§5.5: Ekman Pumping
Effective Depth of Ekman Layer.
Defining $\gamma = \sqrt{f/2K}$, we derived the solution

$$u = u_g (1 - e^{-\gamma z} \cos \gamma z)$$
$$v = u_g e^{-\gamma z} \sin \gamma z$$

corresponding to the **Ekman spiral**.
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With $f = 10^{-4} \text{ s}^{-1}$ and $K = 5 \text{ m}^2 \text{s}^{-1}$ we have

$$D = \frac{\pi}{\gamma} = \pi \sqrt{\frac{2K}{f}} = \pi \sqrt{\frac{2 \times 5}{10^{-4}}} = 993 \text{ m} \approx 1 \text{ km}$$

Thus, the effective depth of the Ekman boundary layer is about one kilometre.
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Thus, the effective depth of the Ekman boundary layer is about one kilometre.

Note that $D$ depends on the values of $f$ and $K$ so the particular value 1 km is more an indication of the scale that a sharp quantitative estimate.
Remarks on the Ekman Spiral
• The Ekman theory predicts a cross-isobar flow of 45° at the lower boundary. This is not in agreement with observations.
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The modified Ekman Layer is discussed on Holton (§5.3.6). We will not discuss it here.
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We will now calculate the vertical velocity at the top of the Ekman layer.
First, consider a purely zonal geostrophic flow. So the isobars are oriented in an east-west direction.
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The cross-isobar mass transport through a column of unit width extending through the entire PBL is the vertical integral of $\rho_0 v$ through the layer $z = 0$ to $z = D = \pi/\gamma$:

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The result is thus

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Here we have used the fact that

\[ e^{-\pi} \approx 0.0432 \ll 1 \]
Exercise: Show that

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Solution:

- Evaluate the integral analytically
- Consult a Table of Integrals (e.g., GR2.663)
- Evaluate by numerical integration (MATLAB)
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For example, it can be done by means of integration by parts (twice), or by expressing the \( \sin \)-function in terms of complex exponentials.
Next, integrate the continuity equation through the PBL:

\[ \int_0^D \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \, dz = \int_0^D \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \, dz + [w(D) - w(0)] = 0 \]
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We assume the surface is flat, so that \( w(0) = 0 \). Then

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Now recall the Ekman solution

\[ u = u_g [1 - e^{-\pi z/D} \cos(\pi z/D)] , \quad v = u_g e^{-\pi z/D} \sin(\pi z/D) \]
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Substituting this into the equation for $w(D)$ gives

$$w(D) = -\frac{\partial u_g}{\partial y} \int_0^D e^{-\pi z/D} \sin(\pi z/D) \, dz = -\frac{1}{2} \left( \frac{D}{\pi} \right) \frac{\partial u_g}{\partial y}$$
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We now note that the **geostrophic vorticity** is given by

\[ \zeta_g = \left( \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} \right) = -\frac{\partial u_g}{\partial y} \]
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Thus,

\[ w(D) = -\frac{1}{2} \left( \frac{D}{\pi} \right) \frac{\partial u_g}{\partial y} = \left( \frac{D}{2\pi} \right) \zeta_g \]
This is the so-called **Ekman Pumping** formula:

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w(D) = \left( \frac{1}{2\pi} \right) D\zeta_g
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Note on **Dines Mechanism** to be added later.
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\[ w(D) = \left( \frac{1}{2\pi} \right) \times 10^3 \times (5 \times 10^{-5}) = \frac{5 \times 10^{-2}}{2\pi} \approx 8 \text{ mm s}^{-1} \sim 1 \text{ cm s}^{-1} \]
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If it is sufficient to lift air to its LCL, then latent heat release allows stronger updrafts within the convective clouds.
Storms in Teacups

Standing waves in a tea cup, induced by the propeller rotation of an airoplane.
Cyclostrophic Balanced Rotation
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That is, the azimuthal velocity depends linearly on the radial distance:

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U_g = 0, \quad V_g = \omega r
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The centrifugal force is given, as usual, by

