The turbulent equilibration of an unstable baroclinic jet

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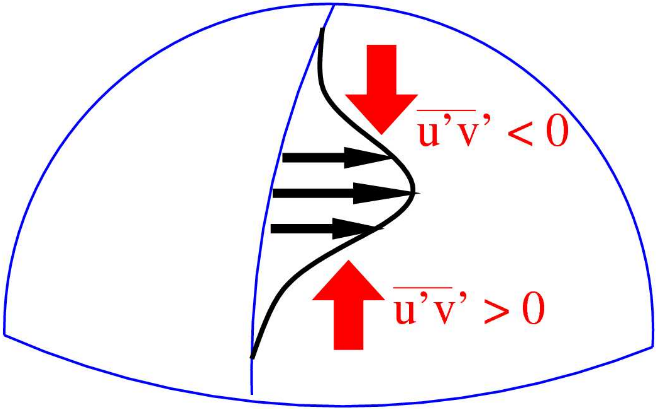
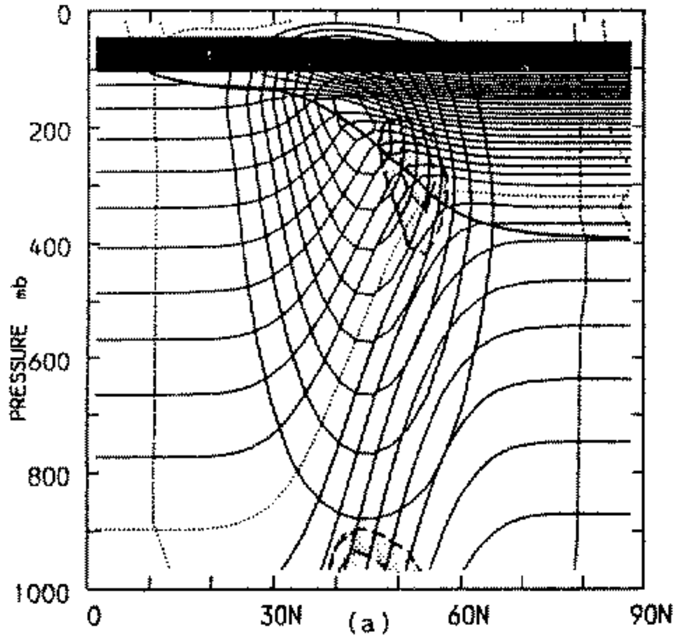
Upgradient momentum fluxes



Eddy momentum fluxes enhance jet

- The extratropical tropospheric jets have long been known to be maintained by `upgradient' momentum fluxes (e.g. Starr, 1968).
- This was a puzzle to early researchers... who sought turbulent closures for the Reynolds' stress, and discovered `negative viscosity'!
- Upgradient momentum fluxes are also a feature of baroclinic lifecycle simulations (Simmons and Hoskins, 1976). The tropospheric jet is accelerated by the action of the eddies.

Primitive Equation Life-Cycles



(from Thorncroft et al., QJRMS, 1993)

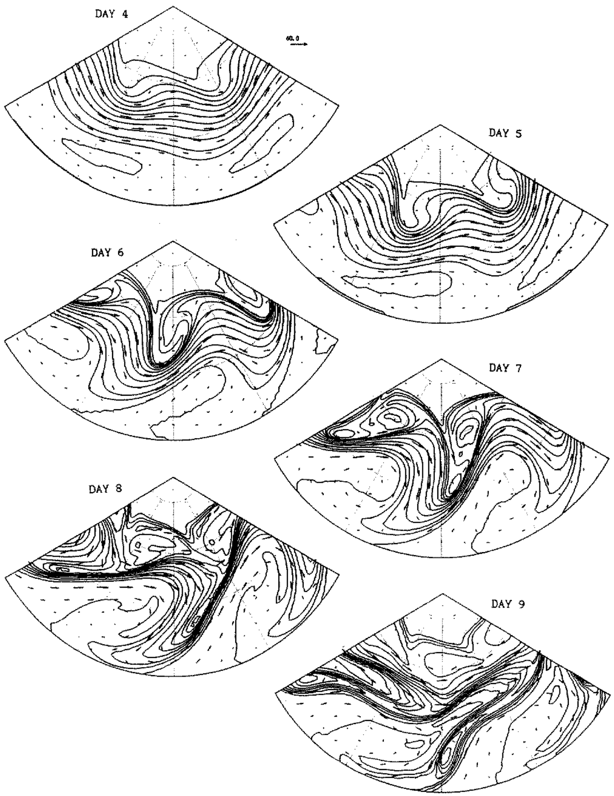


Figure 7. Potential temperature on the PV = 2 PVU surface in LC1 between days 4 and 9. Contours are drawn every 5 K from 290 K to 350 K going equatorward. Wind vectors are included; for days 4 to 7 these have had the phase speed of the normal mode removed to give the relative flow. The arrow scale is as in Fig. 11 (with allowance for the different figure dimensions). Sectors are shown for two wavelengths between latitude 20°N and the pole. Lines of constant latitude and longitude are drawn every 20 and 30 degrees, respectively.

Tropopause potential temperature (from THM93)

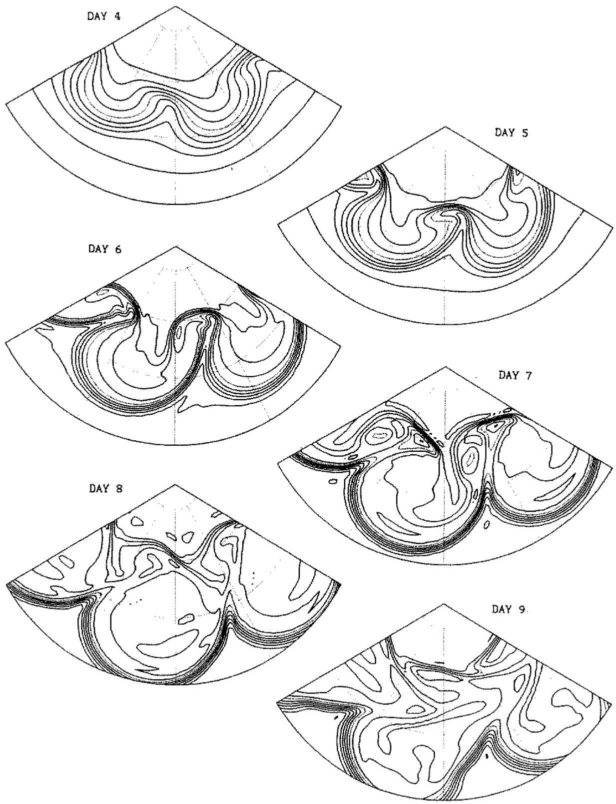
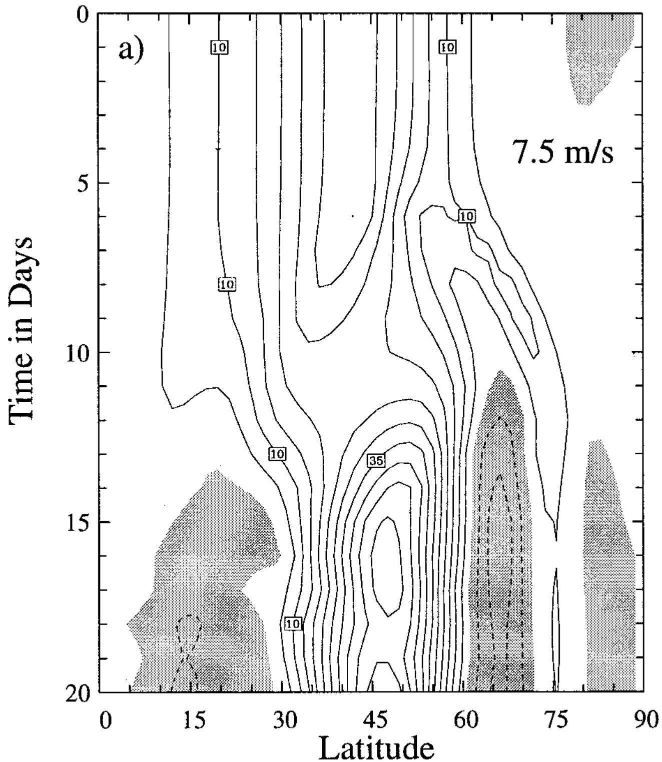


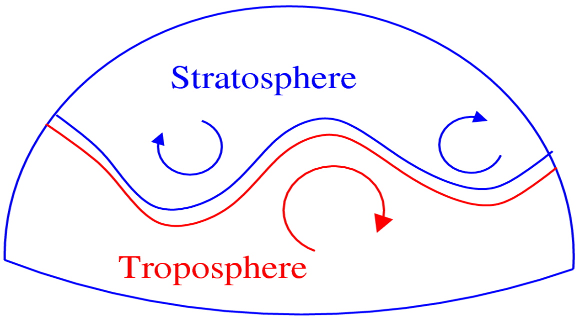
Figure 5. Temperature at $\sigma = 0.967$ in LC1 between days 4 and 9. Contours are drawn every 4 K with the 0 °C contour dotted. Sectors are shown for two wavelengths between latitude 20°N and the pole. Lines of constant latitude and longitude are drawn every 20 and 30 degrees, respectively.

