Spectra and Vortices of Two-Dimensional Turbulence

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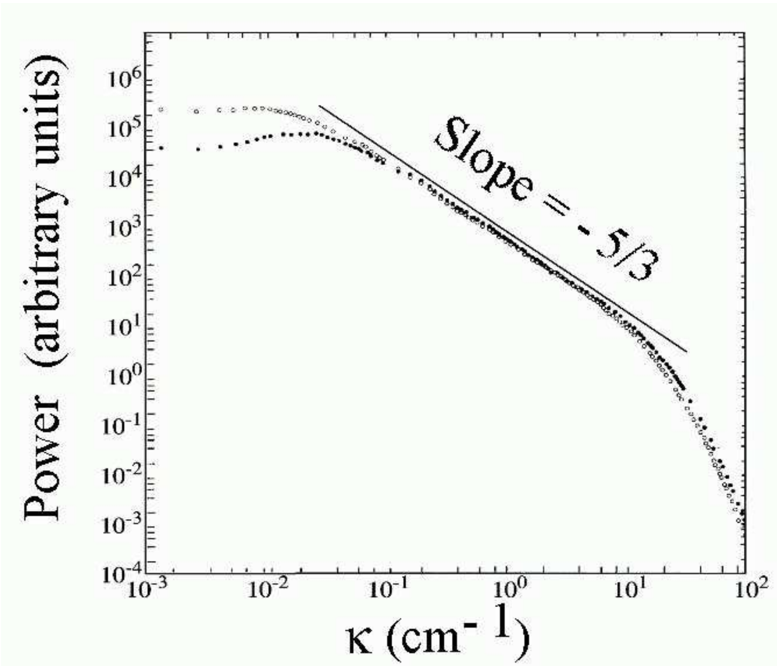
3 Dimensional Turbulence

- Generation of small scale structure from large scale motion.
- Described by Navier-Stokes equations :

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \nu \nabla^2 \vec{u} + \vec{f}$$

$$\nabla \cdot \vec{u} = 0$$

Phenomenology of 3 Dimensional Turbulence in a Nutshell

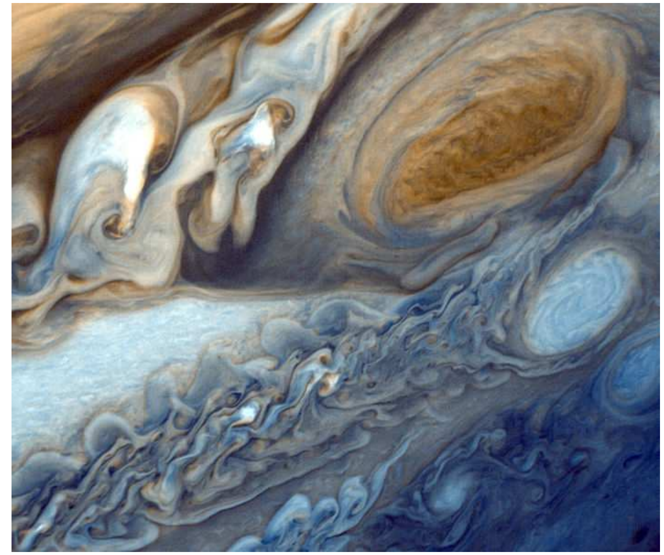


- Reynolds number $R = \frac{LU}{\nu}$.
- Energy injected into large eddies.
- Energy removed from small eddies at viscous scale.
- Transfer by interaction between eddies.
- Concept of *inertial range*

K41 : In the limit of ∞ R, all small scale statistical properties depend only on the local scale, k, and the energy dissipation rate, ϵ. Dimensional analysis :

$$E(k) = c\epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$ Kolmogorov spectrum

Two dimensional turbulence



- Generation of large scale structure from small scale motion
- Notion of an inverse cascade
- Two dimensional turbulence looks really different to three dimensions.
- Why?

Special Role of Vorticity in Two Dimensions

Vorticity, $\omega = \nabla \times \vec{u}$, is a *scalar* in 2D:

$$\frac{\partial \omega}{\partial t} + (\vec{u} \cdot \nabla)\,\omega = \text{forcing} + \text{dissipation}$$

where

$$\vec{u} = (u_x, u_y) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)$$

with stream-function, ψ obtained from solving

$$\omega = -\nabla^2 \psi.$$

In 2D, unlike 3D, both total energy, $E = \int |\vec{u}|^2 \, d\vec{x}$ and total enstrophy, $H = \int \omega^2 d\vec{x}$ are conserved. This fact profoundly alters the physics.

Kraichnan-Leith-Batchelor Phenomenology

A stationary state requires *two* cascades with two independent dissipation mechanisms with *two* fluxes :

- Direct cascade of enstrophy from the forcing scale to small scales. K41 Hypothesis gives

$$E(k) \sim \zeta^{\frac{2}{3}} k^{-3}.$$

- Inverse cascade of energy from the forcing scale to large scales.

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}.$$

- This picture assumes infinite inertial ranges for both cascades. In reality there are finite size effects which are more potent in 2D than in 3D.

Are the KLB spectra realised?

- Clean numerical or experimental observations of the KLB dual cascade are difficult but possible (recent large simulations of Boffetta, Celani et al.)
- A problem occurs when the inverse cascade is blocked by the largest scale - a fact which Kraichnan himself was aware of.
- What is the true stationary spectrum of two dimensional turbulence in a finite box?
- Several contradictory scenarios in the literature.
  - Smith and Yakhot ("condensation" at large scales)
  - Borue (k^{-3} spectrum)
  - Tran and Bowman (k^{-3} and $k^{-5/3}$)

$k^{-5/3}$ spectrum at early times

L. Smith and V. Yakhot Phys. Rev. Lett. 1993:

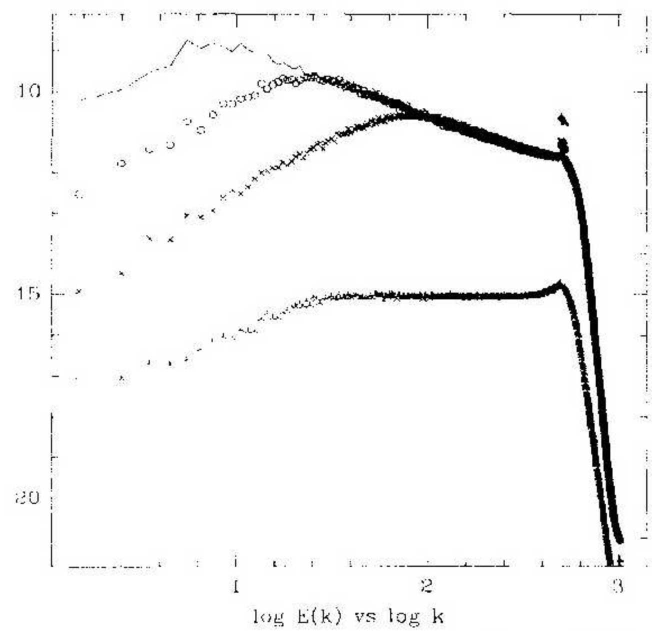


FIG. 1. Time evolution (increasing upward) of $E(k)$ for the 2048^2 run.

