Fundamentals of Atmospheric Modelling

Peter Lynch, Met Éireann

Mathematical Computation Laboratory (Opp. Room 30)
Dept. of Maths. Physics, UCD, Belfield.
January–April, 2004.

Lecture 9

Mixed Rossby & Gravity Waves

Introduction

- ***We continue our investigation of the linear solutions of the shallow water equations.***
- ***First, we compare the properties of the simple wave solutions already found.***
- ***Then we consider the case where gravity waves and Rossby waves occur simultaneously as solutions.***
- ***Finally,, we introduce the concept of filtering the equations.***

Comparison of Wave Speeds

We can estimate the *relative sizes* of the phase speeds of the two types of pure wave solutions for parameter values typical of the atmosphere. In the absence of a mean flow, the phase speeds are

$$c_{\mathrm{G}} = \sqrt{\bar{\Phi}} = \sqrt{gH}\,; \qquad c_{\mathrm{R}} = -\frac{\beta}{k^2} = -\frac{\beta L^2}{4\pi^2}\,.$$

Comparison of Wave Speeds

We can estimate the *relative sizes* of the phase speeds of the two types of pure wave solutions for parameter values typical of the atmosphere. In the absence of a mean flow, the phase speeds are

$$c_{\mathrm{G}} = \sqrt{\bar{\Phi}} = \sqrt{gH}\,; \qquad c_{\mathrm{R}} = -\frac{\beta}{k^2} = -\frac{\beta L^2}{4\pi^2}\,.$$

Taking approximate values

$$g = 10\,\mathrm{m\,s}^2\,, \qquad H = 10^4\,\mathrm{m}\,,$$
$$\beta = 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}\,, \qquad L = 10^6\,\mathrm{m}$$

the phase speeds are

$$c_{\mathrm{G}} \approx 300\,\mathbf{m/s}\,; \qquad c_{\mathrm{R}} \approx 0.5\mathbf{m/s}\,.$$

Comparison of Wave Speeds

We can estimate the *relative sizes* of the phase speeds of the two types of pure wave solutions for parameter values typical of the atmosphere. In the absence of a mean flow, the phase speeds are

$$c_{\mathrm{G}} = \sqrt{\bar{\Phi}} = \sqrt{gH}\,; \qquad c_{\mathrm{R}} = -\frac{\beta}{k^2} = -\frac{\beta L^2}{4\pi^2}\,.$$

Taking approximate values

$$g = 10\,\mathrm{m\,s}^2\,, \qquad H = 10^4\,\mathrm{m}\,,$$
$$\beta = 10^{-11}\,\mathrm{m}^{-1}\,\mathrm{s}^{-1}\,, \qquad L = 10^6\,\mathrm{m}$$

the phase speeds are

$$c_{\mathrm{G}} \approx 300\,\mathbf{m/s}\,; \qquad c_{\mathrm{R}} \approx 0.5\mathbf{m/s}\,.$$

Thus the gravity waves travel much faster than the Rossby or planetary waves:

$$|c_{\mathrm{R}}| \ll |c_{\mathrm{G}}|$$

Even Rossby waves with wavelength $L = 10^7\,\mathbf{m}$, *i.e.,* comparable in size to the earth's radius, have a phase speed of about $50\,\mathbf{m/s}$, still much slower than the gravity waves).

Even Rossby waves with wavelength $L = 10^7$ m , *i.e.*, comparable in size to the earth's radius, have a phase speed of about 50 m/s, still much slower than the gravity waves).

The other crucial distinction between the two types of waves:

- ***For gravity waves, the divergence is vital for the dynamics.***
The <u>vorticity vanishes</u>.
- ***For Rossby waves, the vorticity is the important quantity.***
The <u>divergence vanishes</u>.

Mixed Gravity & Rossby Waves

We consider now a more complicated case in which both gravity waves and Rossby waves are present. To keep things as simple as possible, *we ignore variations with latitude* ($i.e.$, in the y-direction).

Mixed Gravity & Rossby Waves

We consider now a more complicated case in which both gravity waves and Rossby waves are present. To keep things as simple as possible, *we ignore variations with latitude* ($i.e.$, in the y-direction).

Then the <u>divergence, vorticity and continuity equations</u> for the linearized perturbation variables are:

$$\alpha\left(\frac{\partial\delta}{\partial t}+\bar{u}\frac{\partial\delta}{\partial x}\right)-f\zeta+\beta u+\Phi_{xx}=0$$

$$\left(\frac{\partial\zeta}{\partial t}+\bar{u}\frac{\partial\zeta}{\partial x}\right)+f\delta+\beta v=0$$

$$\left(\frac{\partial\Phi}{\partial t}+\bar{u}\frac{\partial\Phi}{\partial x}\right)+v\frac{\partial\bar{\Phi}}{\partial y}+\bar{\Phi}\frac{\partial u}{\partial x}=0\,.$$

Here $\delta=u_x$, $\zeta=v_x$ and the basic state satisfies $f\bar{u}=-\bar{\Phi}_y$.