Surface temperature (from THM93)



Zonal wind evolution (from Hartmann and Zuercher, JAS, 1998)

Potential Vorticity Homogenization



~330K Isentropic Surface
PV mixing on flanks of tropopause

- A modern view (e.g. Dritschel and McIntyre, 2008) is that downgradient mixing (or homogenization) of potential vorticity can account for upgradient momentum fluxes.
- Potential vorticity (PV)

$$P = \rho^{-1}(2\Omega + \zeta) . \nabla \theta$$

is conserved following fluid parcels. Eddies act to mix PV, hence downgradient turbulent closures based on PV make sense.

- Nevertheless, turbulent closures based on these ideas are not entirely successful. PV mixing in geophysical flows is often highly localised, and a problem is determining in advance where mixing will occur.

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A simple but relevant model?

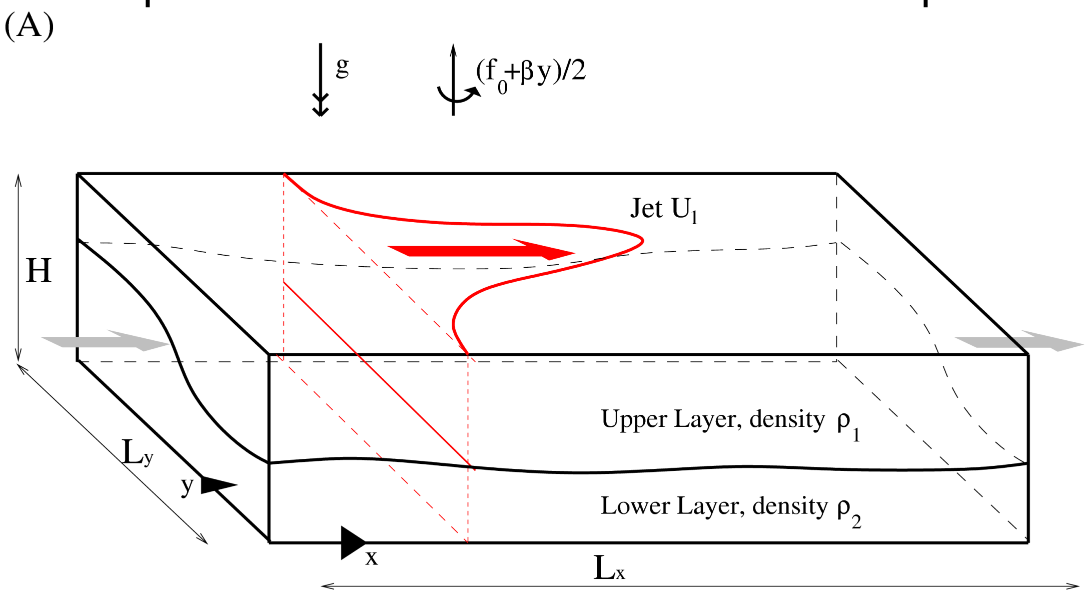
- **Aim.** Try to understand the flow in a simple model: An unforced baroclinic lifecycle in a two-layer β–channel model.

- **Objectives - I.** Discover what controls the locations of the regions of potential vorticity mixing in the channel.

- **Objectives - II.** Derive a physical principle to explain the model behaviour. Use the physical principle to predict the eddy momentum fluxes as a function of the model parameters.

Phillips' Two-Layer Model

A periodic channel centred on the extratropics.



A cheap general circulation model!

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Equations of Motion

The non-dimensional model equations are

$$\frac{D_i q_i}{Dt} = \kappa \nabla^4 \psi_i, \qquad \text{where}$$

$$q_i = \beta y + \nabla^2 \psi_i + \frac{(-1)^i}{2}(\psi_1 - \psi_2), \qquad i = 1, 2$$

With velocity in each layer $\boldsymbol{u_i} = (-\psi_{iy}, \psi_{ix})$

And advective derivative $D_i / Dt = \partial_t - \psi_{iy} \partial_x + \psi_{ix} \partial_y$

i=1 Upper Layer, *i=2* Lower layer.

Boundary Conditions:

$\psi_{ix} = 0$, on $y = \pm\frac{L_y}{2}$ (no normal flow on boundaries),

$\overline{\psi}_{iyt} = 0$, on $y = \pm\frac{L_y}{2}$ (no acceleration of mean flow along boundaries).

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Model Parameters

Equations are nondimensionalised using the internal Rossby radius as a length scale,

$$L_D = \frac{\sqrt{g'H}}{\sqrt{2}f_0}, \qquad \left(g' = g\frac{(\rho_2 - \rho_1)}{\rho}\right).$$

This results in the following nondimensional parameters that characterise the flow development.

$$\beta = \frac{\hat{\beta} L_D^2}{U}, \qquad \sigma = \frac{W}{L_D}, \qquad \kappa = \frac{\nu}{f_0 L_D^2}$$

Inverse Criticality — Jet Width — Rossby-Reynolds no.

Here f_0 is the Coriolis parameter, $\hat{\beta}$ is the planetary vorticity gradient, ν the molecular diffusivity and *U* and *W* are derived from the initial flow $u_1 = U\,sech^2(y/W),\ \ u_2 = 0.$

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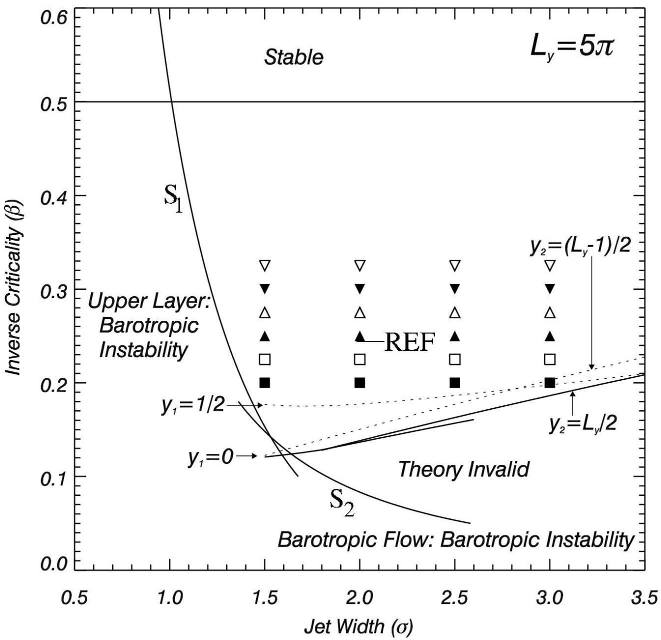
Parameter Space: Regime diagram



- Simulations independent of κ in the limit κ–> 0.
- Typical baroclinic lifecycles occur for jet width σ in the range [1.5,3].
- We examine behaviour for inverse criticalities β in the range [0.2,0.35]. β=0.5 is the stability boundary.

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Numerical Implementation

- The equations are solved using a standard Fourier (x-direction) / grid point (y-direction) model with (equivalent) resolution 2048 x 640 grid points, on a domain 20π x 5π Rossby radii.
- A small perturbation is added to the initial jet and the model is run for $t = 250 f_0^{-1}$ (approximately 25 days) by which time the lifecycle is complete.
- A parameter sweep in (β, σ) space is performed.