- Everyone(?) agrees that the $k^{-5/3}$ scaling is correct during the *establishment* of the inverse cascade.
- For an infinite system this is the full story.

"Condensation" of energy at the large scales

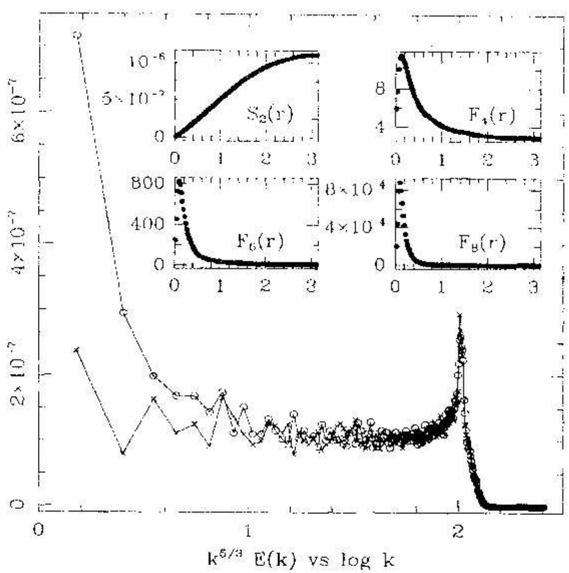


FIG. 3. Compensated spectra at an early time t_1 (crosses) and a later time t_2 (circles) after condensate formation. Inset: S_2 and F_{2n}, $n=2$–4 at t_2.

Later times

- For a finite system the front cannot continue indefinitely.
- No (or very little) large scale dissipation
- Inverse cascade eventually reaches the largest scale, k_0.
- Compensated spectra show energy pile-up at largest scale
- Emergence of "condensate" $E(k) \sim E_c(t)\,\delta(k-k_0) + \widetilde{E}(k)$.

Alternatively : k^{-3} at later times

V. Borue Phys. Rev. Lett. 1994

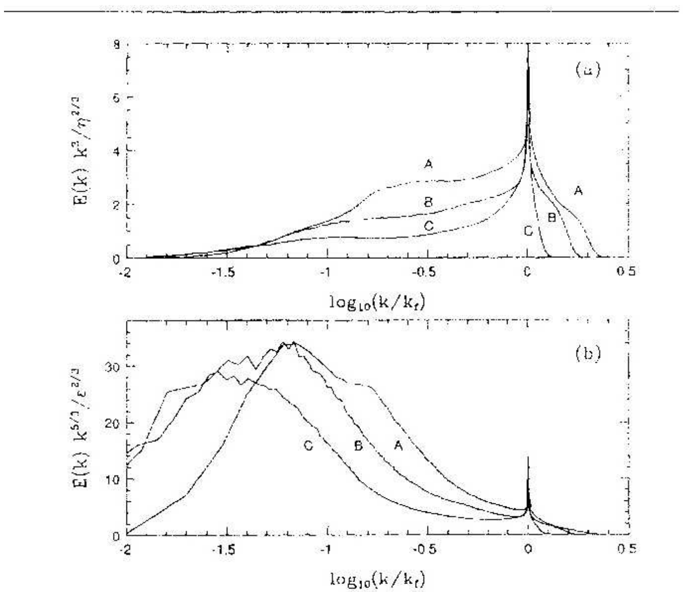


FIG. 1. Energy spectra normalized as (a) $E(k)k^3/\eta^{2/3}$ and (b) $E(k)k^{5/3}/\varepsilon^{2/3}$ as the function of $\log_{10}(k/k_f)$ for three runs labeled according to Table I.

Late times

- Hypo-viscosity at large scales.
- Inverse cascade probably reaches the largest scale, k_0, but weakly.
- Compensated stationary spectra show k^{-3} scaling (almost) everywhere.
- No sign of condensation in the sense of Smith and Yakhot.

Or both spectra can co-exist?

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Tran and Bowman Phys. Rev. E 2004 :

Late times

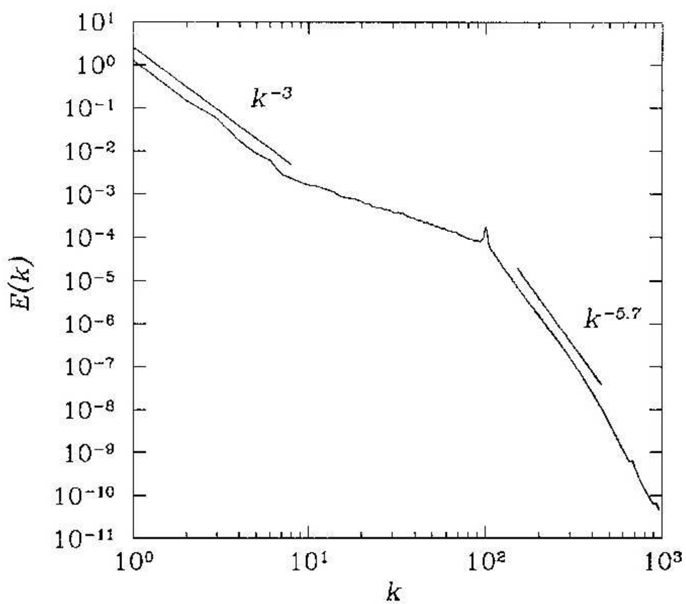


FIG. 5. The energy spectrum $E(k)$ vs k at $t=720$.

