Nonlinear Conservation Laws

- Many conservation laws, such as those of gas dynamics, are of the form

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \qquad (1)$$

i.e. the *flux* is a function of $u(x,t)$. This can also be re-written as

$$\frac{\partial u}{\partial t} + v(u)\frac{\partial u}{\partial x} = 0 \quad \text{where} \quad v(u) \equiv \frac{\partial f}{\partial u} \qquad (2)$$

- One important example is Burger's equation for inviscid fluid flow

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \qquad (3)$$

Recall that the Navier-Stokes equation for one dimensional flow is

$$u_t + uu_x = -\frac{1}{\rho}p_x + \mu u_{xx}$$

where μ is the coefficient of viscosity. Therefore the Burger's equation describes the flow of a pressureless, inviscid fluid. It's solution is, nevertheless, non-trivial.

- The characteristics for this equation are defined by

$$\frac{dx}{dt} = u\left(x(t), t\right)$$

and, since u is constant on each characteristic, the characteristics are straight lines with their slopes determined by the initial data (see exercises), i.e. given the initial condition $u(x,0) = u_0(x)$ the characteristic passing through the point $(x_*, 0)$ has the slope

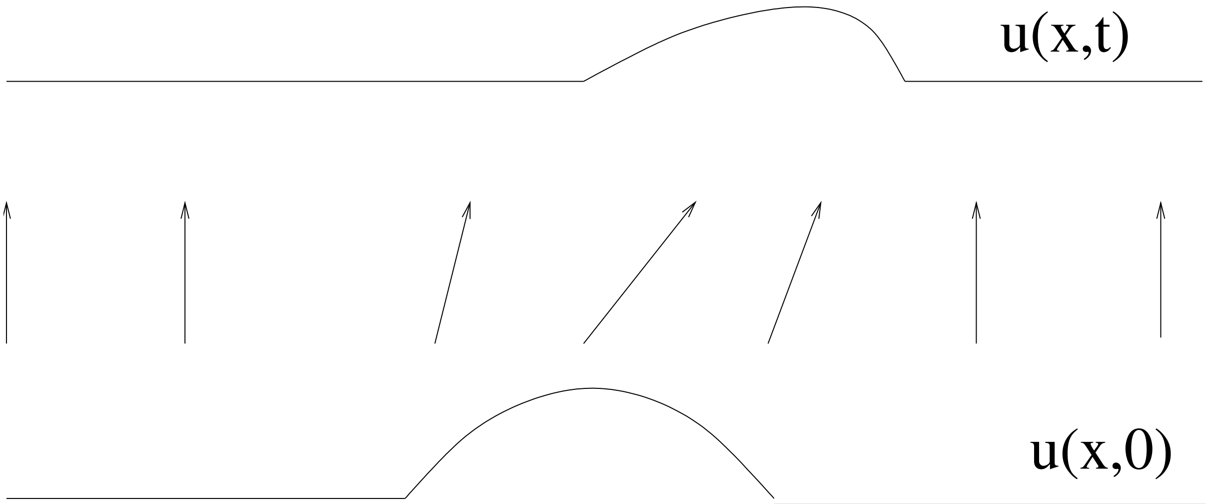
$$\frac{dt}{dx} = \frac{1}{u_0(x_*)}$$

on the (t,x) plane.

- When the general solution is smooth, at (x,t) we can draw the straight characterstic through that point back to $t = 0$ so that the solution can be written in implicit form,

