Fundamentals of Atmospheric Modelling

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Lecture 10

Rossby Wave Packets
Introduction

First, we consider wave interactions, and introduce the concept of group velocity.

Then we define Rossby wave packets and study their behaviour.

Finally, we illustrate the importance of group velocity for Rossby waves, using real atmospheric data.
Interference of Two Waves

The simplest case to study is the superposition of two waves. We assume the two components have equal amplitudes and approximately the same wavenumbers and frequencies:

$$\psi(x, t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t).$$

The components move with respective phase speeds

$$c_1 = \frac{\omega_1}{k_1} \quad \text{and} \quad c_2 = \frac{\omega_2}{k_2}.$$
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By elementary trigonometry, \( \psi \) may be written

\[ \psi(x, t) = 2 \cos \left( k_1 - k_2 \right) x - \omega_1 - \omega_2, \frac{t}{2} \right) \cdot \cos \left( k_1 + k_2 \right) x - \omega_1 + \omega_2, \frac{t}{2} \right) \). \]
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$$c_1 = \omega_1 / k_1 \quad \text{and} \quad c_2 = \omega_2 / k_2.$$ 

By elementary trigonometry, \(\psi\) may be written

$$\psi(x, t) = 2 \cos \left( \frac{k_1 - k_2}{2} x - \frac{\omega_1 - \omega_2}{2} t \right) \cdot \cos \left( \frac{k_1 + k_2}{2} x - \frac{\omega_1 + \omega_2}{2} t \right).$$

We write the mean values and the differences

$$\bar{k} = (k_1 + k_2)/2 \quad \text{and} \quad \bar{\omega} = (\omega_1 + \omega_2)/2,$$

$$\Delta k = (k_1 - k_2)/2 \quad \text{and} \quad \Delta \omega = (\omega_1 - \omega_2)/2.$$ 

Then the wave combination is

$$\psi(x, t) = 2 \cos (\Delta k \cdot x - \Delta \omega \cdot t) \cdot \cos (\bar{k} x - \bar{\omega} t).$$
Again:

\[ \psi(x, t) = 2 \cos(\Delta k \cdot x - \Delta \omega \cdot t) \cdot \cos(\bar{k}x - \bar{\omega}t) \]

\[ = 2 \cos \left[ \Delta k \left( x - \frac{\Delta \omega}{\Delta k} \cdot t \right) \right] \cdot \cos \left[ \bar{k} \left( x - \frac{\bar{\omega}}{\bar{k}} t \right) \right]. \]

The second term here represents a wave with wavenumber \( \bar{k} \) moving with phase speed

\[ \bar{c} = \bar{\omega}/\bar{k} \]

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which is close to the phase speeds of the two components. The first term is slowly varying in space: it has wavenumber \( \Delta k \) and frequency \( \Delta \omega \), and it moves with a speed \( c_g \), called the group velocity,

\[ c_g = \frac{\Delta \omega}{\Delta k}. \]

The group velocity may be radically different from the phase velocity \( \bar{c} \), and of the opposite sign!
Two wave components of approximately equal wavelength. The envelope amplitude of the sum is clear.
The MATLAB program `gv1.m` shows the evolution of this waveform in time.
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The group velocity for a pair of waves was defined

\[ c_g = \frac{\Delta \omega}{\Delta k}. \]

More generally, there is a dispersion relation

\[ \omega = \omega(k), \]

and the group velocity is defined by

\[ c_g = \frac{\partial \omega}{\partial k}. \]
We consider only a simple case here. A much more detailed discussion may be found in Pedlosky (2003). For nondivergent quasigeostrophic flow on a beta plane of a wave which is independent of the $y$-coordinate, the Rossby phase speed is

$$c = \bar{u} - \frac{\beta}{k^2}.$$ 

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We have the surprising result that the group velocity is directed towards the east (relative to the mean flow) whereas the phase velocity is towards the west.
More generally, a Rossby wave may be travelling in a direction other than westward. If we assume