- Almost no dissipation at large scales.
- Inverse cascade reaches the largest scale, k_0.
- Observe a clean cross-over from k^{-3} to $k^{-\frac{5}{3}}$.
- Non-stationary.
- No "condensate".

Forcing and dissipation

We force and dissipate the vorticity, ω, directly. The forcing and dissipation are modeled in $\vec{k}$-space as follows :

$$\dot{\omega}_{\vec{k}} + \mathrm{NL}\left[\omega_{\vec{k}}, \psi_{\vec{k}}\right] = f_{\vec{k}} - \gamma_i(\vec{k})\omega_{\vec{k}} - \gamma_u(\vec{k})\omega_{\vec{k}}$$

with

$$f_{\vec{k}} = \sqrt{F(k)}\zeta(\vec{k}, t)$$ Intermediate scale forcing

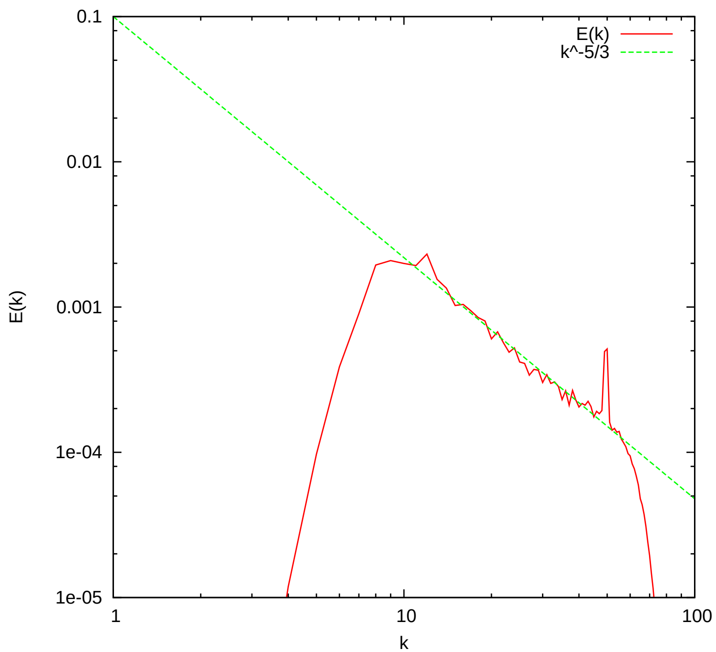
$$\gamma_u(\vec{k}) = \nu_u\, k^{\alpha_u}$$ Small scale dissipation

$$\gamma_i(\vec{k}) = \nu_i\, k^{-\alpha_i}$$ Large scale dissipation

$< \zeta(\vec{k}_1, t_1)\zeta(\vec{k}_2, t_2) >= \delta(\vec{k}_1 - \vec{k}_2)\delta(t_1 - t_2)$ Physical values are $\alpha_i = 0$ (friction) and $\alpha_u = 2$ (viscous dissipation)

Realisation of the "normal" inverse cascade

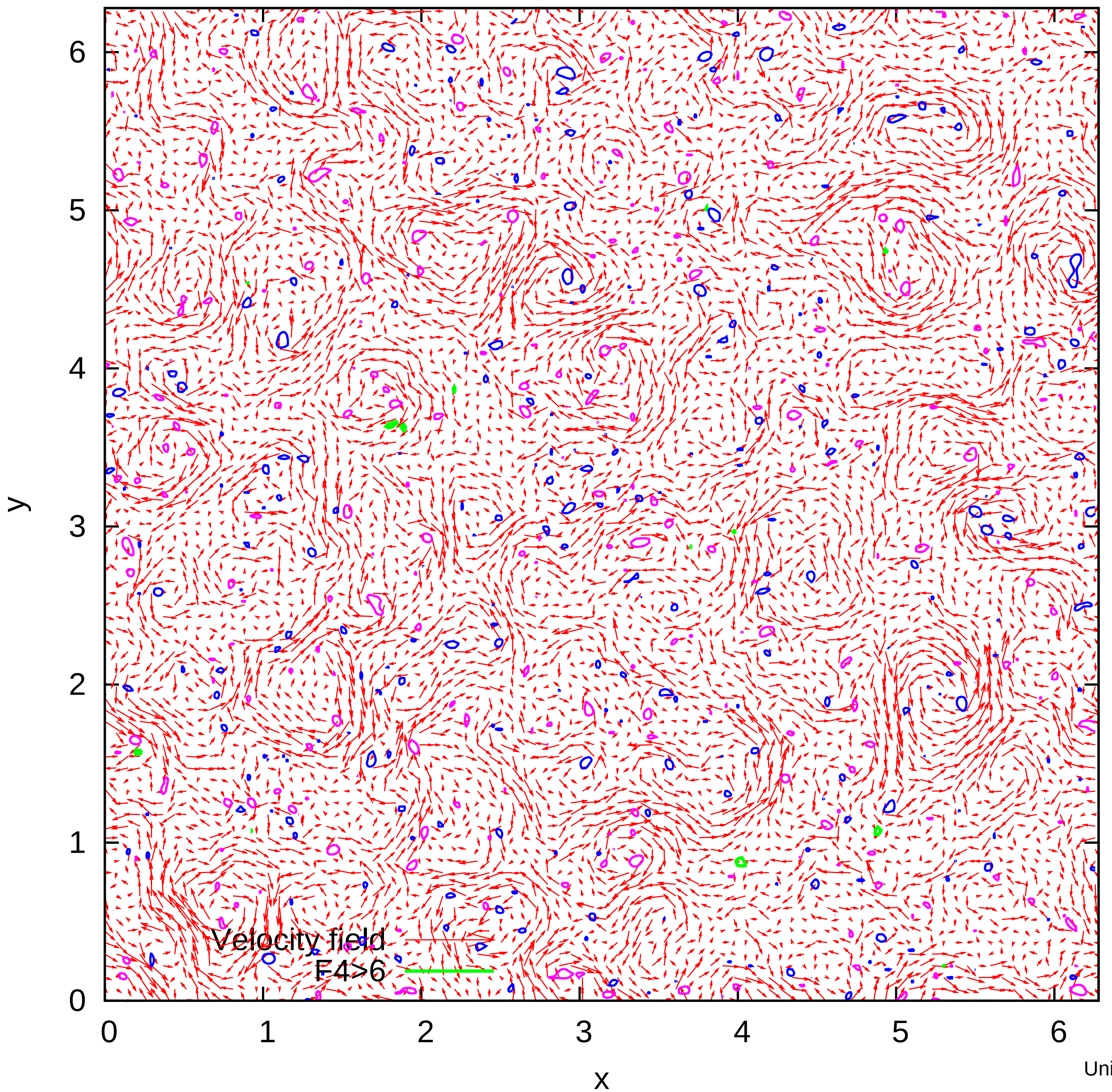
If the large scale dissipation is strong enough we should be able to arrest the $k^{-5/3}$ before it reaches k_0.