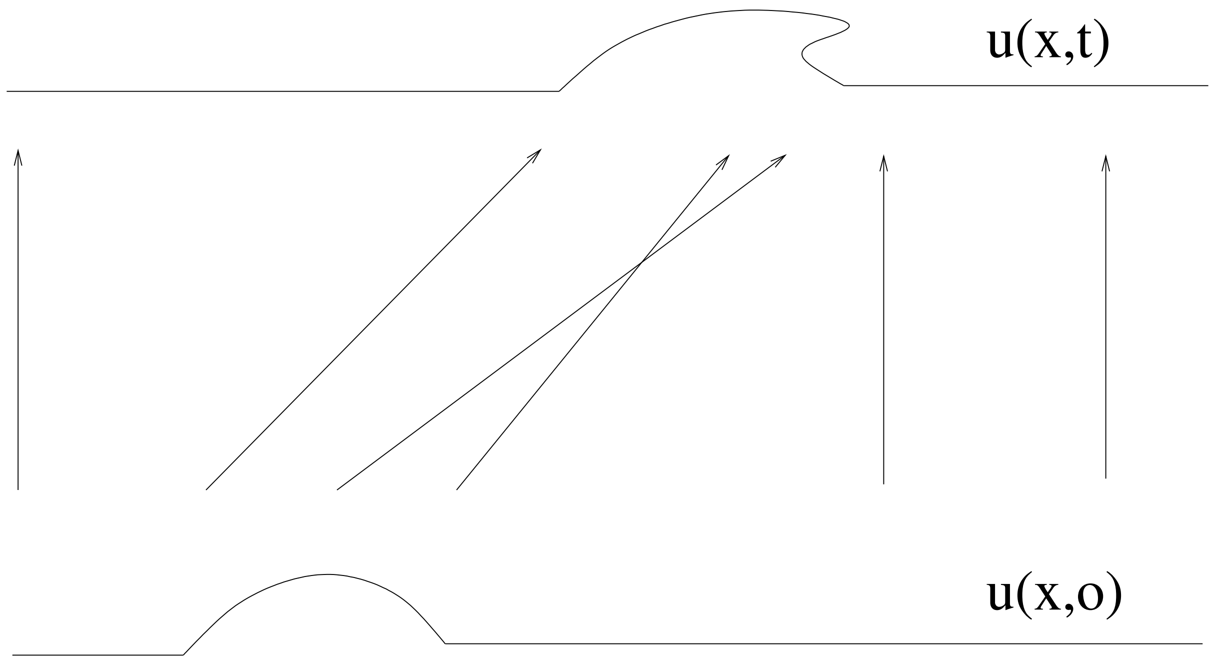
$$u(x,t) = u_0\left(x - u(x,t)t\right)$$

as shown in the diagram below for small t. See the exercises for solving this equation using the method of characteristics.



Note that since $v(u) = u$, the greater the amplitude of the wave the faster it moves. Therefore, parts of the wave which have an initial negative gradient will steepen until...

- Characteristics eventually cross at which point a singularity appears in the solution where it becomes multiple valued.



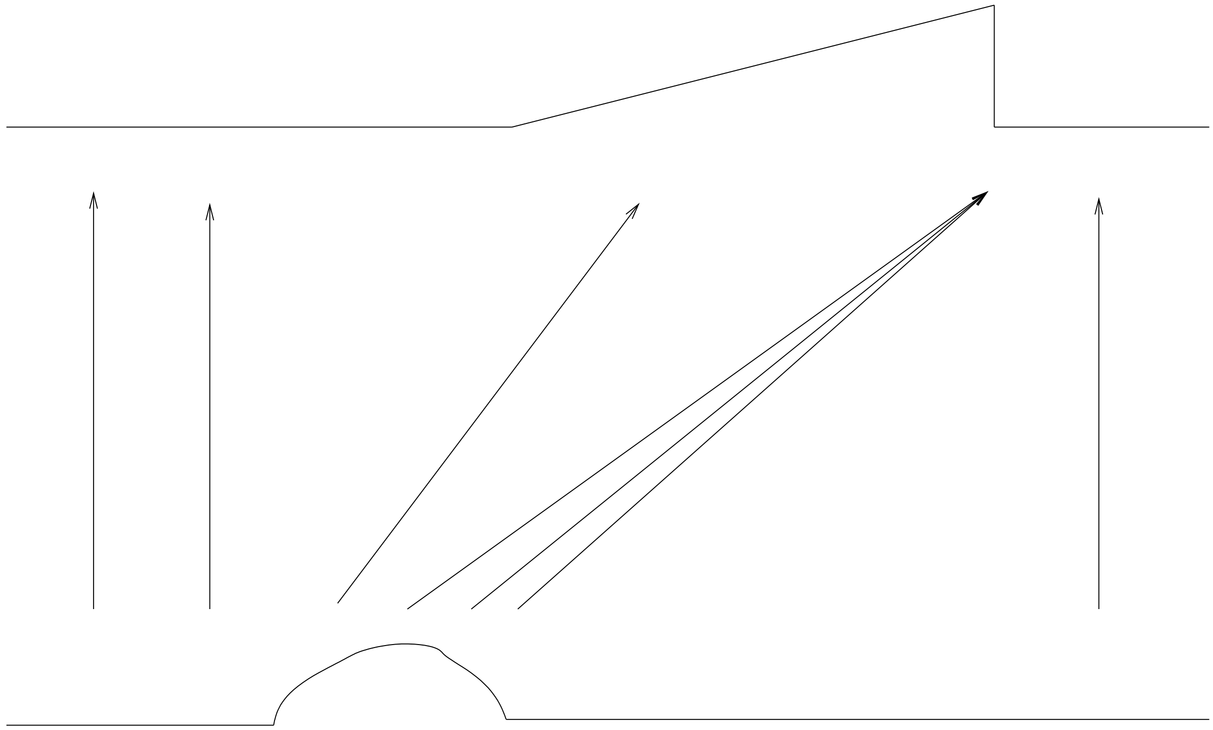
In the exercises you are asked to show that with an initial profile $u_0(x)$ where u_0' is somewhere negative, that characteristics will first cross at the time

$$t = -\frac{1}{\min u_0'(x)}$$

- The problem is that the mathematical model for gas flow breaks down at this point since viscosity becomes important as steep gradients develop and we should be solving the equation

$$u_t + uu_x = \mu u_{xx}$$

This leads to the formation of a shock or discontinuity in the flow beyond which the flow can't steepen further. This is shown schematically in the following plot.