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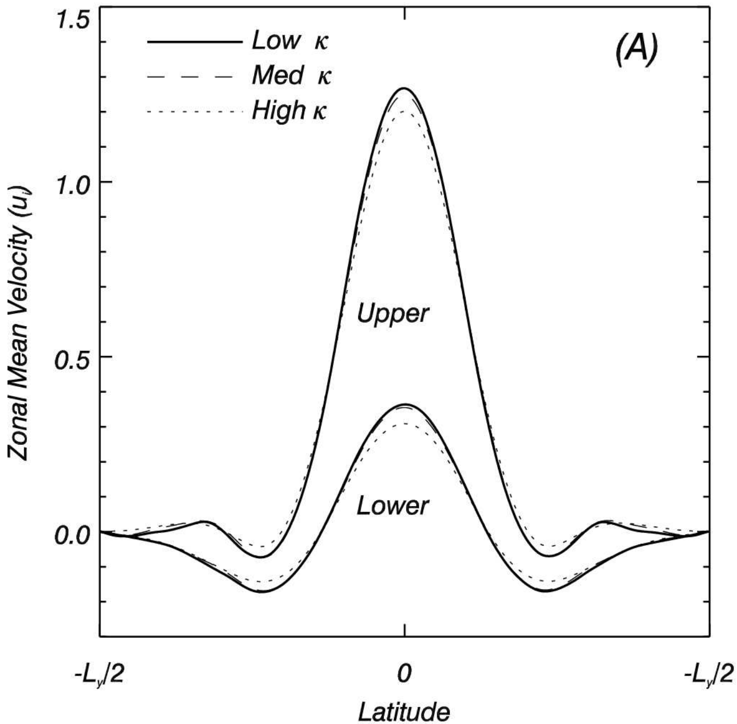
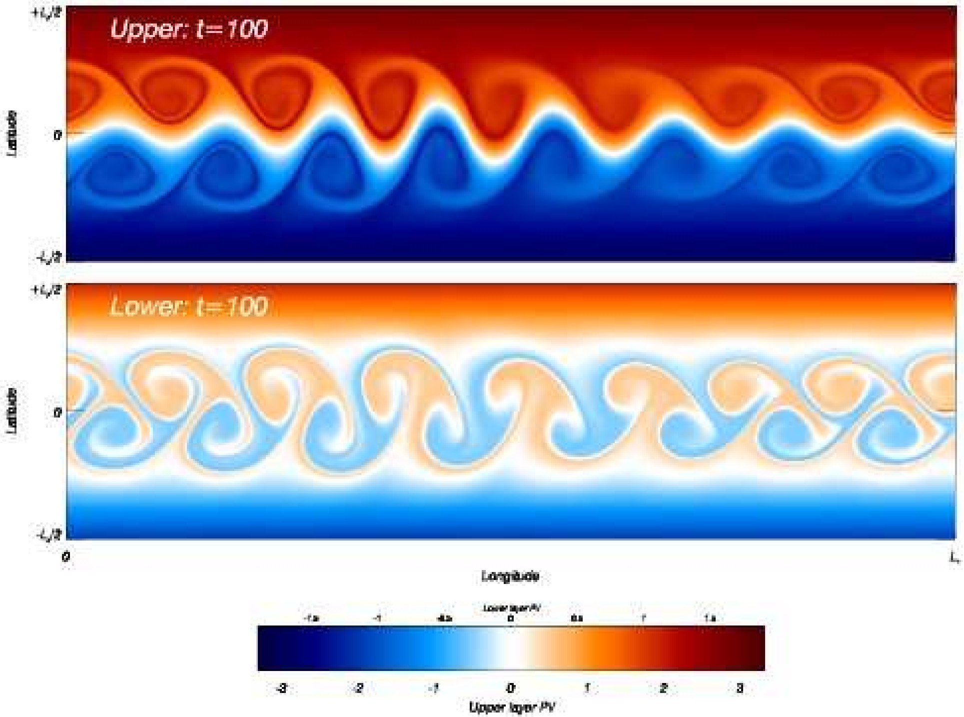
Final Zonal Mean Wind: Independence of κ



- By $t = 250\, f_0^{-1}$ the energy transfers between mean flow and eddies have ceased and the flow is almost steady.
- This final state can be demonstrated to be independent of κ.

Potential vorticity: Early in lifecycle

Reference Simulation $(\beta,\sigma)=(0.25,2)$



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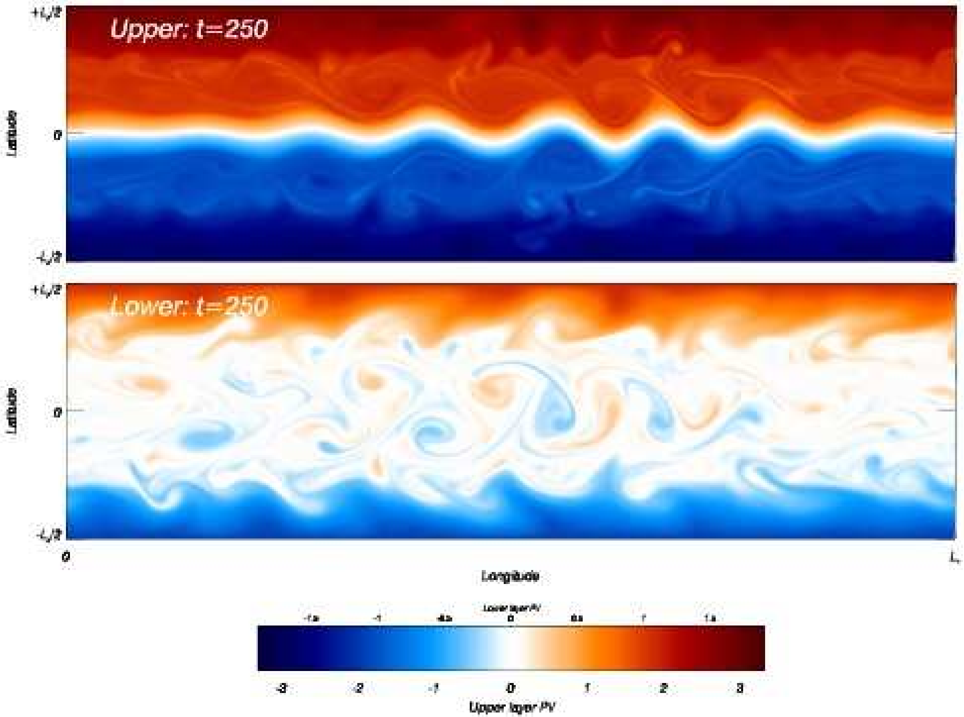
Potential vorticity: Late in lifecycle

Reference Simulation $(\beta,\sigma)=(0.25,2)$



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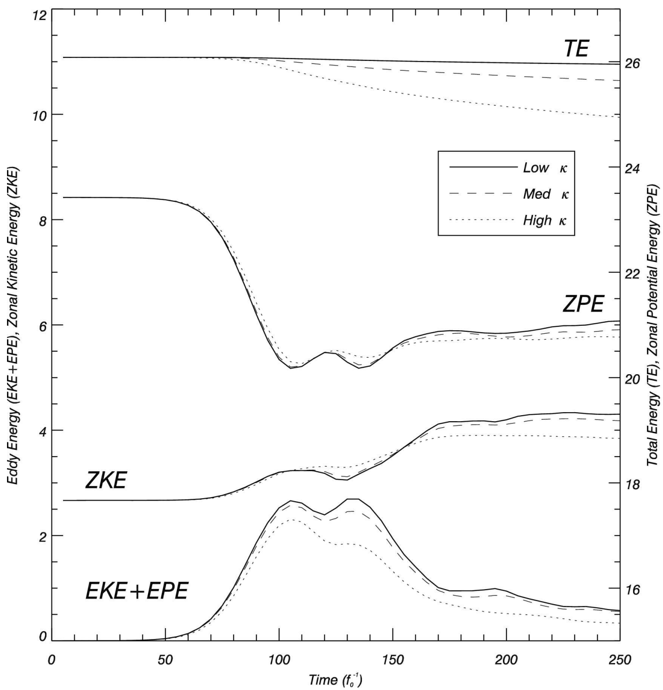
Energetics Cycle



- During the lifecycle: Available mean potential energy is first converted to eddy energy.
- Eddy energy is then converted to zonal mean kinetic energy.
- By the end, both total momentum and zonal mean energy (approximately) are equal to their initial values.

$$M = M_0, \qquad \overline{E} \approx E \approx E_0.$$

Conserved Quantities

The evolving flow conserves zonal momentum *M* and (to a good approximation) energy *E.*

$$M = \int u_1 + u_2 \, d^2\mathbf{x},$$

$$E = T + V = \frac{1}{2}\int |\mathbf{u_1}|^2 + |\mathbf{u_2}|^2 + \frac{1}{2}(\psi_1 - \psi_2)^2 \; d^2\mathbf{x}.$$

In the limit $\kappa \to 0$ functionals of potential vorticity, “Casimir Integrals” are also conserved.

$$\mathcal{C}[q_i] = \int \sum_i C_i(q_i) d^2\mathbf{x},$$

Even if diffusion is present these last constraints are important... no `new' PV values can appear.