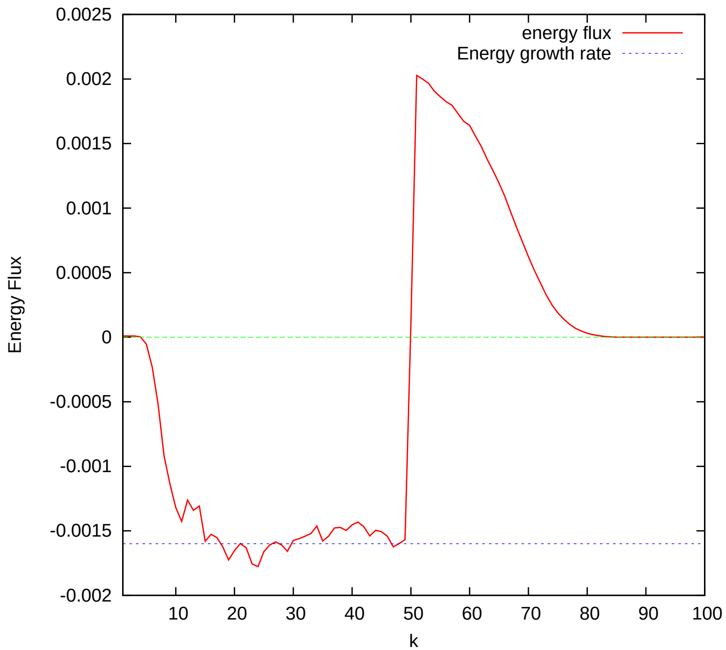
Late times

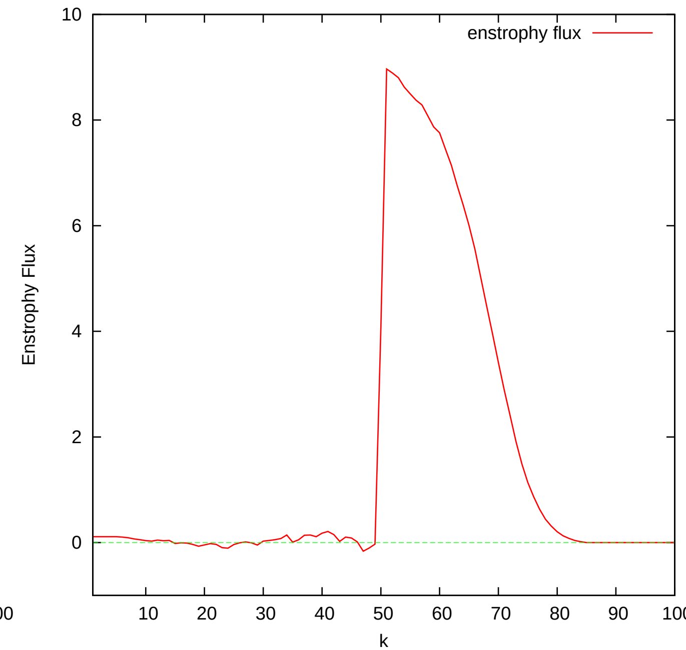
- Input energy flux ≈ 0.004.
- Kolmogorov constant ≈ 5.5.
- Very little direct cascade.
- About 50% of energy goes upscale and 50% downscale
- Clean, stationary $k^{-5/3}$ spectrum.

Flow field in physical space



Energy and Enstrophy Fluxes





- Fluxes are computed using standard expressions (eg Kraichnan, 1967)
- Behave as expected

Critical value the large scale dissipation

- Characteristic velocity fluctuation at scale r can be related to the exponent of the energy spectrum :

$$(\delta v)_r \sim (\epsilon r)^{\frac{x-1}{2}} \qquad E(k) \sim k^{-x} \;\; (1 < x < 3)$$

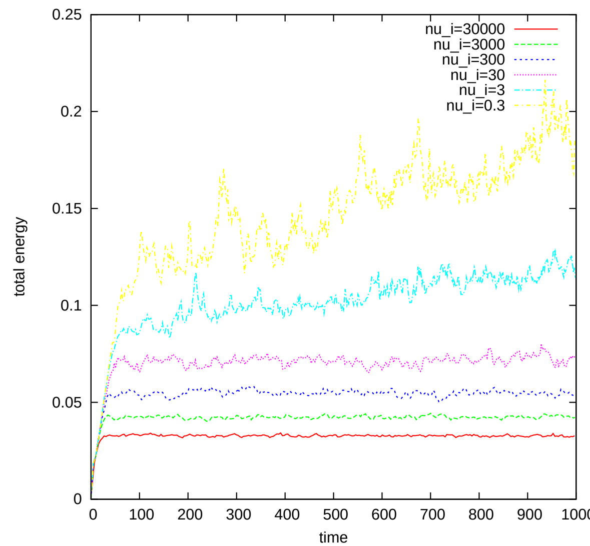
- K-L-B spectra : $x = 5/3$ gives $(\delta v)_r \sim (\epsilon r)^{1/3}$, $x = 3$ gives $(\delta v)_r \sim (\epsilon r)$
- Dissipation (large) scale, ξ, can be estimated by balancing terms :

$$\vec{v} \cdot \nabla \vec{v} \sim \nu_i \nabla^{-\alpha_i} \vec{v} \Rightarrow \xi = \left(\nu_i \epsilon^{-\frac{1}{3}}\right)^{-\frac{3}{2+3\alpha_i}}$$

