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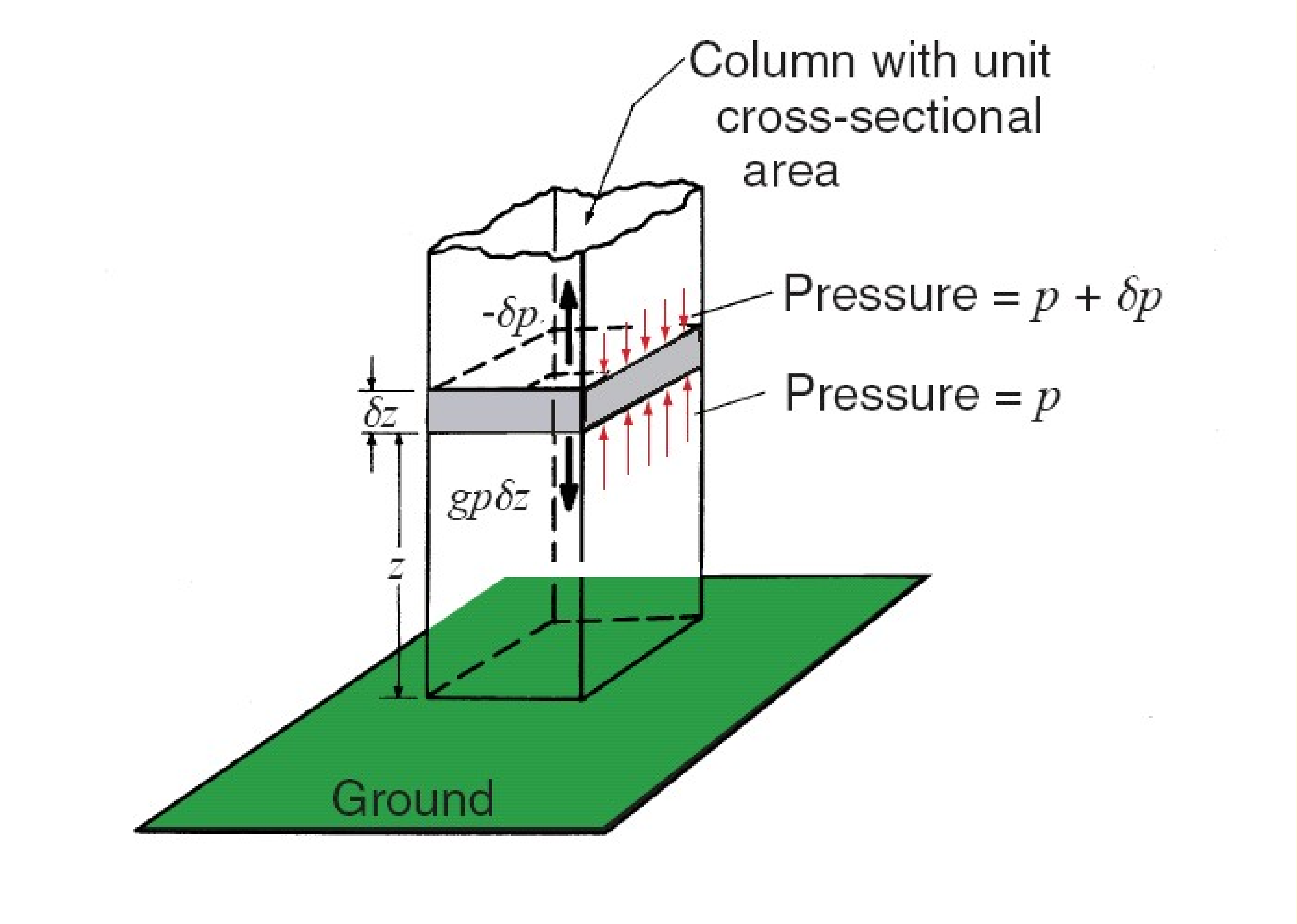
If the net upward pressure force on the slab is equal to the downward force of gravity on the slab, the atmosphere is said to be in *hydrostatic balance*.



Balance of vertical forces in an atmosphere in hydrostatic balance.

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The upward pressure on the lower face of the shaded block must be slightly greater than the downward pressure on the upper face of the block.