$$\psi = \psi_0 \exp[i(kx + \ell y - \omega t)]$$

the dispersion relation is

$$\omega = k\bar{u} - \frac{k\beta}{k^2 + \ell^2}.$$ 

and the phase speed for wavenumber $k$ is thus

$$c(k) = \bar{u} - \frac{\beta}{k^2 + \ell^2}.$$ 

The mean flow $\bar{u}$ simply transports wave patterns eastward (for $\bar{u} > 0$) at a constant speed, so we will ignore this effect by assuming $\bar{u} = 0$. 
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The mean flow \( \bar{u} \) simply transports wave patterns eastward (for \( \bar{u} > 0 \)) at a constant speed, so we will ignore this effect by assuming \( \bar{u} = 0 \).
The components of group velocity in the \( x \) and \( y \) directions are:
\[
c_{gx} = \frac{d\omega}{dk} = + \left( \frac{k^2 - \ell^2}{k^2 + \ell^2} \right) \frac{\beta}{k^2 + \ell^2}
\]
\[
c_{gy} = \frac{d\omega}{d\ell} = + \left( \frac{2k\ell}{k^2 + \ell^2} \right) \frac{\beta}{k^2 + \ell^2}
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The group velocity in the \( x \)-direction may be eastward or westward, depending on the sign of \( k^2 - \ell^2 \): for waves which are large-scale in \( x \) (small \( k \)) \( c_{gx} \) is negative; for waves which are small-scale (large \( k \)) it is positive.
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The group speed in the \( y \)-direction depends on the sign of \( k\ell \). However, the phase speed is \( c_y = \omega/\ell \), so the ratio is

\[ \frac{c_{gy}}{c_y} = -\frac{2\ell^2}{k^2 + \ell^2} < 0. \]

Thus the group velocity in the \( y \)-direction is in the opposite sense to the phase velocity.

**Exercise:** Plot the phase and group speeds as functions of the wavenumbers \( k \) and \( \ell \).
Extraction of the Envelope

The envelope of a wave packet may be extracted using ideas based on the Hilbert transform. For full details, see Bracewell (1978, pp. 267–272). Several applications of this technique are presented in Zimin, et al., (2003).

Let $\psi(\lambda)$ be a function on a periodic domain $0 \leq \lambda < 2\pi$. We perform the following operations in sequence:

- **Compute the Fourier coefficients**: $\hat{\psi}_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\lambda}\psi(\lambda) \, d\lambda$.
- **Set the coefficients to zero for negative index**: $\tilde{\psi}_k = H_k \hat{\psi}_k$ where $H_k$ is the Heaviside sequence.
- **Compute the inverse transform**: $\Psi(\lambda) = \sum_{k=-\infty}^{\infty} \tilde{\psi}_k e^{ik\lambda}$.
- **Double and take the absolute value**: $A(\lambda) = 2|\Psi(\lambda)|$.

In words, we calculate the Fourier series, throw away the negative frequencies, invert, double and take the absolute value.
A simple example illustrates the technique. Suppose $\psi(\lambda) = A \cos n\lambda$. There are just two nonvanishing terms in the Fourier series: $A \cos n\lambda = \frac{1}{2} A [\exp(n\lambda) + \exp(-n\lambda)]$. Elimination of the negative frequency part leaves $\frac{1}{2} A \exp(n\lambda)$ and twice the absolute value of this is $A$, as expected.

The generalization for a function $\psi(x)$ which is not periodic is straightforward: the Fourier series is replaced by the Fourier transform:

- Compute the Fourier transform: $\hat{\psi}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x} \psi(x) \, dx$.
- Set the transform to zero for negative $\omega$: $\tilde{\psi}(\omega) = H(\omega) \hat{\psi}(\omega)$ where $H(\omega)$ is the Heaviside function.
- Compute the inverse transform: $\Psi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \tilde{\psi}(\omega) \, d\omega$.
- Double and take the absolute value: $A(x) = 2|\Psi(x)|$. 
The theoretical explanation of the envelope extraction method is given in Bracewell (*loc. cit.*).
The envelope extraction may be combined with low-pass or band-pass filtering by replacing the Heaviside function by a suitable masking function, for example

\[
M(\omega) = \begin{cases} 
1, & \omega_L \leq \omega \leq \omega_H \\
0, & \text{otherwise}
\end{cases}
\]

which eliminates all components except in the frequency band \([\omega_L, \omega_H]\).
Gaussian Wave-packet
Suppose that we may express the streamfunction at the initial time $t = 0$ as

$$\psi(x, 0) = A \exp[-\frac{1}{2}x^2/\sigma_0^2] \exp(ik_0x),$$

that is, as a rapidly varying wave function whose amplitude envelope varies slowly with $x$.