- Inverse cascade is blocked when $\xi \sim L$. Occurs when the dissipation is too weak. Critical value :

$$\nu_i^* = \epsilon^{\frac{1}{3}} L^{-\frac{2+3\alpha_i}{3}}.$$

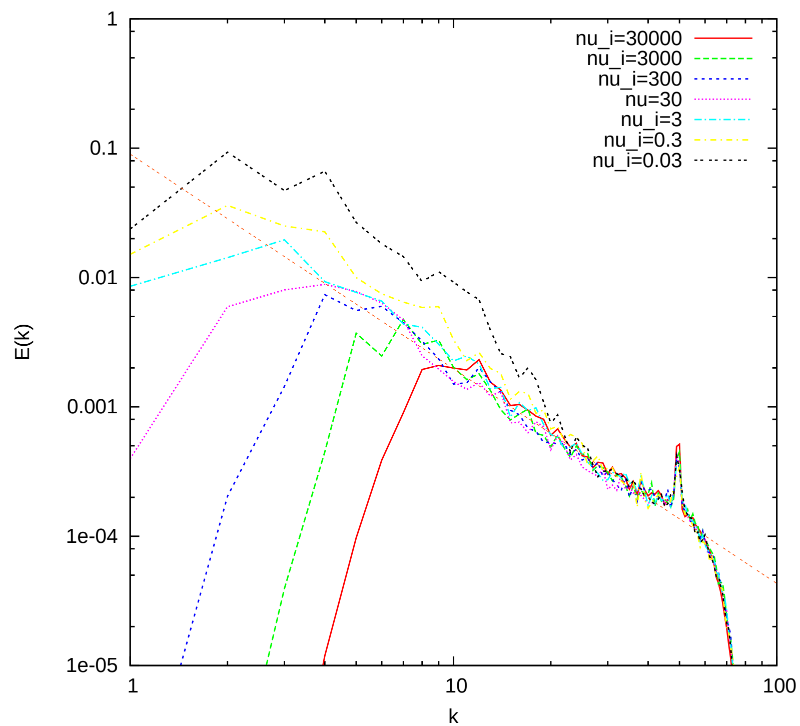
Stationary states for decreasing ν_i



- As ν_i decreases it takes longer for the system to reach a stationary state.
- Relative size of the fluctuations increases as the energy gets concentrated at larger scales.