Therefore, the net vertical force on the block due to the vertical gradient of pressure is upward and given by the positive quantity $-\delta p$ as indicated in the figure.

For an atmosphere in hydrostatic equilibrium, the balance of forces in the vertical requires that

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The negative sign ensures that the pressure decreases with increasing height.

Since $\rho = 1/\alpha$, the equation can be rearranged to give

$$g\,dz = -\alpha\,dp$$

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$$-\int_{p(z)}^{p(\infty)} dp = \int_{z}^{\infty} g\rho\, dz$$

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If the mass of the Earth's atmosphere were uniformly distributed over the globe, the pressure at sea level would be 1013 hPa, or 1.013×10^5 Pa, which is referred to as 1 *atmosphere* (or 1 atm).

Geopotential

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The work (in joules) in raising 1 kg from z to $z + dz$ is $g\,dz$. Therefore

$$d\Phi = g\,dz$$

or, using the hydrostatic equation,

$$d\Phi = g\,dz = -\alpha\,dp$$

The geopotential $\Phi(z)$ at height z is thus given by

$$\Phi(z) = \int_0^z g\,dz\,.$$

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We define the *geopotential height* Z as

$$Z = \frac{\Phi(z)}{g_0} = \frac{1}{g_0}\int_0^z g\, dz$$

where g_0 is the globally averaged acceleration due to gravity at the Earth's surface.

Geopotential height is used as the vertical coordinate in most atmospheric applications in which energy plays an important role. It can be seen from the Table below that the values of Z and z are almost the same in the lower atmosphere where $g \approx g_0$.

Table 3.1

Values of the geopotential height (Z), and acceleration due to gravity (g), at 40° latitude for geometric height (z)

z(km)	Z(km)	g(m s^{-2})
0	0	9.81
1	1.00	9.80
10	9.99	9.77
100	98.47	9.50
500	463.6	8.43

The Hypsometric Equation

In meteorological practice it is not convenient to deal with the density of a gas, ρ, the value of which is generally not measured. By making use of the hydrostatic equation and the gas law, we can eliminate ρ:

$$\frac{\partial p}{\partial z} = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}$$

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Integrating between pressure levels p_1 and p_2, with geopotentials Z_1 and Z_1 respectively,

$$\int_{\Phi_1}^{\Phi_2} d\Phi = -\int_{p_1}^{p_2} R_d T_v \frac{dp}{p}$$

or

$$\Phi_2 - \Phi_1 = -R_d \int_{p_1}^{p_2} T_v \frac{dp}{p}$$

Dividing both sides of the last equation by g_0 and reversing the limits of integration yields

$$Z_2 - Z_1 = \frac{R_d}{g_0} \int_{p_2}^{p_1} T_v \, \frac{dp}{p}$$

The difference $Z_2 - Z_1$ is called the geopotential *thickness* of the layer.

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If the virtual temperature is constant with height, we get

$$Z_2 - Z_1 = H \int_{p_2}^{p_1} \frac{dp}{p} = H \log\left(\frac{p_1}{p_2}\right)$$

or

$$p_2 = p_1 \exp\left[-\frac{Z_2 - Z_1}{H}\right]$$

where $H = R_d T_v / g_0$ is the *scale height*. Since $R_d = 287\,\mathbf{J\,K^{-1}kg^{-1}}$ and $g_0 = 9.81\,\mathbf{m\,s^{-2}}$ we have, approximately, $H = 29.3\,T_v$.

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If we take a *mean* value for virtual temperature of $T_v = 255\,\mathbf{K}$, the scale height H for air in the atmosphere is found to be about $7.5\,\mathbf{km}$.