Shock formation is a very difficult and specialised part of computational gas dynamics (AND NOT VERY IMPORTANT IN METEOROLOGY !!) and we will not study it in detail. Physically of course a shock is not a mathematical discontinuity but has a width determined by viscous dissipation. See *Further Reading*

Numerical Methods for Conservation laws

- Much of the groundwork for solving the inviscid Burger's equation has been done in the previous chapter. Here we take that work to guide us in what follws.
- Firstly, the FTCS method for Burger's equation is every bit as useless as it was for linear advection equation.
- For those methods that do work we still have the constraint that the timestep cannot exceed the CFL limit, but now the advection speed is determined by the local value of $u(x, t)$.
- The upwind scheme can also be generalised to the conservation law to give

$$u_j^{n+1} = \begin{cases} u_j^n - \frac{\tau}{h}\Delta_{+x} f_j^n & \text{if } v_{j+1/2}^n < 0 \\ u_j^n - \frac{\tau}{h}\Delta_{-x} f_j^n & \text{if } v_{j-1/2}^n > 0 \end{cases} \qquad (4)$$

where $f_j^n \equiv f(u_j^n)$ and $v_{j\pm 1/2} \equiv v[(u_j^n + u_{j\pm 1/2}^n)/2]$. We will see this in practice below for Burger's equation.

- There is an interesting variant of the Lax Wendroff method which is discussed in *further reading* and is called the two-step method. Here provisional values of u are calculated at half step values $(x_{j+1/2}, t_{n+1/2})$,

$$u_{j+1/2}^{n+1/2} = \frac{1}{2}\left(u_j^n + u_{j+1}^n\right) - \frac{\tau}{2h}\left(f(u_{j+1}^n) - f(u_j^n)\right)$$

which are then used to calculate the fluxes,

$$u_j^{n+1} = u_j^n - \frac{\tau}{h}\left(f(u_{j+1/2}^{n+1/2}) - f(u_{j-1/2}^{n+1/2})\right)$$

- Finally, the Leapfrog method is simply

$$u_j^{n+1} = u_j^{n-1} - \frac{\tau}{h}\left(f_{j+1}^n - f_{j-1}^n\right)$$

MATLAB Programme *burgers.m*

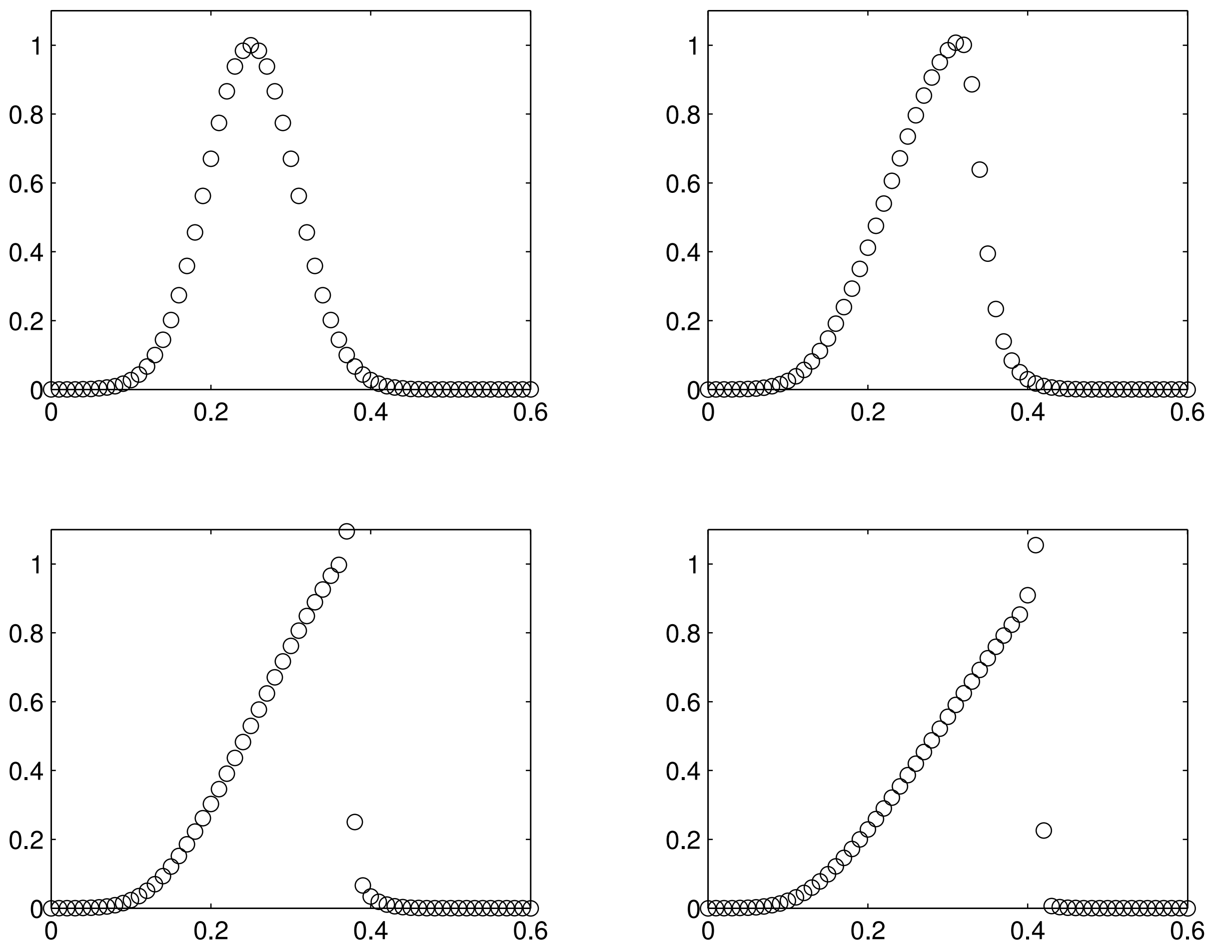
- The plots shown here are intended to show a difficult example to solve numerically. See the next bullet point below for a simpler test.