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\frac{V^2}{r} = \omega^2 r
\]
For steady flow, the centrifugal force is balanced by the pressure gradient force:

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\frac{1}{\rho_0} \frac{\partial p}{\partial r} = \omega^2 r
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This can be integrated immediately to give

$$p = p_0 + \frac{1}{2}\rho_0 \omega^2 r^2$$

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As a result, there is radial inflow near the bottom. By continuity of mass, this must result in upward motion near the centre.
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Furthermore, outflow must occur in the fluid above the boundary layer.
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The geostrophic vorticity is given, in cylindrical coordinates, by

$$\zeta_g = \mathbf{K} \cdot \nabla \times \mathbf{V}_g = \frac{1}{r} \left[ \frac{\partial(rV_g)}{\partial r} - \frac{\partial U_g}{\partial \theta} \right] = \frac{1}{r} \frac{\partial (\omega r^2)}{\partial r} = 2\omega$$
Thus, the Ekman pumping is

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We may compare this to the azimuthal velocity. At \( r = 5 \text{ cm} \) we have

\[ V_g = \omega r = 2\pi \text{s}^{-1} \times 5 \text{ cm} \approx 30 \text{ cm s}^{-1} \]

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**Exercise:** Create a storm in a teacup:
Stir your tea (no milk) and observe the leaves.
Exercise:

- Calculate the mass influx through the sides of a cyclone.
- Equate this to the upward flux through the top of the boundary layer.
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From the Ekman solution, the mean inward velocity is

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\bar{V}_{\text{inward}} = \frac{1}{2}ug \left( \frac{D}{\pi} \right)
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The horizontal inward mass transport is

$$M_H = \rho_0 \bar{V}_{\text{inward}} = \frac{1}{2} \rho_0 u g \left( \frac{D}{\pi} \right) \times 2\pi R = \rho_0 u g D R$$
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These must be equal, so

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For solid body rotation, \( u_g = \omega R \) and the geostrophic vorticity is \( \zeta_g = 2\omega \), so

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w(D) = \frac{D}{2\pi} \zeta_g
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We estimate the characteristic spin-down time of the secondary circulation for a barotropic atmosphere.
Spin-Down

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The barotropic vorticity equation is

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\frac{d\zeta_g}{dt} = -f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = f \frac{\partial w}{\partial z}
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Assuming \( w(H) = 0 \) and substituting the Ekman pumping for \( W(D) \) we get

\[
\frac{d\zeta_g}{dt} = -\frac{f}{(H - D)} \left( \frac{1}{2\pi} \right) D\zeta_g
\]
Assuming $H \gg D$, this is

$$\frac{d\zeta_g}{dt} = -\frac{fD}{2\pi H} \zeta_g$$
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If we define the time-scale

$$\tau_{Ekman} = \frac{2\pi H}{fD}$$

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The size of $\tau_{\text{Ekman}}$ may be estimated for typical values:
\[
\tau_{\text{Ekman}} = \frac{2\pi H}{fD} = \frac{2\pi \times 10^4}{10^{-4} \times 10^3} \approx 6 \times 10^5 \text{s}
\]
which is about seven days.
We may compare this to the time-scale for eddy diffusion. The diffusion equation is

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$$\frac{U}{\tau_{\text{Diff}}} = \frac{KU}{H^2}$$
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For the values already assumed (\( K = 5 \text{ m}^2 \text{ s}^{-1} \) and \( H = 10 \text{ km} \)) we get

\[
\tau_{\text{Diff}} = \frac{H^2}{K} \approx \frac{10^8}{5} = 2 \times 10^7 \text{ s}
\]

which is of the order of 225 days, about 30 times longer than \( \tau_{\text{Ekman}} \).
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However, cumulonimbus convection can produce rapid transport of heat and momentum through the entire troposphere.
Ekman spiral in the ocean.
Typical La Niña Pattern

Mean sea surface temperature, eastern Pacific Ocean
5 September to 5 October, 1998.
End of §5.5