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***Exercise:* Check these statements.**

The temperature and vapour pressure of the atmosphere generally vary with height. In this case we can define a mean virtual temperature $\bar{T}_v$ (see following Figure):

$$\bar{T}_v = \frac{\int_{\log p_2}^{\log p_1} T_v d \, \log p}{\int_{\log p_2}^{\log p_1} d \, \log p} = \frac{\int_{\log p_2}^{\log p_1} T_v d \, \log p}{\log(p_1/p_2)}$$

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Using this in the thickness equation we get

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This is called the *hypsometric equation*:

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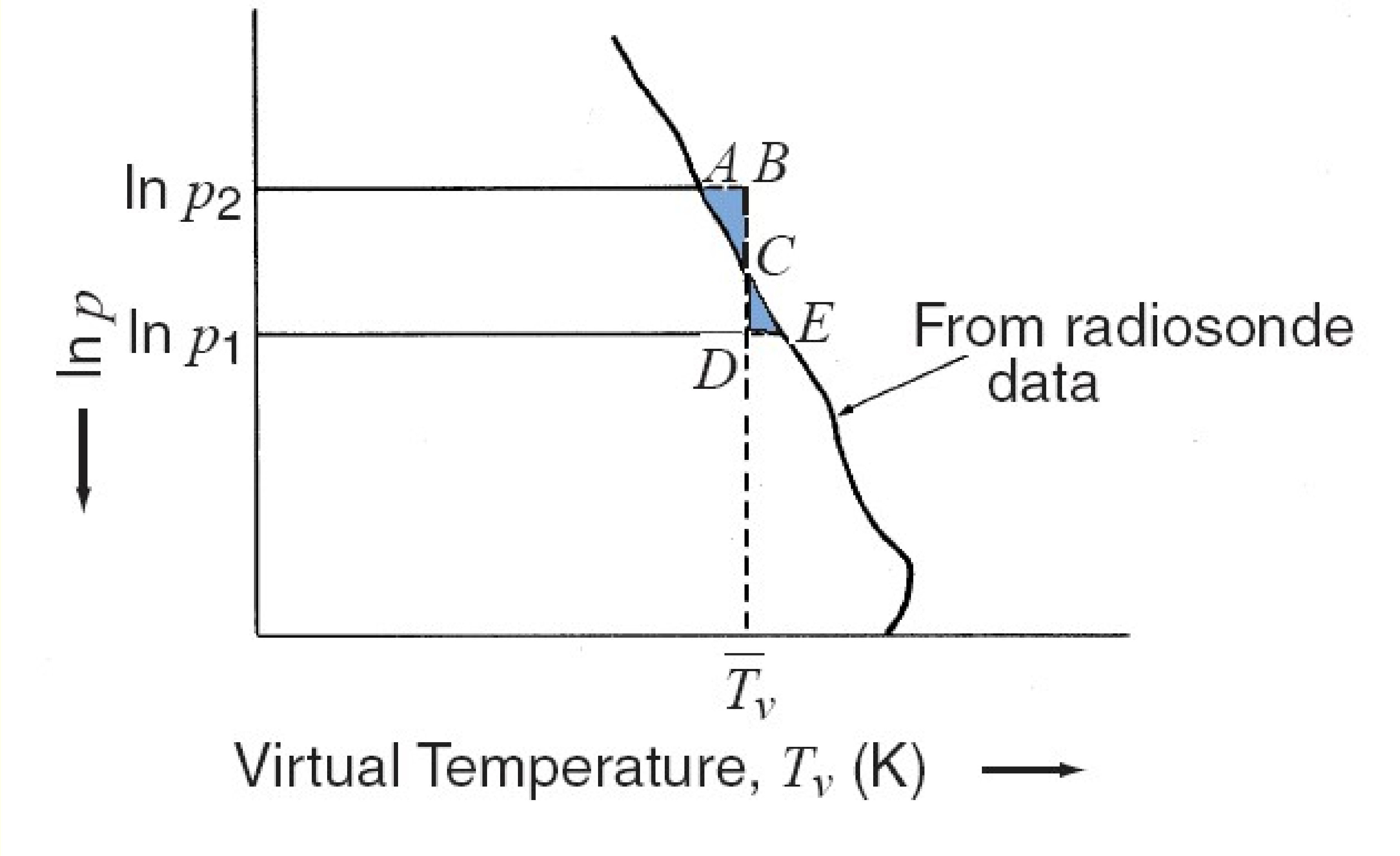

Figure 3.2. Vertical profile, or sounding, of virtual temperature. If area ABC is equal to area CDE, then $\bar{T}_v$ is the mean virtual temperature with respect to $\log p$ between the pressure levels p_1 and p_2.