  This file solve the burger's equation using an initial gaussian profile. The process of shock formation can be seen qualitatively from the plots. Note that the Lax Wendroff term contains essentially a second derivative term which is called artificial viscosity which numerically plays the same role as viscosity in the Navier Stokes equations. The upwind scheme, which is not shown here and left as an exercise, actually resolves the shock better than Lax Wendroff.

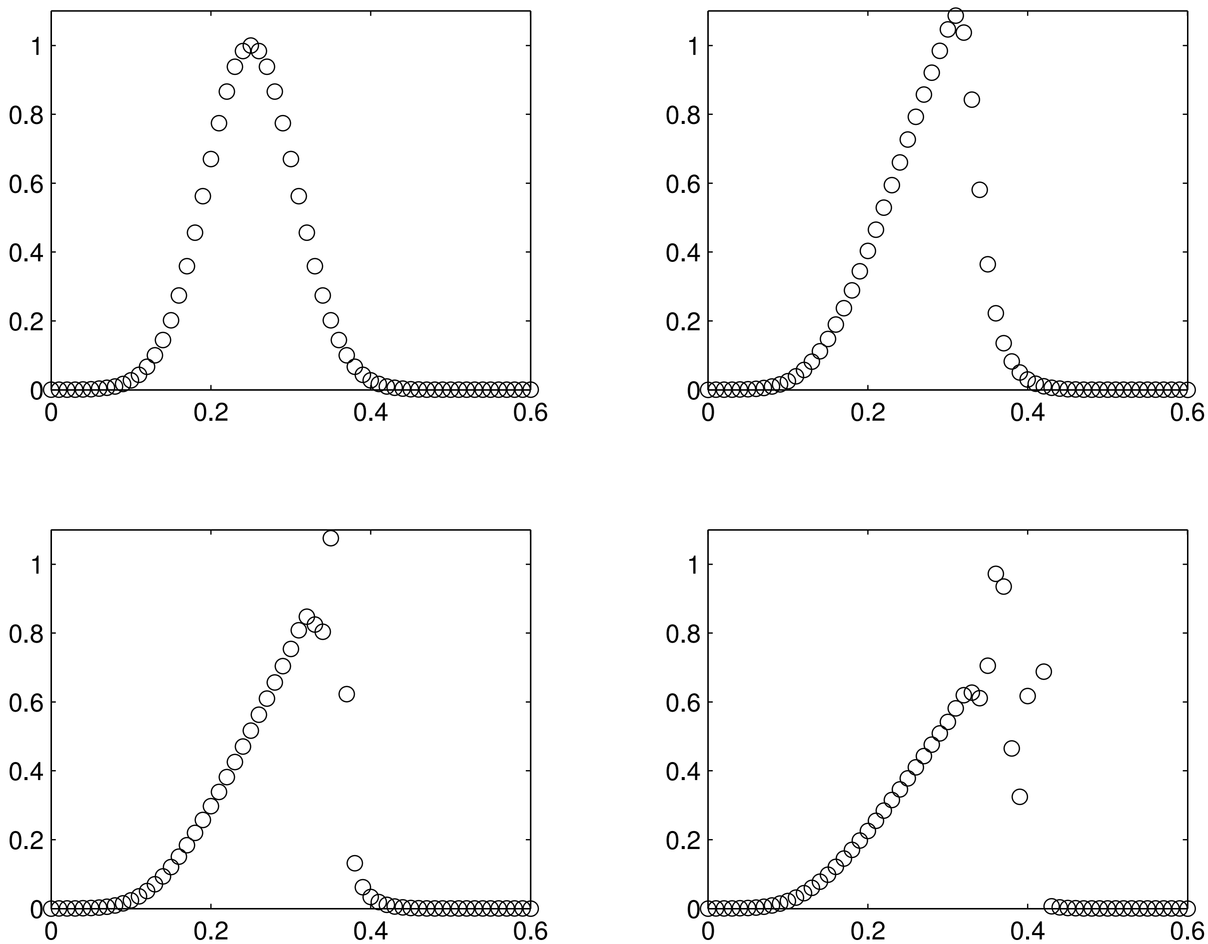
- **N.B.** As an important exercise use the initial condition of a slight bump on a uniform background,