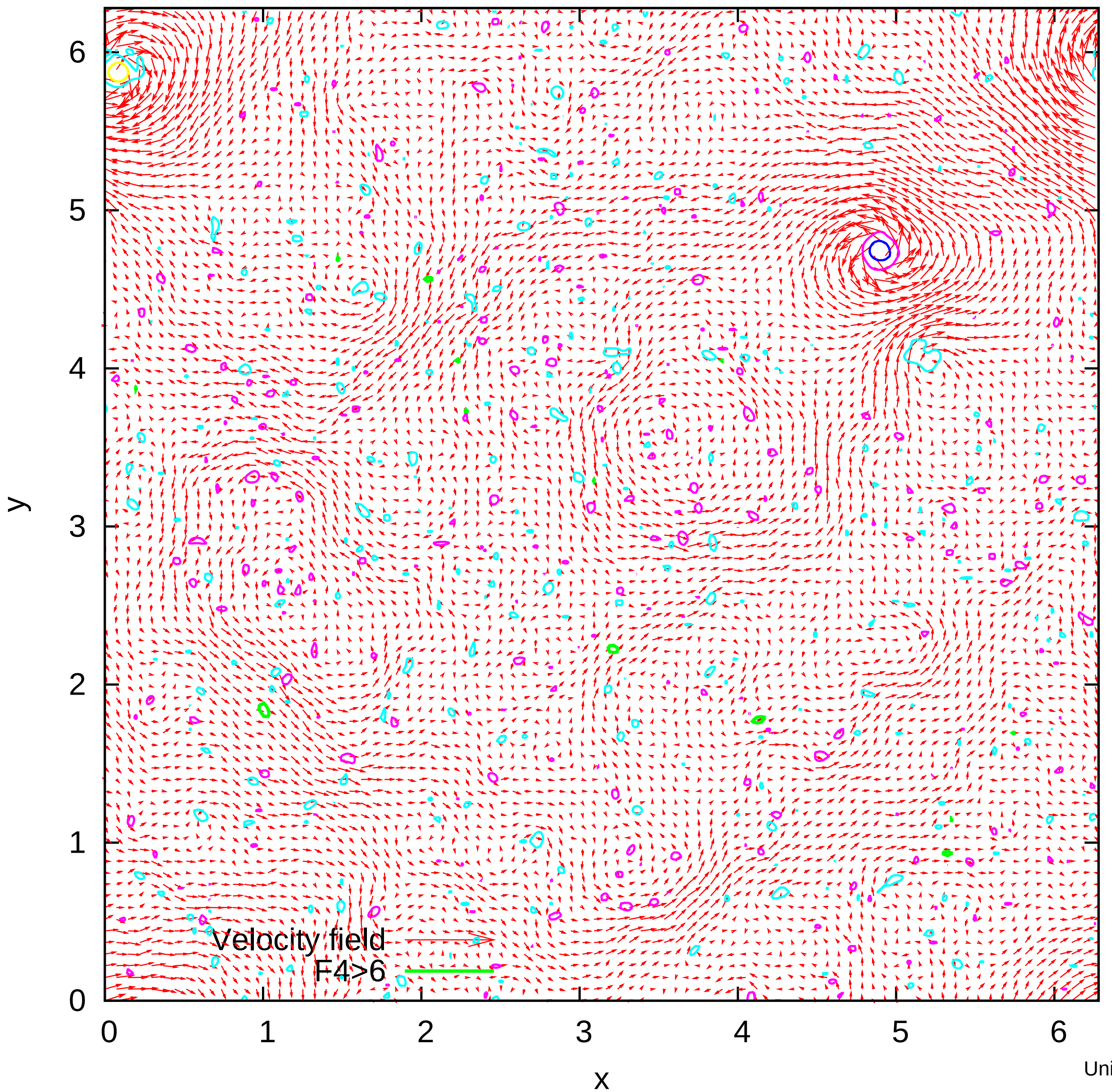
Click here to view the movie

Spectra with decreasing ν_i

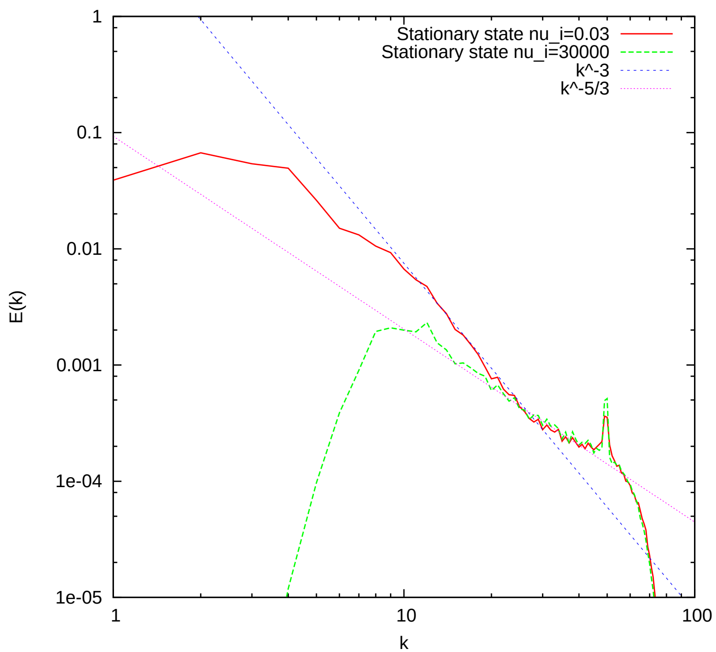


- For large dissipation you see a stationary $5/3$ spectrum as expected (range is small)
- Deviations occur when the dissipation weakens.
- Reasonable agreement with estimated value of ν_i^*.

Flow field in physical space



Emergence of a k^{-3} regime



- Comparison of stationary states for $\nu_i = 30000$ and $\nu_i = 0.03$.
- Both $-5/3$ and -3 (stationary) spectra can be observed depending on the efficiency of large scale dissipation.
- Evidence is qualitative.

"Strength" of the coherent component of the flow depends on the relative efficiencies of the forcing and large scale damping.

Experimental measurements by Shats et al.

PHYSICAL REVIEW E **71**, 046409 (2005)

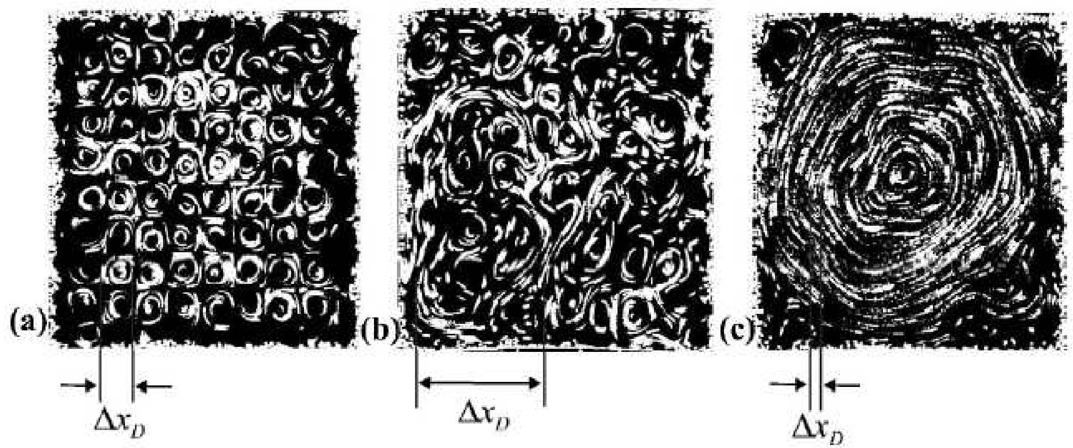


FIG. 2. Evolution of turbulence in a thin layer of electrolyte in a cell during spectral condensation. Trajectories of the tracer particles averaged over 12 frames of recorded video are shown. (a) The initial (linear) stage, $t=3$ s. (b) The inverse cascade stage, $t=25$ s. (c) The condensate stage, $t=60$ s. Δx_D represents the spatial scale of the trace particle transport during three stages of the flow evolution.

Experimental measurements by Shats et al.

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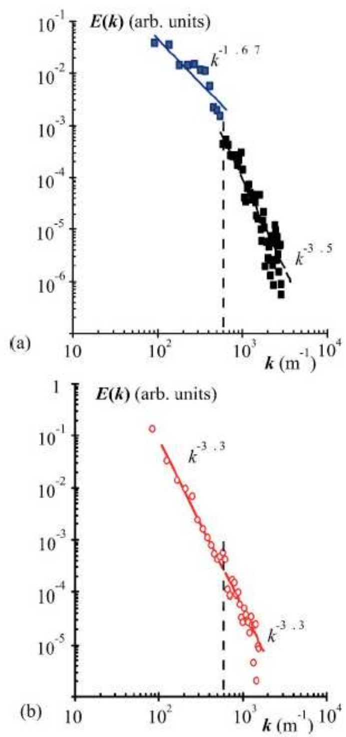
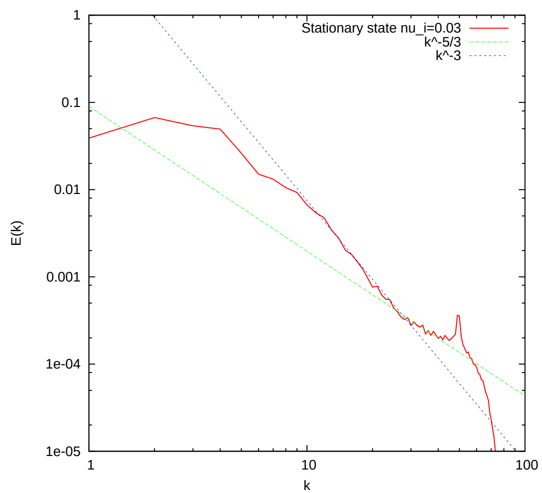


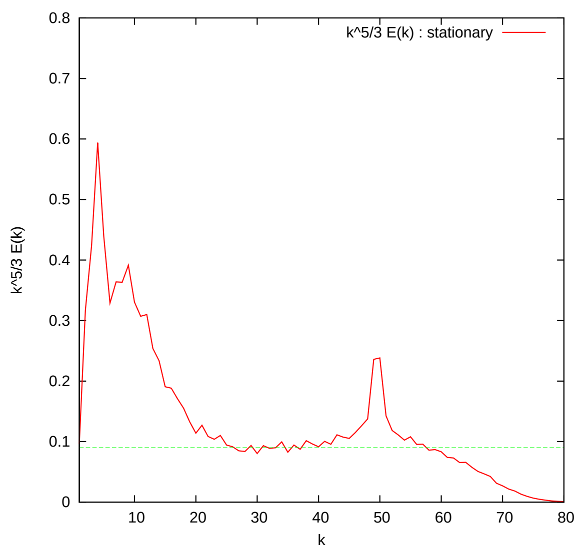
FIG. 4. (Color online) Energy spectra of the fluid velocities (a) during the inverse cascade stage of the flow development and (b) after the condensate has formed. The injection scale k_i is shown by the vertical dashed line.