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A Physical Principle for Baroclinic Equilibration

Statement of Hypothesis: Baroclinic eddies act to alter the mean state to minimize the available potential energy, subject to the dynamical constraints, by complete homogenization of potential vorticity within well-delineated regions.

Formulation of Problem: Given any unstable initial jet, with PV profile $Q_i(y)$, and parameters (β, σ), find permissible boundaries of the mixing regions y_1, y_2, y_3 that serve to minimize the potential energy.

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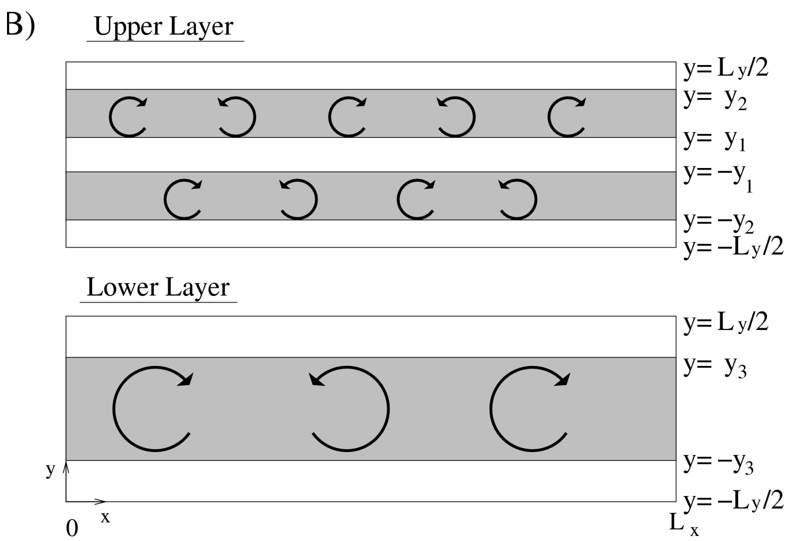
Assumptions underpinning theory



- PV Mixing pattern is “as illustrated”, with just the edge locations of the mixing regions unknown.
- Negligible energy is dissipated.
- All but negligible energy returns to the zonal mean

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Equilibrated Flow

The final PV q_i is assumed to be related to the initial PV Q_i by

$$q_1(y) = \begin{cases} Q_1(y) & y_2 < y < \frac{L_y}{2} \\ Q_m & y_1 < y < y_2 \\ Q_1(y) & -y_1 < y < y_1 \\ -Q_m & -y_2 < y < -y_1 \\ Q_1(y) & -\frac{L_y}{2} < y < -y_2 \end{cases}, \qquad q_2(y) = \begin{cases} Q_2(y) & y_3 < y < \frac{L_y}{2} \\ 0 & -y_3 < y < y_3 \\ Q_2(y) & -\frac{L_y}{2} < y < -y_3 \end{cases}$$

where

$$Q_m = \frac{1}{y_2 - y_1} \int_{y_1}^{y_2} Q_1(y)\, dy.$$

It is straightforward to obtain the final streamfunction ψ_i and velocity u_i from q_i.

Aim: Predict the boundaries of the mixing regions y_1, y_2, y_3

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Nonlinear Equations

To minimize potential energy, while constraining total energy and momentum to be equal to their initial values, form the function,

$$F\left(y_1, y_2, y_3; Q_i(y)\right) = \overline{V} + \lambda \overline{E} + \mu M$$

with λ,μ Lagrange multipliers. To minimize *V*, find the critical points of *F*, giving 5 nonlinear equations

$$\frac{\partial F}{\partial y_1} = \frac{\partial F}{\partial y_2} = \frac{\partial F}{\partial y_3} = 0, \quad \overline{E} = E_0, \quad M = M_0,$$

in the 5 unknowns $(y_1, y_2, y_3, \lambda, \mu)$. This leads to...

$$\frac{\partial F}{\partial y_1} = (Q_m - Q_1(y_1))\left(-2(\phi(y_1) - \phi_m) + 4\lambda(\psi_1(y_1) - \psi_m) + \mu(y_2 - y_1)\right)$$

$$\frac{\partial F}{\partial y_2} = (Q_m - Q_1(y_2))\left(2(\phi(y_2) - \phi_m) - 4\lambda(\psi_1(y_2) - \psi_m) + \mu(y_2 - y_1)\right)$$

$$\frac{\partial F}{\partial y_3} = Q_2(y_3)\left(2\phi(y_3) + 4\lambda\psi_2(y_3) - 2\mu y_3\right)$$

$$\phi_{yy} - \phi = (\psi_1 - \psi_2), \qquad \phi_y = 0, \quad \text{on} \quad y = \pm\frac{L_y}{2},$$

$$\phi_m = \frac{1}{y_2 - y_1}\int_{y_1}^{y_2} \phi \, dy, \qquad \psi_{im} = \frac{1}{y_2 - y_1}\int_{y_1}^{y_2} \psi_i \, dy, \qquad i = 1, 2.$$

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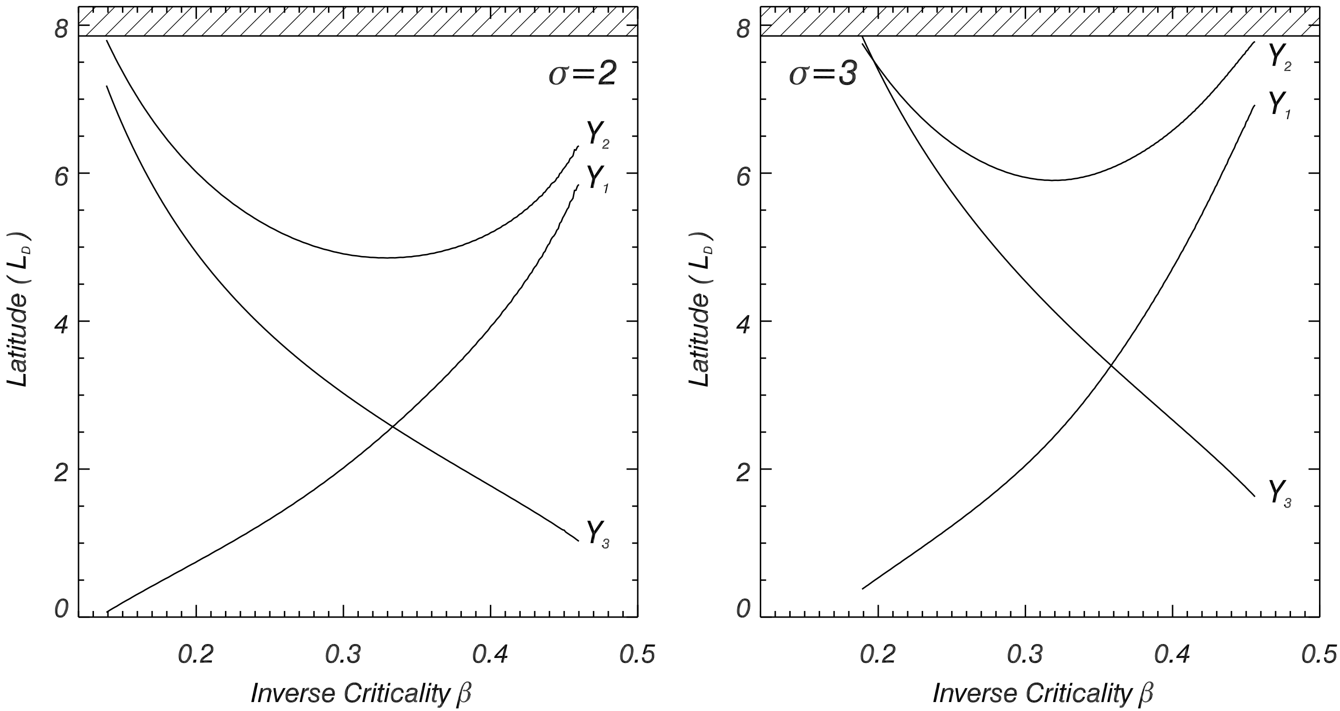
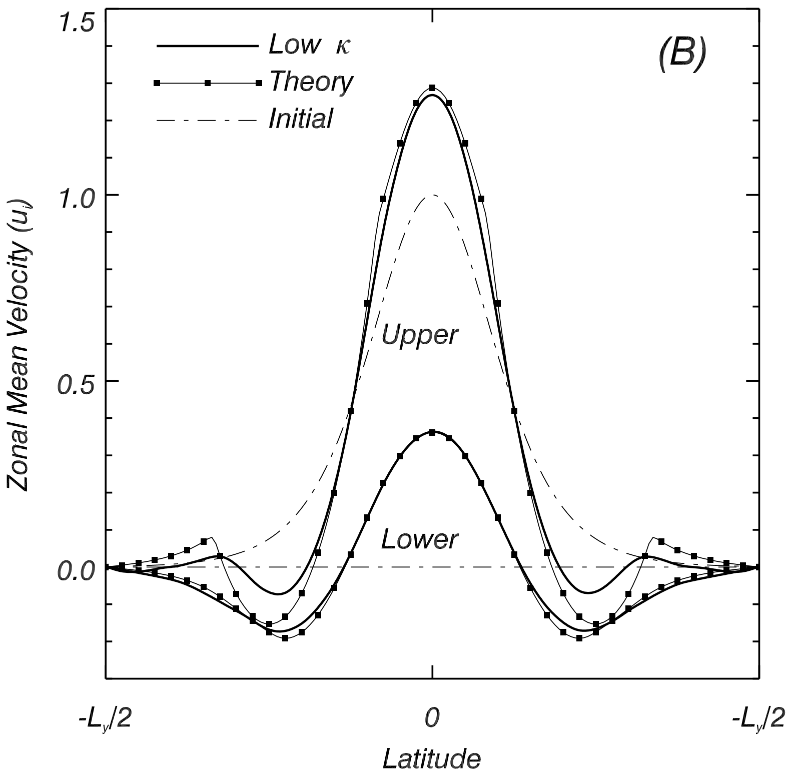
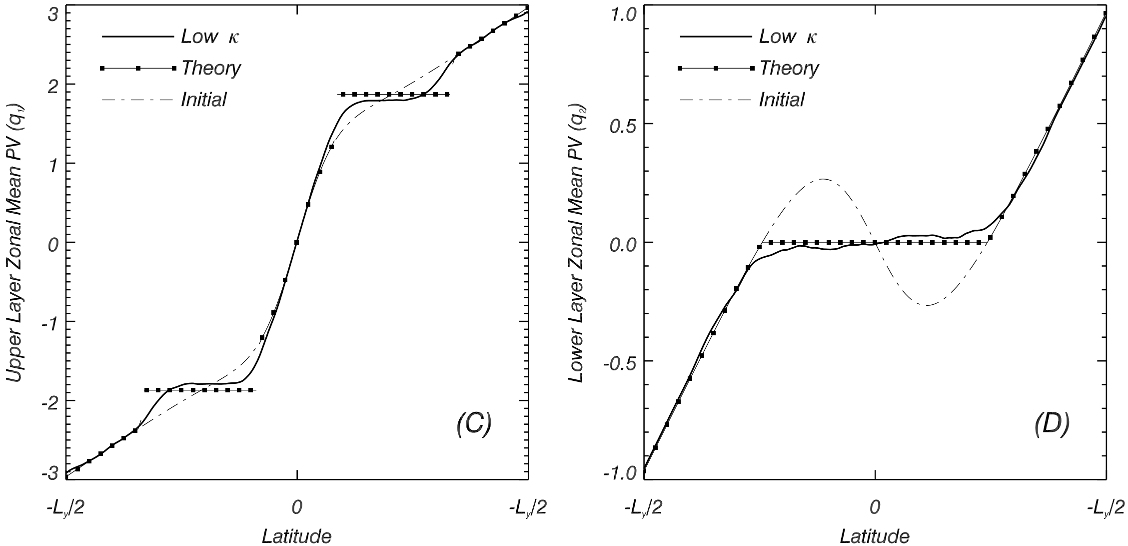