A straightforward application of Fourier’s Theorem allows us to write this as

$$\psi(x, 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx)\hat{\psi}(k, 0) \, dk$$

where the spectral transform is given by

$$\hat{\psi}(k, 0) = \int_{-\infty}^{\infty} \exp(-ikx)\psi(k, 0) \, dk = \sqrt{2\pi}\sigma_0 A \exp[-\frac{1}{2}\sigma_0^2(k - k_0)^2].$$

Assume that the mode with wavenumber $k$ has frequency $\omega(k)$, given by the Rossby wave dispersion relation. Then the phase velocity is

$$c = \bar{u} - \beta \frac{\beta}{k^2}.$$  

(1)
The group velocity is

\[ c_g = \frac{d(kc)}{dk} = \bar{u} + \frac{\beta}{k^2}. \]  

(2)

We suppose that the governing equation for \( \psi(x, t) \) is linear. Then each Fourier component will evolve independently of the others. So the solution may be written

\[ \psi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(k, 0) \exp[i(kx - \omega t)] \, dk. \]  

(3)

For large \( \sigma \), the transform \( \hat{\psi} \) is concentrated near \( k = k_0 \) and we can approximate the frequency \( \omega \) using the Taylor series

\[ \omega(k) \approx \omega(k_0) + \left[ \frac{d\omega}{dk} \right]_{k_0} (k - k_0) + \frac{1}{2} \left[ \frac{d^2\omega}{dk^2} \right]_{k_0} (k - k_0)^2. \]

or, more briefly, with obvious notation,

\[ \omega \approx \omega_0 + \omega'_0(k - k_0) + \frac{1}{2} \omega''_0(k - k_0)^2. \]
Substituting this into (3) and evaluating the integral (about a page of calculus) we get

\[
\psi(x, t) = \left( \frac{A\sigma_0}{\sqrt{\sigma_0^2 + i\omega''t}} \right) \exp \left[ -\frac{(x - \omega_0't)^2}{2(\sigma_0^2 + i\omega''t)} \right] \exp[i(k_0x - \omega_0t)].
\]

(4)

This solution has several points of interest. The first term shows that, for large time, the amplitude decreases as \( t^{-1/2} \). The second term is the envelope, which we examine presently. The last term represents an oscillation with wavenumber \( k_0 \) and frequency \( \omega_0 \), which travels with phase speed \( c_0 = \omega_0/k_0 \). The middle term on the right of (4) may be written

\[
\exp \left[ -\frac{(x - \omega_0't)^2}{2(\sigma_0^2 + i\omega''t)} \right] = \exp \left[ -\frac{(x - \omega_0't)^2}{2\sigma_0^2(1 + \tau^2)} \right] \exp \left[ i \left( \tau(x - \omega_0't)^2 \right) \right],
\]

where \( \tau = (\omega''/\sigma_0^2)t \) is re-scaled time.
The first component is a Gaussian envelope, centered at $x = \omega'_0 t$, whose width is given by

$$\sigma^2 = \sigma^2_0 (1 + \tau^2)$$

so it moves with the group velocity $c_g = \omega'_0$ and spreads as time increases. The second term is a chirp-function: its local wavenumber is zero at $x = \omega'_0 t$ and increases linearly with distance from this point. The factor $\tau/(1 + \tau^2)$ vanishes at $t = 0$ and for large time, reaching its maximum at $\tau = 1$. The full solution is now written as a product of four components:

$$\psi(x, t) = \left( \frac{A\sigma_0}{\sqrt{\sigma^2_0 + i\omega''_0 t}} \right) \exp \left[ -\frac{\xi^2}{2\sigma^2} \right] \exp \left[ i \left( \frac{\tau \xi^2}{2\sigma^2} \right) \right] \exp[i k_0 (x - c_0 t)],$$

where $\xi = x - \omega'_0 t$. 
The properties of the solution

\[
\psi(x, t) = \left( \frac{A \sigma_0}{\sqrt{\sigma_0^2 + i \omega'' t}} \right) \exp \left[ -\frac{\xi^2}{2\sigma^2} \right] \exp \left[ i \left( \frac{\tau \xi^2}{2\sigma^2} \right) \right] \exp[i k_0 (x - c_0 t)],
\]

may be summarised as follows

- Individual wave crests move with the phase velocity \(c_0\).
- The overall amplitude decays as \(O(t^{-1/2})\).
- The envelope moves with the group velocity \(c_g = \omega'_0\).
- The spread of the envelope grows as \(\sigma^2 = \sigma_0^2 (1 + \tau^2)\).
- What to say about the chirp part?
References
