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(A) Growth by Condensation
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(A) Growth by Condensation

We saw from Kelvin’s Equation that, if the supersaturation is large enough to activate a droplet, the droplet will continue to grow. We will now consider the \textit{rate} at which such a droplet grows by condensation.
Consider first an isolated droplet, with radius $r$ at time $t$, in a supersaturated environment in which the water vapour density at a large distance from the droplet is $\rho_v(\infty)$ and the water vapour density adjacent to the droplet is $\rho_v(r)$. 
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Then, the rate of increase in the mass of the droplet at time $t$ is equal to the flux of water vapour across any spherical surface of radius $R$ centered on the droplet.
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We define the **diffusion coefficient** $D$ of water vapour in air as the rate of mass flow of water vapour across a unit area in the presence of a unit gradient in water vapour density.
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Then the rate of increase in the mass \( M \) of the droplet is given by

\[
\frac{dM}{dt} = 4\pi R^2 D \frac{d\rho_v}{dR}
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Since, under steady-state conditions, \( dM/dt \) is independent of \( R \), the above equation can be integrated as follows

\[ \frac{dM}{dt} \int_{R=r}^{R=\infty} \frac{dR}{R^2} = 4\pi D \int_{\rho_v(r)}^{\rho_v(\infty)} d\rho_v \]
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This gives

\[ \frac{1}{r} \frac{dM}{dt} = 4\pi D [\rho_v(\infty) - \rho_v(r)] \]
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Substituting \( M = \frac{4}{3} \pi r^3 \rho_\ell \), where \( \rho_\ell \) is the density of liquid water, into this last expression, we obtain

\[ r \frac{dr}{dt} = \frac{D}{\rho_\ell} [\rho_v(\infty) - \rho_v(r)] \]
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Here \( e(\infty) \) is the water vapour pressure in the ambient air well removed from the droplet and \( e(r) \) is the vapour pressure adjacent to the droplet.

If \( e \) is not too different from \( e_s \), then
\[ \frac{e(\infty) - e(r)}{e(\infty)} \approx \frac{e(\infty) - e_s}{e_s} = \left( \frac{e(\infty)}{e_s} - 1 \right) = S \]

where \( S \) is the supersaturation of the ambient air (expressed as a fraction rather than a percentage).
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Hence, we get

\[ r \frac{dr}{dt} = G_\ell S \]

where \( G_\ell = D \rho_v(\infty)/\rho_\ell \).
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It can be seen from this that, for fixed values of \( G_\ell \) and the supersaturation \( S \), the rate of increase \( dr/dt \) is inversely proportional to the radius \( r \) of the droplet.
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We write

\[ r \, dr = G_\ell S \, dt \]

which can be integrated immediately to give

\[ r = \sqrt{2G_\ell St} \quad \text{so that} \quad r \propto t^{1/2} \]
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Thus, droplets growing by condensation initially increase in radius very rapidly but their rate of growth diminishes with time (see following figure).
Schematic curves of droplet growth (a) by condensation from the vapour phase (blue curve) and (b) by collection of droplets (red curve).
Since the rate of growth of a droplet by condensation is inversely proportional to its radius, the smaller activated droplets grow faster than the larger droplets.
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Consequently, in this simplified model, the sizes of the droplets in the cloud become *increasingly uniform with time* (that is, the droplets approach a *monodispersed distribution*).
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Consequently, in this simplified model, the sizes of the droplets in the cloud become increasingly uniform with time (that is, the droplets approach a monodispersed distribution).

Comparisons of cloud droplet size distributions measured a few hundred meters above the bases of non-precipitating warm cumulus clouds with droplet size distributions computed assuming growth by condensation for about 5 min show good agreement (figure follows).
Cloud droplet size distribution measured 244 m above the base of a warm cumulus cloud (red) and the corresponding computed droplet size distribution assuming growth by condensation only (blue).
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For a cloud droplet $10 \mu m$ in radius to grow to a raindrop $1 mm$ in radius, an increase in volume of **one millionfold** is required! However, only about one droplet in a million (about $1 \text{ liter}^{-1}$) in a cloud has to grow by this amount for the cloud to rain.
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The enormous increases in size required to transform cloud droplets into raindrops is illustrated by the next diagram.
Relative sizes of cloud droplets and raindrops; \( r \) is the radius in micrometers, \( n \) the number per liter of air, and \( v \) the terminal fall speed in centimeters per second.
(B) Growth by Collection

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Since the terminal fall speed increases with the size of the droplet, larger droplets have a higher than average terminal fall speed.