Constant Pressure Surfaces

Since pressure decreases monotonically with height, pressure surfaces never intersect. It follows from the hypsometric equation that that the thickness of the layer between any two pressure surfaces p_2 and p_1 is proportional to the mean virtual temperature of the layer, $\bar{T}_v$.

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Essentially, the air between the two pressure levels expands and the layer becomes thicker as the temperature increases.

Exercise: Calculate the thickness of the layer between the 1000 hPa and 500 hPa pressure surfaces, (a) at a point in the tropics where the mean virtual temperature of the layer is 15°C, and (b) at a point in the polar regions where the mean virtual temperature is -40°C.

Solution: **From the hypsometric equation,**

$$\Delta Z = Z_{500} - Z_{1000} = \frac{R_d \bar{T}_v}{g_0} \ln\left(\frac{1000}{500}\right) = 20.3\,\bar{T}_v\,\textbf{metres}$$

Therefore, for the tropics with virtual temperature $\bar{T}_v = 288\,\mathbf{K}$ **we get**

$$\Delta Z = 5846\,\mathbf{m}\,.$$

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In operational practice, thickness is rounded to the nearest 10 m and expressed in decameters (dam). Hence, answers for this exercise would normally be expressed as 585 dam and 473 dam, respectively.

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- **Cold-core upper low**

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- Warm-core hurricane
- Cold-core upper low
- Extratropical cyclone

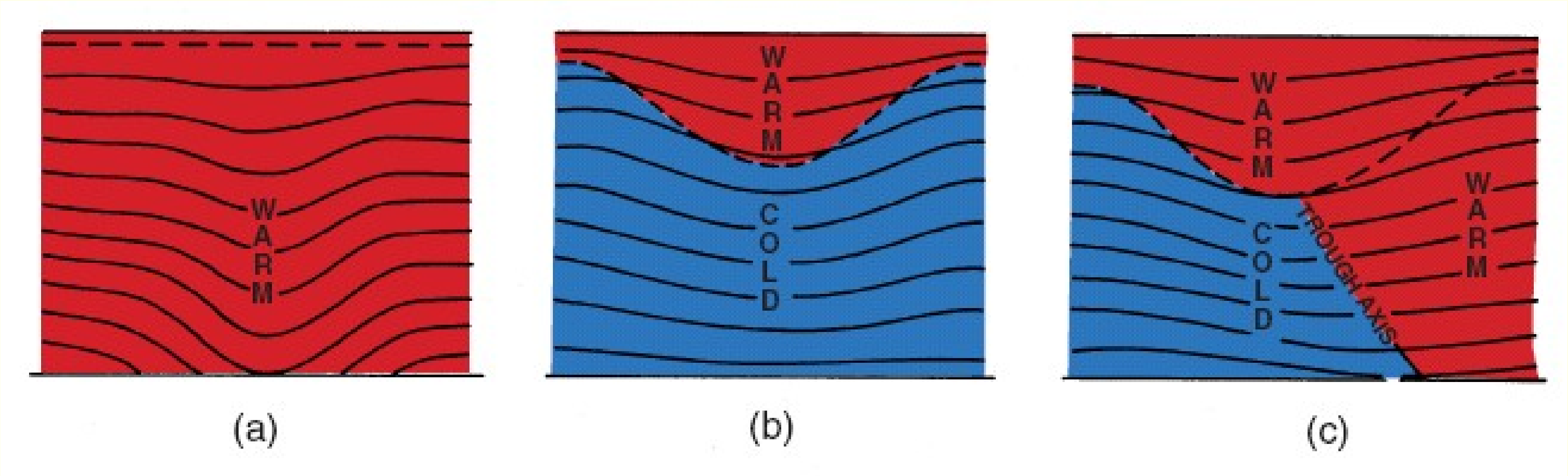

Figure 3.3. Vertical cross-sections through (a) a hurricane, (b) a *cold-core* upper tropospheric low, and (c) a middle-latitude disturbance that tilts westward with height.

Reduction of Pressure to Sea Level

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Let Z_g and p_g be the geopotential and pressure at ground level and Z_0 and p_0 the geopotential and pressure at sea level ($Z_0 = 0$).