$$u(x,0) = u_0 + u_1(x,t)$$

where $u_1 \ll u_0$ and u_0 is a constant. What happens now ? See the last problem in the exercises.



• The two-step method, for comparison, with Lax Wendroff.



Systems of Equations

- All of the above methods can easily be generalised to systems of equations. For example the wave equation describing the propagation of waves on a string is

$$\frac{\partial^2 A}{\partial t^2} = v^2 \frac{\partial^2 A}{\partial x^2}$$

where $A(x,t)$ is the wave amplitude and v the wave speed. Using the definitions

$$u_1 \equiv \frac{\partial A}{\partial t} \quad \text{and} \quad u_2 \equiv v \frac{\partial A}{\partial x}$$

the wave equation can be written as a pair of linear advection equations,

$$\frac{\partial u_1}{\partial t} = v \frac{\partial u_2}{\partial x} \quad \text{and} \quad \frac{\partial u_2}{\partial t} = v \frac{\partial u_1}{\partial x}$$

which can be solved simultaneously using the numerical methods above.

- Another example is the Euler equations of gas dynamics, which were mentioned earlier, and can be written in the form

$$\mathbf{u}_t + (\mathbf{f}(\mathbf{u}))_x = 0$$

where the arrays **u** and **f** are given by

$$\mathbf{u} = \begin{pmatrix} \rho \\ \rho v \\ E \end{pmatrix} \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

and

$$\mathbf{f} = \begin{pmatrix} \rho v \\ \rho v^2 + p \\ v(E+p) \end{pmatrix} = \begin{pmatrix} u_2 \\ u_2^2/u_1 + p(u) \\ u_2(u_3 + p(u))/u_1 \end{pmatrix}$$

and $p(u)$ is the equation of state. Note that this is a nonlinear system of equations.

- The numerical difference equations for u_1, u_2 and u_3 are then solved simultaneously. For the Lax-Wendroff method we would need to calculate the Jacobian matrix $\mathbf{f}'(\mathbf{u})$. For the two-step method this is not necessary which is one of it's biggest advantages.

Exercises

1. Verify that the solution to the inviscid Burger's equation $u_t + uu_x = 0$ with initial condition $u(x, 0) = f(x)$ is given implicitly by $u = f(x - ut)$.
2. Using the method of characteristics show that with initial condition $u(x, 0) = x$, the solution to the inviscis Burger's equation is

$$u(x,t) = \frac{x}{t+1}$$

3. Repeat the previous question but with $u(x, 0) = x^2$.
4. Is it possible to analytically solve, using characteristics, with initial condition $u(x, 0) = e^x$?
5. Try solving with initial condition $u(x, 0) = 1 - x$. What goes wrong ?
6. Show that if Burger's equation is solved with initial shape $u_0(x)$, where $u_0'(x)$ is somewhere negative, the characteristics will first cross at time

$$t_{\text{cross}} = -\frac{1}{\min\left(u_o'(x)\right)}$$

7. Consider the initial condition

$$u(x,0) = u_0 + u_1(x,t)$$

where $u_1 \ll u_0$ and u_0 is constant. Show that, to lowest order, the solution at later times is just the solution to the linear advection equation with speed u_0.