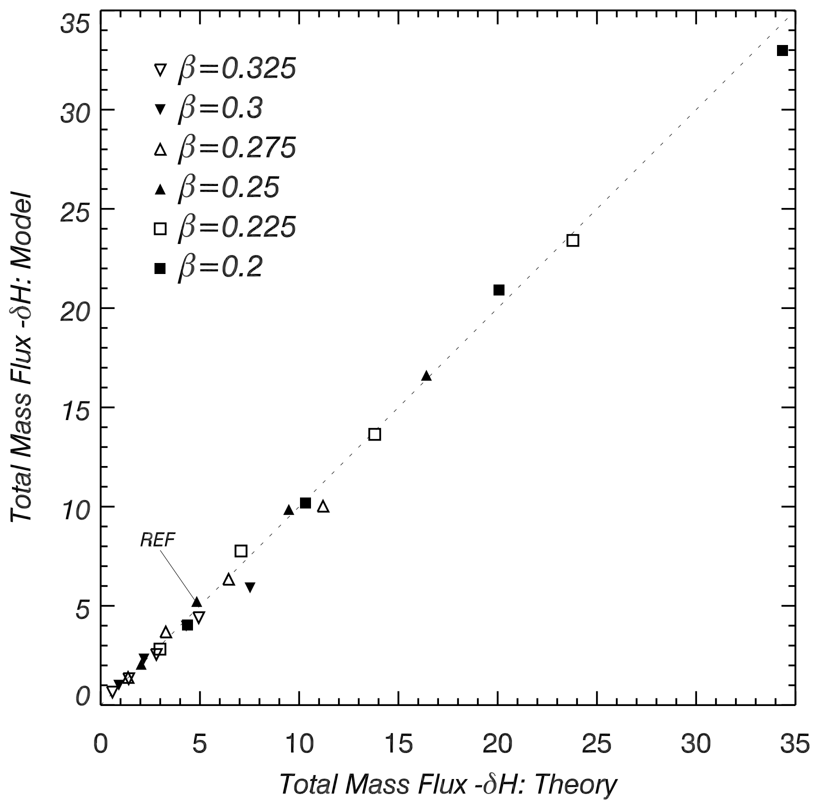
Solutions of nonlinear equations



Upper and Lower Layer Zonal Mean Wind

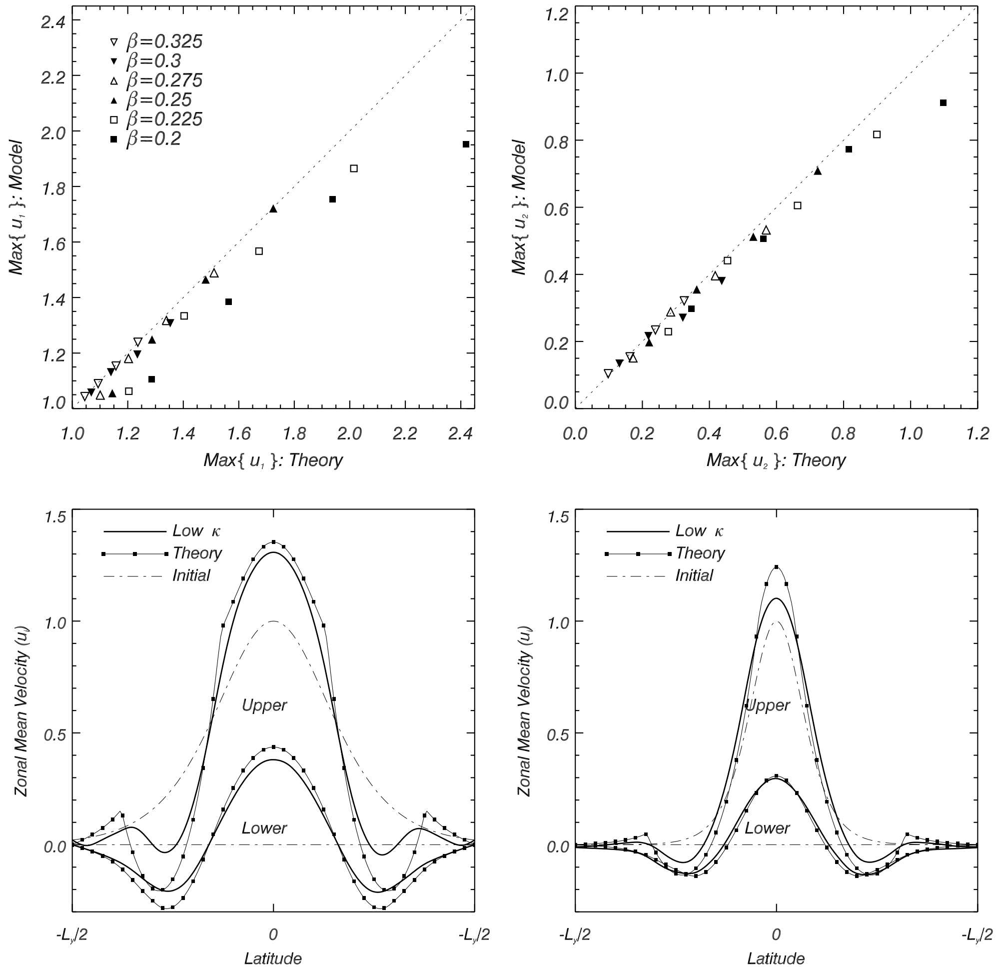


Upper and Lower Layer PV



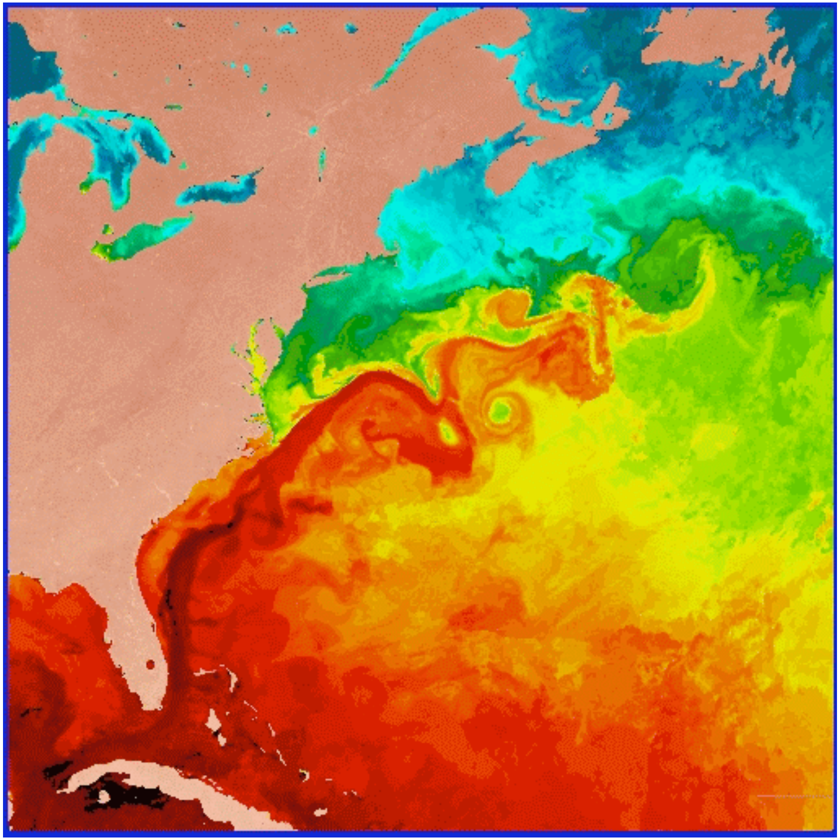
Total mass flux $\delta\mathcal{H}$: $\mathcal{H} = \int \; y(\psi_2 - \psi_1)\; d^2\mathbf{x}$.

Theory vs Simulation: Zonal mean winds



Robust and Leaky Transport Barriers

- What happens when $y_1 \to 0$?
- The width of the barrier to mixing in the upper layer, i.e. the model 'tropopause' or oceanic front, tends to zero.
- This suggests that the barrier, assumed up to this point to be impermeable, must 'leak' fluid across it if the equilibration process is to proceed under the principles outlined above.
- Such leakage, which could take the form of eddy shedding (c.f. Gulf stream rings, tropopause cut-off cyclones), will lead to mixing of fluid originating on opposite sides of the jet. Can we predict how much mixing will take place?