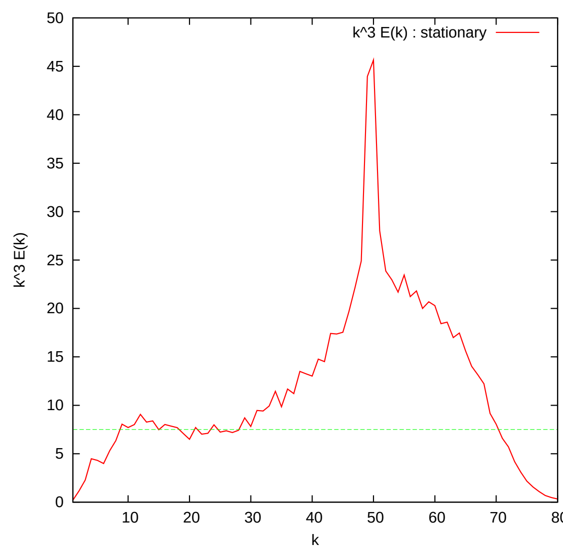
- Experimental measurements of spectra show $k^{-5/3}$ at early times.
- Replaced by $k^{-3.3}$ after the emergence of the large scale coherent part.
- This $k^{-3.3}$ is more robust than the vortices themselves.
- Physical space vorticity configuration is dependent on the boundary conditions.

Is there a cross-over?

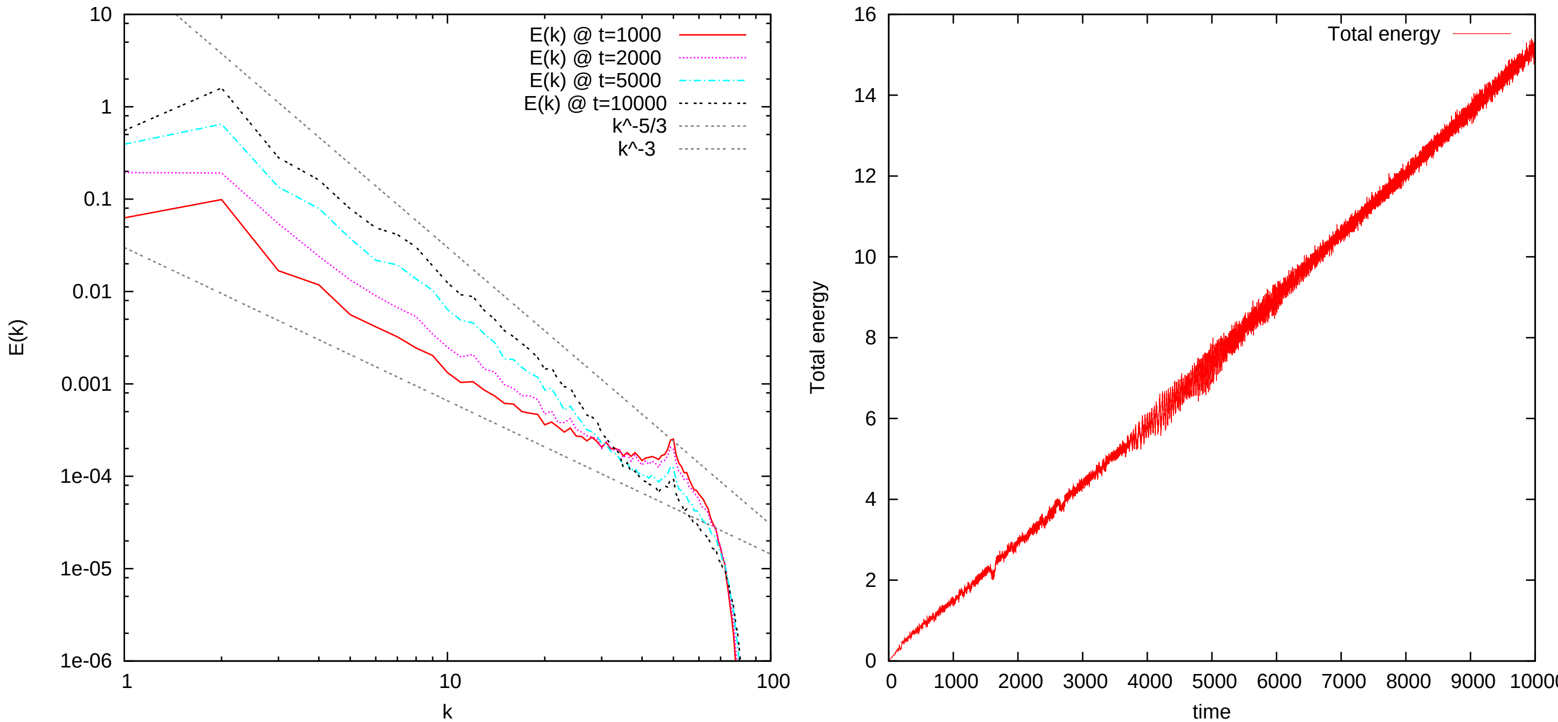


- Plot compensated spectra.
- Answer : probably. Also consistent with previous work.
- Cross-over regime requires tuning of parameters.