8. The Lax Wendroff scheme can be derived in the same manner as for the linear advection equation. As before, the Taylor expansion gives,

$$u(x, t+\tau) = u(x,t) + \tau u_t(x,t) + \frac{1}{2}\tau^2 u_{tt}(x,t) + O(\tau^3)$$

Again we need to replace the t-derivatives with x-derivatives by using the pde and then apply central differencing. Begin by noting that

$$u_t = -f_x \Rightarrow u_{tt} = -f_{xt} = -f_{tx} = -(vu_t)_x = (vf_x)_x$$

so that the Taylor expansion becomes

$$u(x, t+\tau) = u(x,t) - \tau f_x + \frac{1}{2}\tau^2 (vf_x)_x$$

9. Using central differencing gives

$$u_j^{n+1} = u_j^n - \frac{\tau}{h}\Delta_{0x} f(u_j^n) + \frac{1}{2}\left(\frac{\tau}{h}\right)^2 \delta_x \left[v(u_j^n)\delta_x f(u_j^n)\right]$$

and the second order term can be expanded out according to

$$\begin{aligned} \delta_x \left[v(u_j^n)\delta_x f(u_j^n)\right] &= \delta_x \left[v(u_j^n)\left(f(u_{j+1/2}^n) - f(u_{j-1/2}^n)\right)\right] \\ &= v(u_{j+1/2}^n)\left[f(u_{j+1}^n) - f(u_j^n)\right] \\ &\quad - v(u_{j-1/2}^n)\left[f(u_j^n) - f(u_{j-1}^n)\right] \end{aligned}$$

where $u_{j\pm 1/2}^n \equiv (u_j^n + u_{j\pm 1}^n)/2$.

10. Adopting the notation that $v_{j\pm 1/2}^n \equiv v(u_{j\pm 1/2}^n)$ and $f_j^n \equiv f(u_j^n)$, the Lax Wendroff method for conservation laws can be written as,

$$\begin{aligned} u_j^{n+1} = u_j^n &- \frac{\tau}{2h}\left(1 - v_{j+1/2}^n \frac{\tau}{h}\right)\Delta_{+x} f_j^n \\ &- \frac{\tau}{2h}\left(1 + v_{j-1/2}^n \frac{\tau}{h}\right)\Delta_{-x} f_j^n \qquad (5) \end{aligned}$$

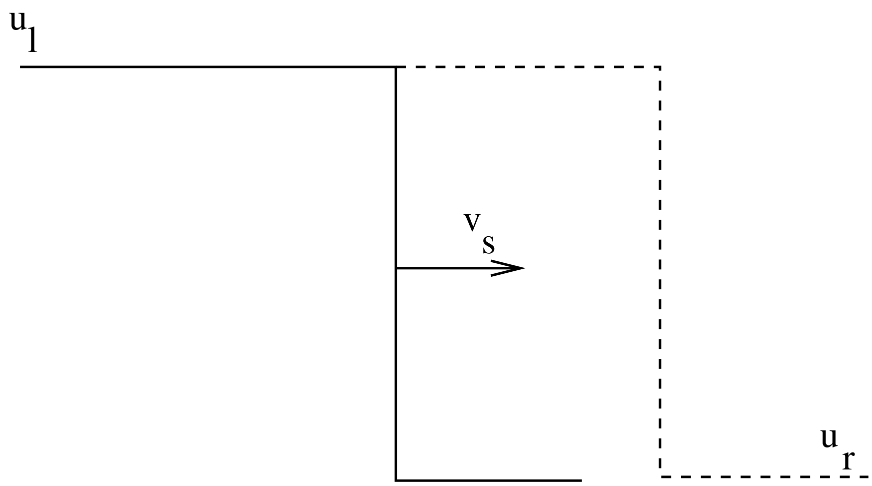
and this scheme works very well so long as the profile is smooth, i.e. doesn't contain shocks.

More Further Reading !
The Riemann problem and shock propagation

11. We can, however, gain insight into the problem of shock propagation by discussing the Riemann problem. Consider a conserved quantity $u(x,t)$ that obeys $u_t + f(u)_x = 0$ where f is convex, i.e. $f''(u) > 0$. Consider the initial condition

$$u(x,0) = \begin{cases} u_r & x < 0 \\ u_l & x \geq 0 \end{cases}$$

where $u_r > u_l$. The discontinuity/shock in the flow will propagate at speed v_s as shown in the diagram.



