The Potential Vorticity Equation
The geopotential tendency equation is

\[
\left[ \nabla^2 + \frac{\partial}{\partial p} \left( \frac{f_0^2}{\sigma} \frac{\partial}{\partial p} \right) \right] \Phi_t = -f_0 \mathbf{V}_g \cdot \nabla \left( \frac{1}{f_0} \nabla^2 \Phi + f \right) + \frac{\partial}{\partial p} \left[ \frac{f_0^2}{\sigma} \mathbf{V}_g \cdot \nabla \left( -\frac{\partial \Phi}{\partial p} \right) \right]
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The second term on the right (Term (C)) may be expanded:

\[-\mathbf{V}_g \cdot \nabla \frac{\partial}{\partial p} \left( \frac{f_0^2}{\sigma} \frac{\partial \Phi}{\partial p} \right) - \frac{f_0^2}{\sigma} \mathbf{V}_g \cdot \nabla \frac{\partial \Phi}{\partial p} \]
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But the thermal wind relationship is

\[f_0 \frac{\partial V_g}{\partial p} = k \times \nabla \frac{\partial \Phi}{\partial p}\]

This is just the \( p \)-derivative of \( f_0 V_g = k \times \nabla \Phi \).
Thus, $\partial V_g/\partial p$ is perpendicular to $\nabla(\partial \Phi/\partial p)$ and the second term above vanishes.
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The remaining term can be combined with term (B) in the tendency equation to give

$$\text{RHS} = -f_0 V_g \cdot \nabla \left[ \frac{1}{f_0} \nabla^2 \Phi + f + \frac{\partial}{\partial p} \left( \frac{f_0 \partial \Phi}{\sigma \partial p} \right) \right] = -f_0 V_g \cdot \nabla q$$
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The quantity in square brackets is called the *quasi-geostrophic potential vorticity*

$$q \equiv \left[ \frac{1}{f_0} \nabla^2 \Phi + f + \frac{\partial}{\partial p} \left( \frac{f_0 \partial \Phi}{\sigma \partial p} \right) \right]$$
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The left side of the tendency equation may be written

$$\text{LHS} = f_0 \frac{\partial}{\partial t} \left[ \frac{1}{f_0} \nabla^2 \Phi + \frac{\partial}{\partial p} \left( \frac{f_0 \partial \Phi}{\sigma \partial p} \right) \right] = f_0 \frac{\partial q}{\partial t}$$

since $f$ does not vary with time.
The tendency equation may now be written in a conservative form called the \textit{quasi-geostrophic potential vorticity equation} or QGPV equation:

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\left( \frac{\partial}{\partial t} + V_g \cdot \nabla \right) q = \frac{dgq}{dt} = 0
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Note that \( q \) is completely determined once the three-dimensional distribution of geopotential \( \Phi \) is given.
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This was the equation

$$\frac{d}{dt}(\zeta + f) = 0$$

for the conservation of *absolute vorticity*. 
Exercise
An idealized geopotential field is given at time $t = 0$ by

$$\Phi = \Phi_0 - f_0 \bar{u}y + A \sin(kx - mp)$$

where $\Phi_0$, $\bar{u}$ and $A$ are functions of $p$. 
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(c) Compute the variations in the temperature field due to the wave.
(d) Compute the vorticity advection and temperature advection.
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(e) Using the geopotential tendency equation, describe how the pressure at a point upstream from a trough and downstream from a ridge is expected to change.
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(f) Using the omega equation, describe the pattern of vertical velocity associated with the wave disturbance.
The ENIAC Integrations

(ENIAC: Electronic Numerical Integrator and Computer)
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Weather forecasting was a scientific problem *par excellence* for solution using a large computer.
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1. Entirely new methods of weather prediction by calculation will have been made possible;

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3. The first step towards influencing the weather by rational human intervention will have been made.
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The ENIAC
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It had:

- 18,000 vacuum tubes
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Power Consumption: 140 kWatts
ENIAC was a decimal machine. No high-level language. Assembly language. Fixed-point arithmetic: $-1 < x < +1$.

10 registers, that is, Ten words of high-speed memory.

Function Tables:
624 6-digit words of “ROM”, set on ten-pole rotary switches.

“Peripheral Memory”: Punch-cards.

Speed: FP multiply: 2ms (say, 500 Flops).

Access to Function Tables: 1ms.

Access to Punch-card equipment: You can imagine!
Evolution of the Project:

- **Plan A: Integrate the Primitive Equations**
  
  *Problems similar to Richardson’s would arise*
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- **Plan B: Integrate baroclinic Q-G System**
  
  Too computationally demanding

- **Plan C: Solve barotropic vorticity equation**
  
  Very satisfactory initial results
Charney, Fjørtoft, von Neumann

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\begin{bmatrix}
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\text{Vorticity}
\end{bmatrix} = 
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The atmosphere is treated as a single layer, and the flow is assumed to be nondivergent. Absolute vorticity is conserved:

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In more detail:

\[
\frac{\partial}{\partial t} [\nabla^2 \psi - F\psi] + \left\{ \frac{\partial \psi}{\partial x} \frac{\partial \nabla^2 \psi}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial \nabla^2 \psi}{\partial x} \right\} + \beta \frac{\partial \psi}{\partial x} = 0
\]
Solution method for QGBVE

\[ \frac{\partial \zeta}{\partial t} = -\mathbf{J}(\psi, \zeta + f) \]
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Solution method for QGBVE

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3. Solve Poisson equation for \(\psi\) (Fourier expansion)
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- Timestep : \( \Delta t = 1 \) hour (2 and 3 hours also tried)
- Gridstep : \( \Delta x = 750 \) km (approximately)
- Gridsize : \( 18 \times 15 = 270 \) points
- Elapsed time for 24 hour forecast: About 24 hours.
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Forecast involved punching about 25,000 cards. Most of the elapsed time was spent handling these.
ENIAC Algorithm

FUNCTION TABLES
- Coriolis parameter
- Map factor
- Scale factors

ENIAC OPERATIONS
1. Time-step extrapolation
2. New height and new vorticity
3. Prepare input deck for Operation 4

PUNCH-CARD OUTPUT
4. Jacobian (vorticity advection)
5. Vorticity tendency
6. Prepare input deck for Operation 8

PUNCH-CARD OPERATIONS
7. First Fourier transform (x)
8. x-transform of vorticity tendency
9. Prepare input deck for Operation 11

8. Second Fourier transform (y)
9. yx-transform of vorticity tendency
10. Prepare input deck for Operation 11

11. Third Fourier transform (y)
12. yyx-transform of vorticity tendency
13. Prepare input deck for Operation 13

14. Fourth Fourier transform (x)
15. Height tendency
16. Prepare input deck for Operation 1

15. Interleave height and vorticity tendencies
16. Height tendency
17. Prepare input deck for Operation 1
ENIAC: First Computer Forecast
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- “This is … an enormous scientific advance on the single, and quite wrong, result in which … [Richardson (1922)] ended.”