Then, for the layer between the Earth's surface and sea level, the hypsometric equation becomes

$$(Z_g - Z_0) = Z_g = \bar{H} \ln \frac{p_o}{p_g}$$

where $\bar{H} = R_d \bar{T}_v / g_0$.

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This can be solved to obtain the sea-level pressure

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The last expression shows how the sea-level pressure depends on the mean virtual temperature between ground and sea level.

If Z_g is small, the scale height $\bar{H}$ can be evaluated from the ground temperature.

Also, if $Z_g \ll \bar{H}$, the exponential can be approximated by

$$\exp\left(\frac{Z_g}{\bar{H}}\right) \approx 1 + \frac{Z_g}{\bar{H}}\,.$$

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With this approximation, we get

$$p_0 \approx p_g\left(1 + \frac{Z_g}{\bar{H}}\right) \qquad \textbf{or} \qquad p_0 - p_g \approx \left(\frac{p_g}{\bar{H}}\right) Z_g$$

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Since $p_g \approx 1000\,$hPa and $\bar{H} \approx$ 8 km, the pressure correction (in hPa) is roughly equal to Z_g (in meters) divided by 8.

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In other words, near sea level the pressure decreases by *about 1 hPa for every 8 m of vertical ascent*.

Exercise: Calculate the geopotential height of the 1000 hPa pressure surface when the pressure at sea level is 1014 hPa. The scale height of the atmosphere may be taken as 8 km.

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Solution: From the hypsometric equation,

$$Z_{1000} = \bar{H} \ln\left(\frac{p_0}{1000}\right) = \bar{H} \ln\left(1 + \frac{p_0 - 1000}{1000}\right) \approx \bar{H}\left(\frac{p_0 - 1000}{1000}\right)$$

where p_0 is the sea level pressure and the approximation

$$\ln(1 + x) \approx x$$

for $x \ll 1$ has been used.

Exercise: Calculate the geopotential height of the 1000 hPa pressure surface when the pressure at sea level is 1014 hPa. The scale height of the atmosphere may be taken as 8 km.

Solution: From the hypsometric equation,

$$Z_{1000} = \bar{H} \ln\left(\frac{p_0}{1000}\right) = \bar{H} \ln\left(1 + \frac{p_0 - 1000}{1000}\right) \approx \bar{H}\left(\frac{p_0 - 1000}{1000}\right)$$

where p_0 is the sea level pressure and the approximation

$$\ln(1 + x) \approx x$$

for $x \ll 1$ has been used.

Substituting $\bar{H} \approx 8000\,\mathbf{m}$ into this expression gives

$$Z_{1000} \approx 8(p_0 - 1000)$$

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Therefore, with $p_0 = 1014\,\mathbf{hPa}$, the geopotential height Z_{1000} of the 1000 hPa pressure surface is found to be 112 m above sea level.

Exercise: Derive a relationship for the height of a given pressure surface p in terms of the pressure p_0 and temperature T_0 at sea level assuming that the *temperature decreases uniformly* with height at a rate Γ K km^{-1}.

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From these equations it follows that

$$\frac{dp}{p} = -\frac{g}{R(T_0 - \Gamma z)}dz$$

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Therefore

$$z = \frac{T_0}{\Gamma} \left[1 - \left(\frac{p}{p_o} \right)^{R\Gamma/g} \right]$$

Altimetry

The *altimetry equation*

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However, the scale of the altimeter is expressed as the height above sea level where z is related to p by the above equation with values for the parameters in accordance with the U.S. Standard Atmosphere:

$$\begin{aligned} T_0 &= 288\,\mathbf{K} \\ p_0 &= 1013.25\,\mathbf{hPa} \\ \Gamma &= 6.5\,\mathbf{K\,km}^{-1} \end{aligned}$$

Exercise (Hard!): Show that, in the limit $\Gamma \to 0$, the altimetry equation is consistent with the relationship

$$p = p_0 \exp\left(-\frac{z}{H}\right)$$

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Solution (Easy!): Use l'Hôpital's Rule.

Note: If you are unfamiliar with l'Hôpital's Rule, either ignore this exercise or, better still, try it using more elementary means.