by the physical response. The dashed red line is $R = 1$, where the numerical solution equals the analytic solution and the dot-dashed blue line is $R = 0$, where the numerical solution is zero due to filtering. The extremes at $m\Delta\lambda/\pi \approx 0.04$ are artefacts, due to the vanishing of the physical solution.

Figure 5. 24-hour height forecasts at $\Delta t = 3600$ s for SLSI (left) and SLLT (right)
Bibliography


Bibliography