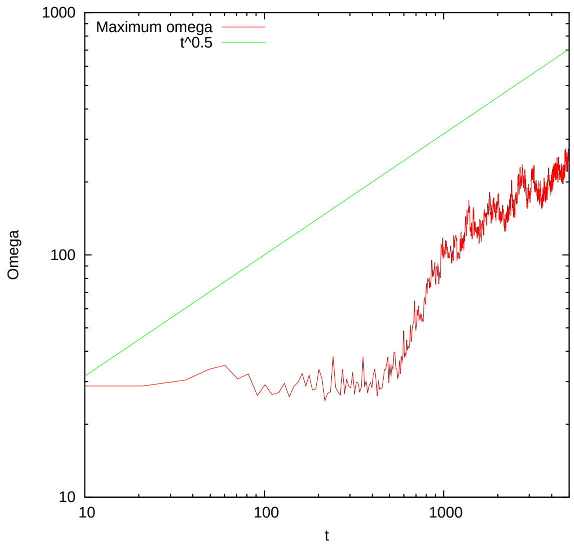


Zero large scale dissipation



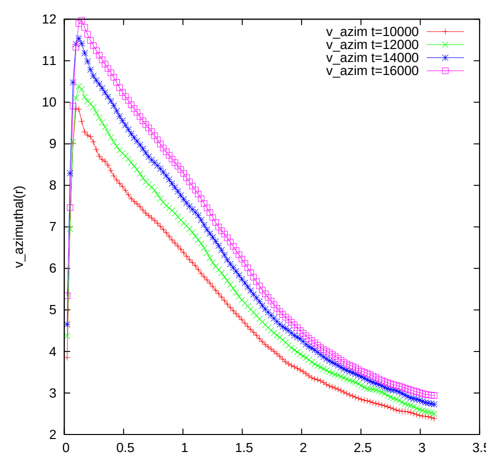
- k^{-3} seems to be established throughout the inertial range.
- Spectrum is completely non-stationary.
- Very large fluctuations in the energy - nonlocality?

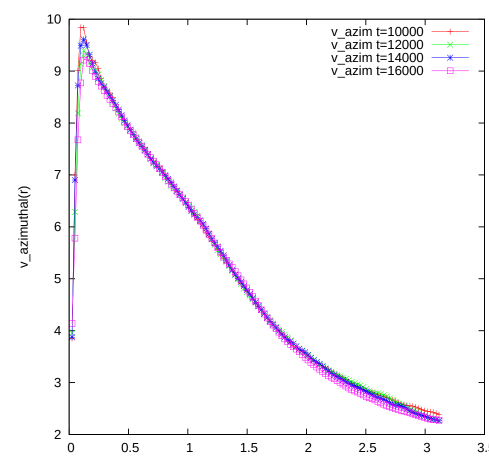
Growth of large scale mean flow



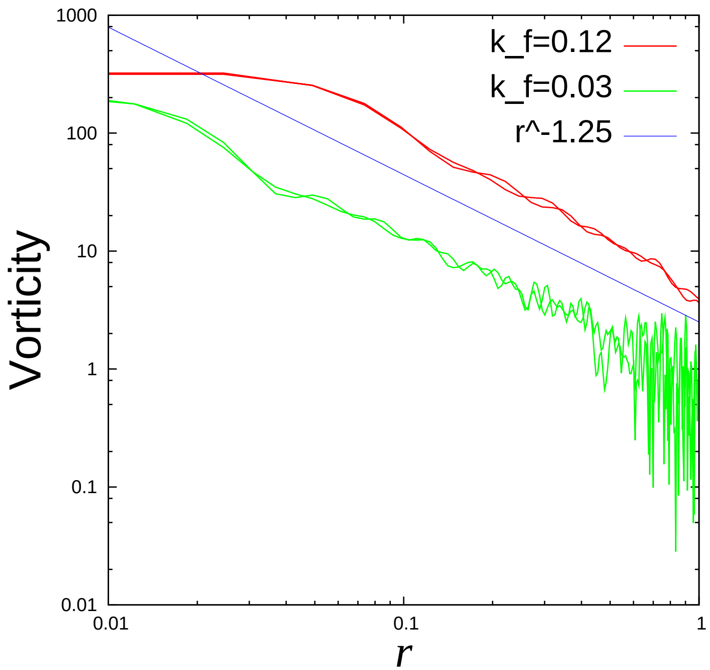
- Large scale flow fluctuates but is not really turbulent.
- Amplitude of vortex dipole grows like $\sqrt{t - t*}$.
- Flow profile within vortices evolves in a self-similar way.

Radial velocity profiles :





Anatomy of the large scale vortices

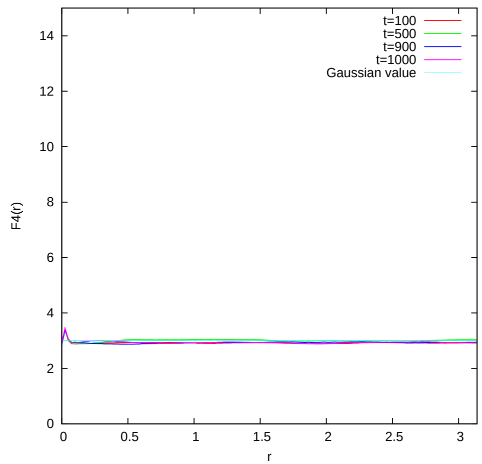


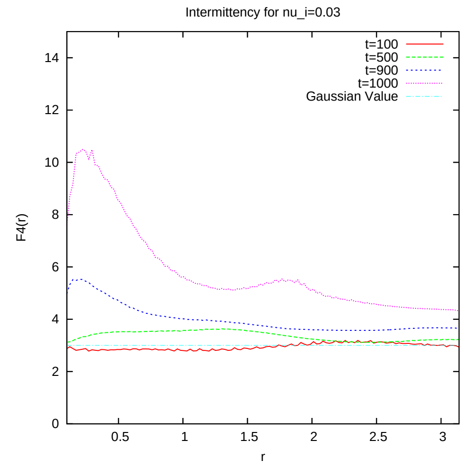
- Vorticity profile within each vortex is approximately

$$\Omega(r) \approx r^{-1.25}$$

- This exponent lacks an obvious theoretical explanation.
- Independent of the forcing scale. Persists when pumping is turned off.

Statistics of velocity increments





Usual inverse cascade   Blocked inverse cascade

Moments of velocity increments (Yakhot et al.):

$$S_n(r) = \langle ||(\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})) \cdot \vec{r}/r||^n \rangle$$

- "Flatness" $F_4(r) = \frac{S_4(r)}{S_2(r)^2}$
- Measures deviation from Guassian statistics.
- "Small scale intermittency" is entirely vortex dominated.

Preliminary conclusions

- There is still some work to be done to understand exactly what is the final stationary spectrum of 2-D turbulence in a finite box in the inverse cascade regime.
- Considering finite size effects and tunable (large scale) dissipation together allows a consistent picture at a qualitative level.
- Configurational details are probably not universal.
- However, the emergence of a smooth coherent flow at large scales might be a common signature of finite size effects.
- Does this have anything to do with atmospheric science?