12. The shock speed can be determined from integrating the conservation equation from $-L$ to $+L$ which gives

$$\frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx = f(u_l) - f(u_r)$$

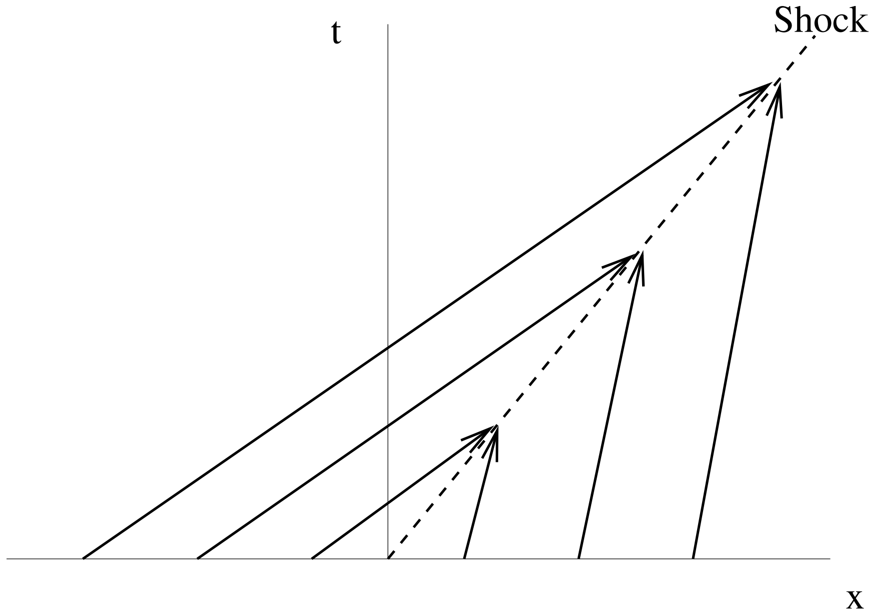
But if $L > v_s t$ we also have

$$\begin{aligned} \int_{-L}^{L} u(x,t)\,dx &= (L+v_st)u_l + (L-v_st)u_r \\ \Rightarrow \quad \frac{d}{dt}\int_{-L}^{L} u(x,t)\,dx &= v_s(u_l - u_r) \end{aligned}$$

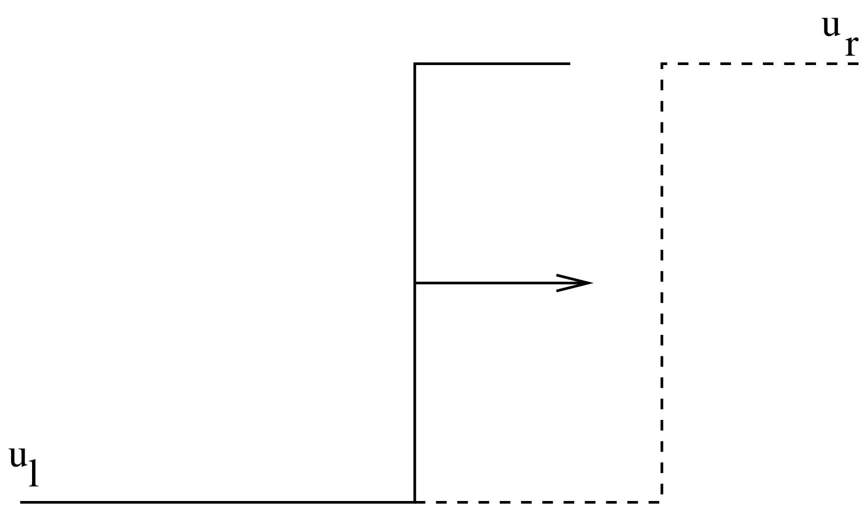
Combining the above gives

$$V_s = \frac{f(u_l) - f(u_r)}{u_l - u_r} \qquad (6)$$

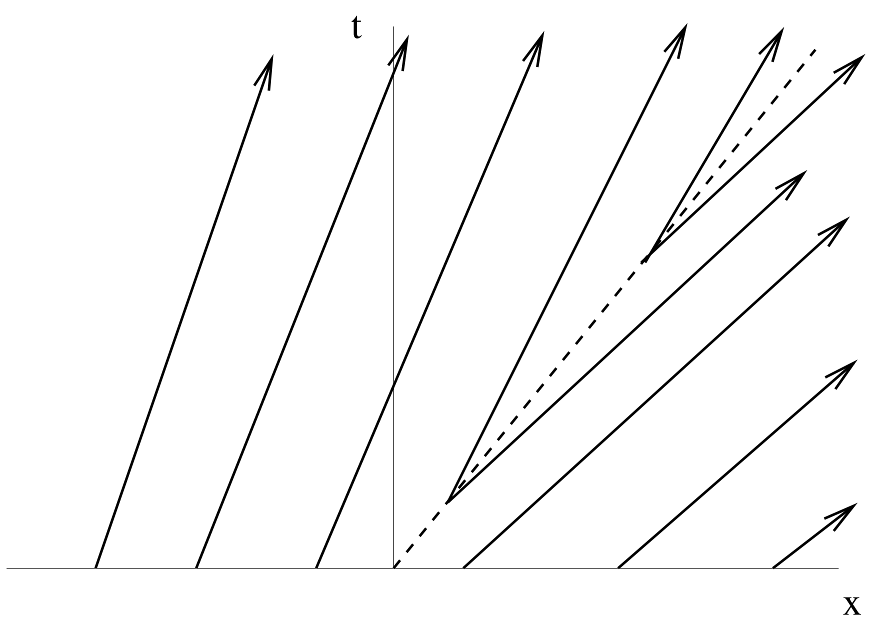
For Burger's equation $f = u^2/2$ so that $v_s = (u_l + u_r)/2$. The characteristics from each region go into the shock $(x = v_st)$ as shown below.