Snapshot: N. Atlantic Sea Surface Temperature

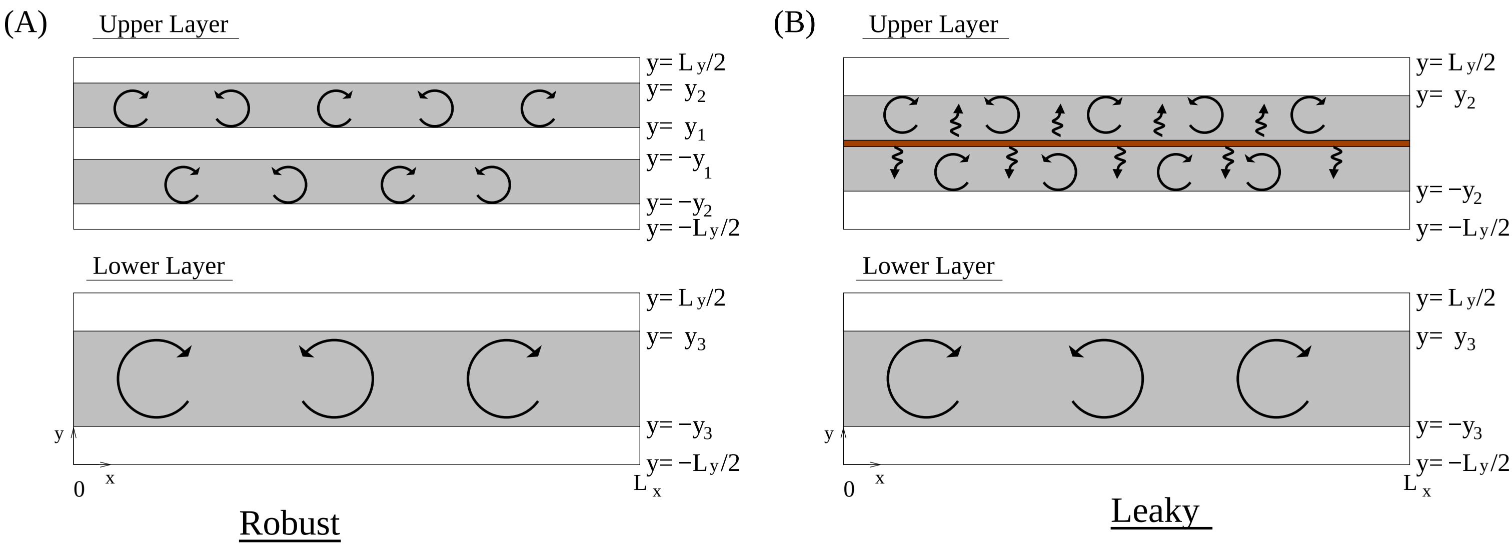
Formulation for a leaky barrier

The leaky barrier formulation assumes that rapid mixing of PV still proceeds to completion on either side of the jet, but that there is also some leakage of fluid across the central barrier at $y = 0$. Taking a finite width scale α for the barrier, the PV profile at the end of the life-cycle can be proposed to have the form:

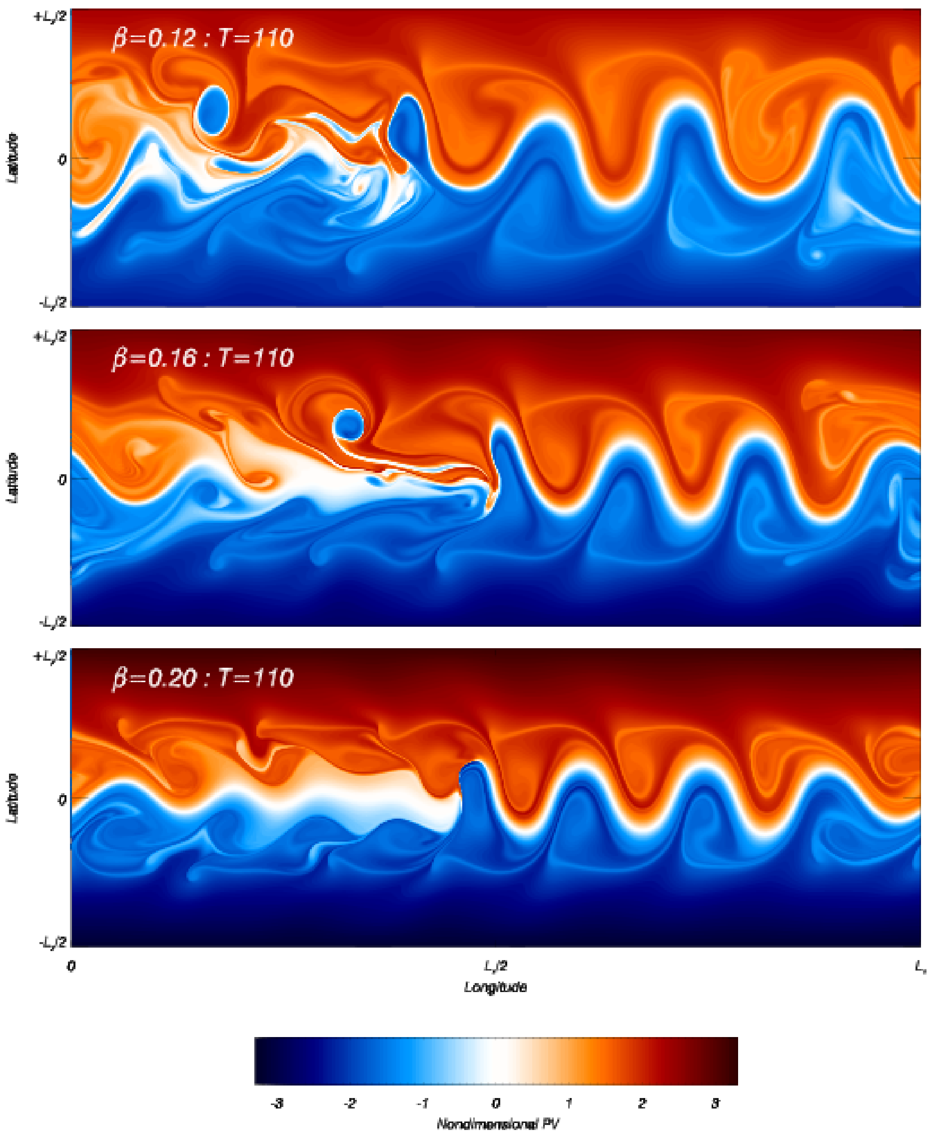
$$q_1(y) = \begin{cases} Q_1(y) & y_2 < y < \frac{L_y}{2} \\ Q_r \tanh\left\{\frac{y}{\alpha}\right\} & -y_2 < y < y_2 \\ Q_1(y) & -\frac{L_y}{2} < y < -y_2 \end{cases}, \quad q_2(y) = \begin{cases} Q_2(y) & y_3 < y < \frac{L_y}{2} \\ 0 & -y_3 < y < y_3 \\ Q_2(y) & -\frac{L_y}{2} < y < -y_3 \end{cases}.$$