Thus, they will collide with smaller droplets lying in their paths.
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In addition, the air exerts a *drag force* $F_{\text{drag}}$ on the body, which acts upwards.
The body will attain a steady terminal fall speed when these three forces are in balance:

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From the above equations, it follows that

\[ v = \frac{2 g (\rho' - \rho) r^2}{9 \eta} \]
If $\rho' \gg \rho$, which it is for liquid and solid objects,

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Water drops of radius 100 $\mu$m, 1 mm and 4 mm have terminal fall speeds of 25.6, 403 and 883 cm s$^{-1}$ respectively, which are very much less than given by the equation.
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This is because as a drop increases in size it becomes increasingly non-spherical and has an increasing wake. This gives rise to a drag force that is much greater than that given above.
Collision and Coalescence

Consider a single drop of radius $r_1$ that is overtaking a smaller droplet of radius $r_2$. 
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The *collision efficiency* $E$ of a droplet of radius $r_2$ with a drop of radius $r_1$ is defined as

$$E = \frac{y^2}{(r_1 + r_2)^2}$$

where $y$ is the distance from the central line for which the droplet just makes a grazing collision with the large drop (see Figure).
Relative motion of a small droplet (blue) with respect to a collector drop (red). $y$ is the maximum impact parameter for a droplet (radius $r_2$) with a collector drop (radius $r_1$).
The next issue is whether or not a droplet is captured (i.e., does *coalescence* occur?) when it collides with a larger drop. Droplets can bounce off one another or off a plane surface of water, as illustrated below.
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This occurs when air becomes trapped between the colliding surfaces, so that they deform without actually touching. In effect, the droplet rebounds on a cushion of air.

\begin{itemize}
  \item \textbf{Left:} A stream of water droplets, about 100 \(\mu\text{m}\) in diameter, rebounding from a plane surface of water.
  \item \textbf{Right:} When the angle between the stream of droplets and the surface of the water is increased beyond a critical value, the droplets coalesce with the water.
\end{itemize}
The *coalescence efficiency* $E'$ of a droplet of radius $r_2$ with a drop of radius $r_1$ is defined as the fraction of collisions that result in a coalescence.
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We will assume that the droplets are uniformly distributed in space and that they are collected uniformly at the same rate by all collector drops of a given size.
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This so-called *continuous collection model* is illustrated in the following diagram.
Schematic to illustrate the continuous collection model for the growth of a cloud drop by collisions and coalescence.
The rate of increase in mass $M$ of the large drop due to collisions is given by

$$\frac{dM}{dt} = \pi r_1^2 (v_1 - v_2) w_\ell E_c$$

where $w_\ell$ is the LWC (in kg m$^{-3}$) of the cloud droplets of radius $r_2$. 
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Substituting \( M = \frac{4}{3} \pi r_1^3 \rho_\ell \) here, where \( \rho_\ell \) is the density of liquid water, we obtain

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If $v_1 \gg v_2$ and we assume that the coalescence efficiency is unity, then

$$\frac{dr_1}{dt} = \frac{v_1 w_\ell E}{4\rho_\ell}$$

Since $v_1$ and $E$ both increase as $r_1$ increases, it follows that $dr_1/dt$ increases with increasing $r_1$; that is, the growth of a drop by collection is an *accelerating process*. 
Schematic curves of droplet growth (a) by condensation from the vapour phase (blue curve) and (b) by collection of droplets (red curve).
This “accelerating” behavior is illustrated by the red curve in the figure above, which indicates negligible growth by collection until the collector drop has reached a radius of about 20\(\mu m\).
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It can be seen from the Figure that for small cloud droplets growth by condensation is initially dominant but, beyond a certain radius, growth by collection dominates and rapidly accelerates.
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Eventually, as the drop grows, $v_1$ becomes greater than the updraft velocity $w$ and the drop begins to fall through the updraft and will eventually pass through the cloud base and may reach the ground as a raindrop.
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Provided that a few drops are large enough to be reasonably efficient collectors (i.e., with radius $\geq 20 \mu m$), and the cloud is deep enough and contains sufficient liquid water, raindrops should grow within reasonable time periods ($\sim 1$ hour).
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Clearly, deep clouds with strong updrafts should produce rain quicker than shallower clouds with weak updrafts.
Computer simulation

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Numerical predictions of the mass spectrum of drops in (a) a warm marine cumulus cloud, and (b) a warm continental cumulus cloud after about one hour of growth.
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It can be seen that the cumulus cloud in marine air develops some drops between 100 and 1000 $\mu$m in radius (that is, *raindrops*), whereas, the continental cloud does not contain any droplets greater than about 20 $\mu$m in radius.
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These markedly different developments are attributable to the fact that the marine cloud contains a small number of drops that are large enough to grow by collection, whereas the continental cloud does not.
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These model results support the observation that a marine cumulus cloud is more likely to rain than a continental cumulus cloud with similar updraft velocity, LWC and depth.
Shape and Size of Raindrops
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Laboratory and theoretical studies indicate that when the bag bursts, it produces a fine spray of droplets and the toroidal ring breaks up into a number of large drops (see Figure to follow).
Sequence of high-speed photographs showing how a large drop in free fall forms a parachute-like shape with a toroidal ring of water around its lower rim. Time interval between photographs = 1 ms.
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Zipf’s Law: In the English language, the probability of encountering the \( n \)th most common word is given roughly by \( P(n) = 0.1/n \) for \( n \) up to 1000 or so. The law breaks down for less frequent words, since the harmonic series diverges.