13. For completeness we should look at the case where $u_l < u < u_r$. In this case there is also a shock solution shown here.

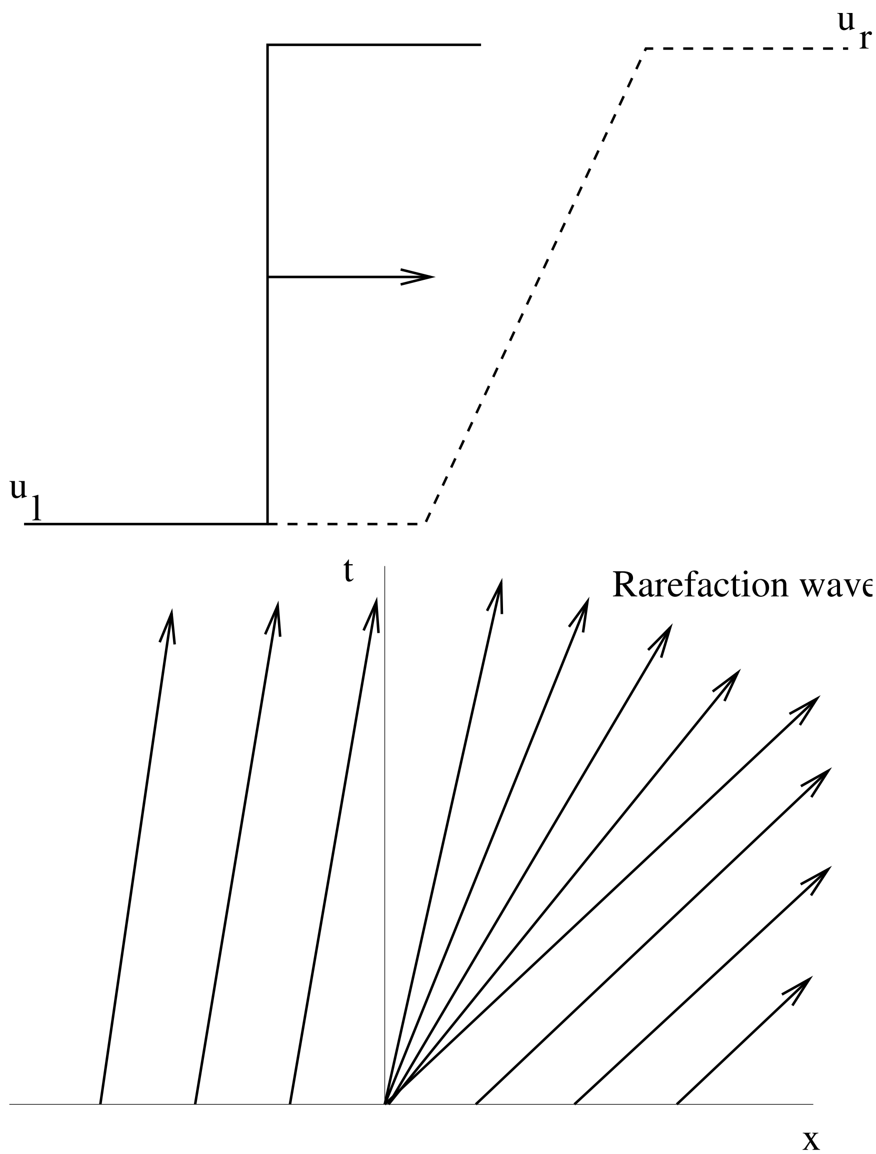


But in this case characteristics come out of the shock.



Therefore, if the data is smeared just a little, through viscosity, this solution becomes unstable. It is known as an entropy-violating shock and its analogue with the Euler equations does not occur in gas dynamics.

14. What actually happens when $u_l < u_r$ is that a rarefaction wave is produced and there is a linear increase from u_l to u_r.



The full solution is (prove this !)

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x/t & u_l t \leq x \leq u_r t \\ u_r & x > u_r t \end{cases}$$