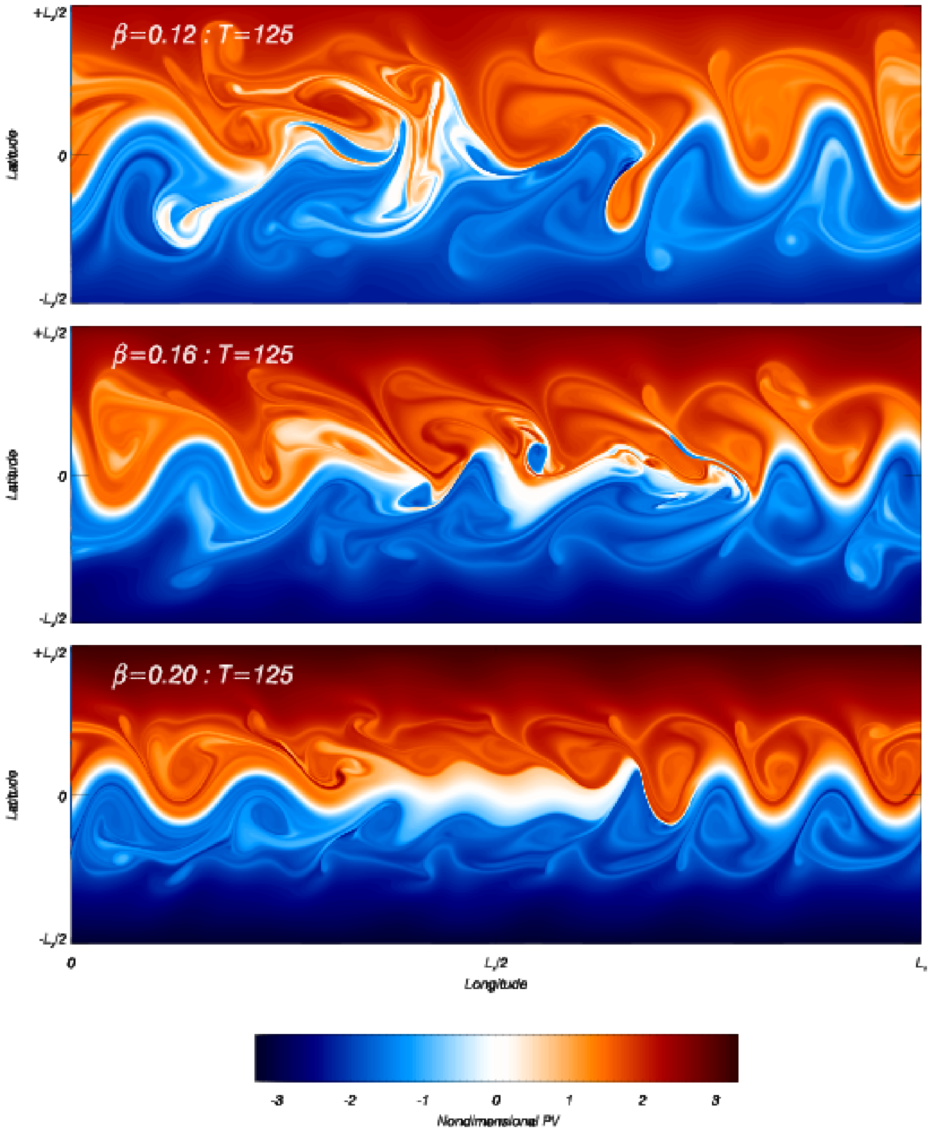
Here Q_r sets the maximum value of the PV in the upper layer within the mixing region. The aim now is to find Q_r, y_2 and y_3 that minimize potential energy V subject to the same constraints on $\overline{E}$ and M as above.

The barrier width α will be set empirically by comparison with the simulations.

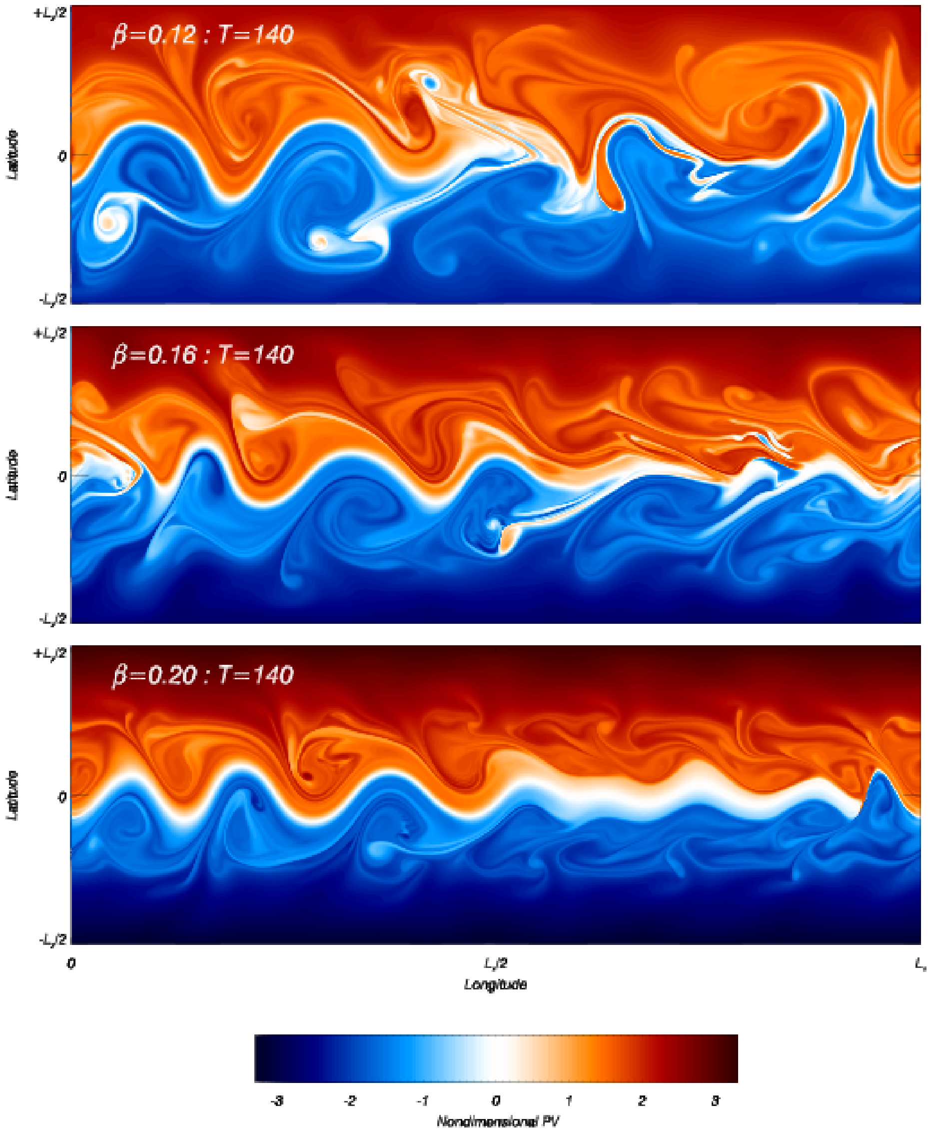


Schematic: Robust and leaky transport barriers


β=0.12 : T=110
β=0.16 : T=110
β=0.20 : T=110
+L_y/2
0
-L_y/2
Latitude
0
L_x/2
L_x
Longitude
-3
-2
-1
0
1
2
3
Nondimensional PV



β=0.12 : T=125
β=0.16 : T=125
β=0.20 : T=125
$+L_y/2$
0
$-L_y/2$
Latitude
0
$L_x/2$
L_x
Longitude
-3
-2
-1
0
1
2
3
Nondimensional PV



β=0.12 : T=140
β=0.16 : T=140
β=0.20 : T=140
+L_y/2
0
-L_y/2
Latitude
0
L_x/2
L_x
Longitude
-3
-2
-1
0
1
2
3
Nondimensional PV

Leaky Nonlinear Equations

As before, there are five unknowns $\{Q_r, y_2, y_3, \lambda, \mu\}$ and five nonlinear equations

$$\frac{\partial F}{\partial Q_r} = \frac{\partial F}{\partial y_2} = \frac{\partial F}{\partial y_3} = 0, \quad \overline{E} = E_0, \quad M = M_0.$$

This time the partial derivatives of F can be evaluated from

$$\begin{aligned} \frac{\partial F}{\partial Q_r} &= 2\phi_m - 4\lambda\psi_{1m} + 2\mu y_m \\ \frac{\partial F}{\partial y_2} &= \left(Q_r \textsf{tanh}\{y_2/\alpha\} - Q_1(y_2)\right)\left(2\phi(y_2) - 4\lambda\psi_1(y_2) + 2\mu y_2\right) \\ \frac{\partial F}{\partial y_3} &= Q_2(y_3)\left(2\phi(y_3) + 4\lambda\psi_2(y_3) - 2\mu y_3\right) \end{aligned}$$

where ϕ is defined as before, and

$$\phi_m = \frac{1}{y_2}\int_0^{y_2} \textsf{tanh}\{y/\alpha\}\ \phi\ dy, \qquad \psi_{1m} = \frac{1}{y_2}\int_0^{y_2} \textsf{tanh}\{y/\alpha\}\ \psi_1\ dy,$$

$$y_m = \frac{1}{y_2}\int_0^{y_2} y\ \textsf{tanh}\{y/\alpha\}\ dy, \qquad i = 1, 2.$$

Leakage across the Mixing Barrier

A measure of the amount of leakage of PV across the upper layer barrier is given by the expression

$$\mathcal{R} = 1 - \frac{\text{Total upper layer PV} > 0\text{: End of Lifecycle}}{\text{Total upper layer PV} > 0\text{: Start of Lifecycle}}.$$

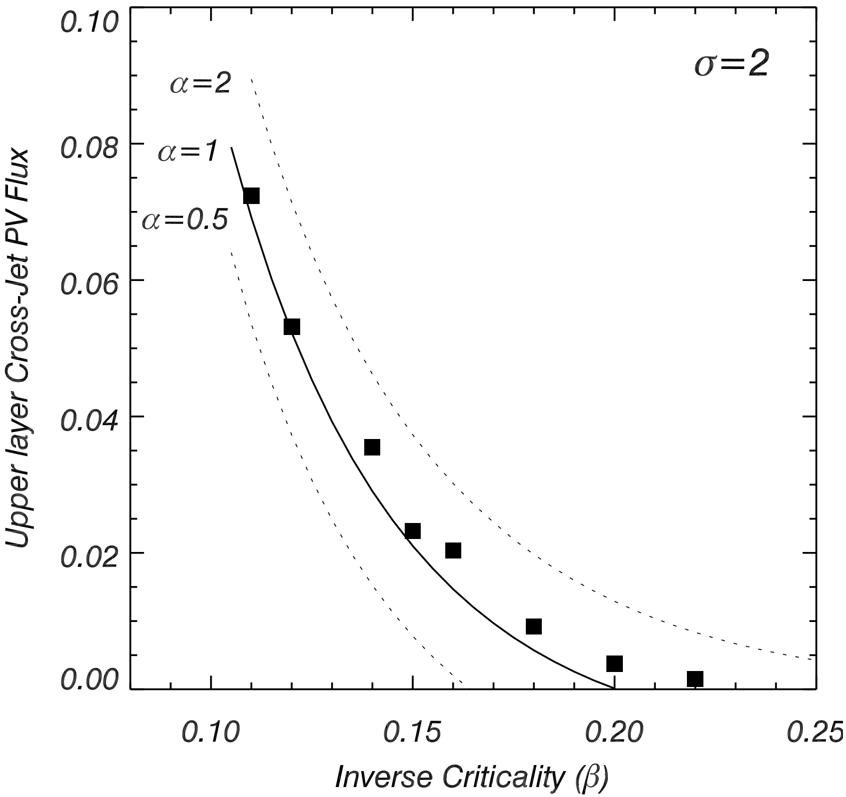
If there is no mixing across the upper layer barrier, $\mathcal{R} = 0$, and if the upper layer of the channel is completely mixed $\mathcal{R} = 1$.

The theory predicts that

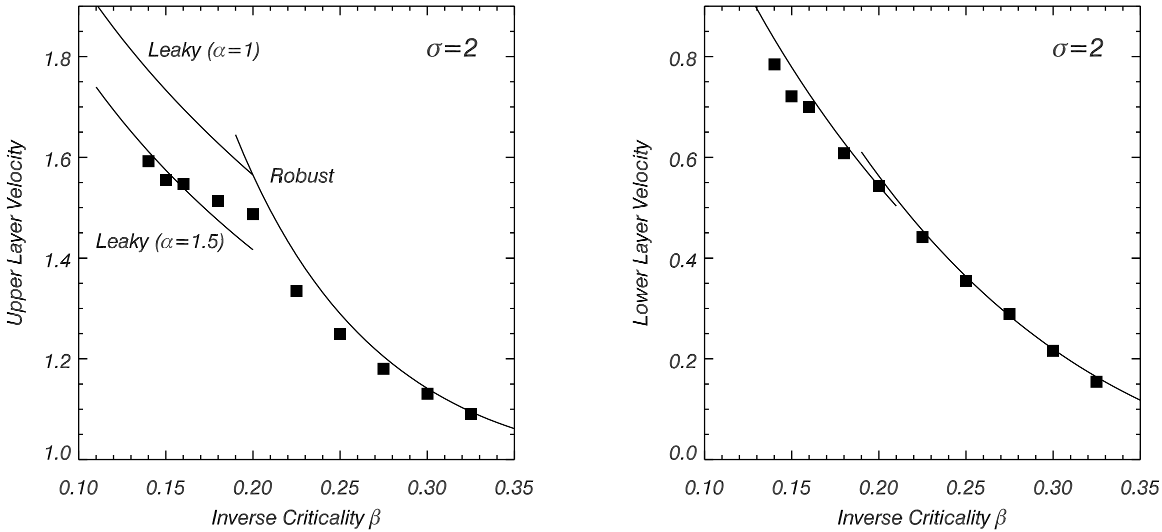
$$\mathcal{R} = \frac{\int_0^{y_2} \; Q_1(y) - Q_r \text{tanh}\{y/\alpha\} \; dy}{\int_0^{L_y/2} \; Q_1(y) \; dy}.$$

The corresponding value of $\mathcal{R}$ from the numerical simulations can be obtained as

$$\mathcal{R} = 1 - \frac{\int_{q_1>0} \; q_1(x, y) \; d^2\mathbf{x}}{L_x \; \int_0^{L_y/2} \; Q_1(y) \; dy}.$$



Cross-jet leakage $\mathcal{R}$: Theory vs Simulations



Upper and lower layer max. zonal mean winds: Leaky and Robust regimes

Conclusions

- 'Equilibration via PV Homogenisation' theory gives an accurate prediction of the structure and strength of the equilibrated jet over a wide range of flow parameters (β, σ).

- Theory predicts the breakdown of its own underlying assumptions as $y_1 \longrightarrow 0$, or $y_2, y_3 \longrightarrow L_y/2$ (walls).

- For distant walls, the limit $y_1 \longrightarrow 0$ corresponds to a transition between a robust and a leaky barrier to transport at the location of the upper jet. PV fluxes across the jet can be predicted.

- Extension to the forced-dissipative situation remains the outstanding challenge.