Mixed Gravity & Rossby Waves

We consider now a more complicated case in which both gravity waves and Rossby waves are present. To keep things as simple as possible, *we ignore variations with latitude* ($i.e.$, in the y-direction).

Then the <u>divergence, vorticity and continuity equations</u> for the linearized perturbation variables are:

$$\alpha\left(\frac{\partial\delta}{\partial t}+\bar{u}\frac{\partial\delta}{\partial x}\right)-f\zeta+\beta u+\Phi_{xx}=0$$

$$\left(\frac{\partial\zeta}{\partial t}+\bar{u}\frac{\partial\zeta}{\partial x}\right)+f\delta+\beta v=0$$

$$\left(\frac{\partial\Phi}{\partial t}+\bar{u}\frac{\partial\Phi}{\partial x}\right)+v\frac{\partial\bar{\Phi}}{\partial y}+\bar{\Phi}\frac{\partial u}{\partial x}=0\,.$$

Here $\delta=u_x$, $\zeta=v_x$ and the basic state satisfies $f\bar{u}=-\bar{\Phi}_y$.

Exercise: Check that these equations are correct.

The coefficient α normally has the value 1. It is a tracer, which is carried through the analysis as a parameter, and allows us to examine the effect of omitting the term $d\delta/dt$ in the divergence equation by giving it the value zero in the final result. (This is just a handy trick to avoid repetition).

The coefficient α normally has the value 1. It is a tracer, which is carried through the analysis as a parameter, and allows us to examine the effect of omitting the term $d\delta/dt$ in the divergence equation by giving it the value zero in the final result. (This is just a handy trick to avoid repetition).

We are interested in wave-like solutions of the form:

$$\begin{pmatrix} u \\ v \\ \Phi \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \Phi_0 \end{pmatrix} \exp[ik(x - ct)]$$

so that $u = \delta/ik$, $v = \zeta/ik$ and the differential operators are:

$$\frac{\partial}{\partial x} \sim ik\,; \quad \frac{\partial^2}{\partial x^2} \sim -k^2\,; \quad \frac{\partial}{\partial t} \sim -ikc\,.$$

The coefficient α normally has the value 1. It is a tracer, which is carried through the analysis as a parameter, and allows us to examine the effect of omitting the term $d\delta/dt$ in the divergence equation by giving it the value zero in the final result. (This is just a handy trick to avoid repetition).

We are interested in <u>wave-like solutions</u> of the form:

$$\begin{pmatrix} u \\ v \\ \Phi \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \\ \Phi_0 \end{pmatrix} \exp[ik(x-ct)]$$

so that $u = \delta/ik$, $v = \zeta/ik$ and the differential operators are:

$$\frac{\partial}{\partial x} \sim ik\,; \quad \frac{\partial^2}{\partial x^2} \sim -k^2\,; \quad \frac{\partial}{\partial t} \sim -ikc\,.$$

The form of the basic state allows us to write

$$v\bar{\Phi}_y = \frac{\zeta}{ik}(-f\bar{u}) = -\left(\frac{f\bar{u}}{ik}\right)\zeta\,.$$

We can now substitute the exponential expression for the dependent variables into the equations, which may then be written in matrix form:

$$\begin{pmatrix} -f & \alpha[ik(\bar{u}-c)]+\dfrac{\beta}{ik} & -k^2 \\ [ik(u-c)+\dfrac{\beta}{ik}] & +f & 0 \\ -\dfrac{f\bar{u}}{ik} & \bar{\Phi} & ik(\bar{u}-c) \end{pmatrix} \begin{pmatrix} \zeta \\ \\ \delta \\ \\ \Phi \end{pmatrix} = \mathbf{0}\,.$$

We can now substitute the exponential expression for the dependent variables into the equations, which may then be written in matrix form:

$$\begin{pmatrix} -f & \alpha[ik(\bar{u}-c)] + \dfrac{\beta}{ik} & -k^2 \\ [ik(u-c) + \dfrac{\beta}{ik}] & +f & 0 \\ -\dfrac{f\bar{u}}{ik} & \bar{\Phi} & ik(\bar{u}-c) \end{pmatrix} \begin{pmatrix} \zeta \\ \delta \\ \Phi \end{pmatrix} = \mathbf{0}\,.$$

Since this is a homogeneous system, there is a solution *iff* the determinant of the coeficient matrix vanishes. Expanding out this gives us the dispersion equation:

$$\left[\alpha(\bar{u}-c) - \frac{\beta}{k^2}\right]^2 (\bar{u}-c) - \left[\frac{f^2}{k^2} + \bar{\Phi}\right](\bar{u}-c) + \left[\frac{f^2}{k^2}\bar{u} + \bar{\Phi}\frac{\beta}{k^2}\right] = 0$$

This is a cubic equation for the phase speed c.

We can now substitute the exponential expression for the dependent variables into the equations, which may then be written in matrix form:

$$\begin{pmatrix} -f & \alpha[ik(\bar{u}-c)] + \dfrac{\beta}{ik} & -k^2 \\ [ik(u-c) + \dfrac{\beta}{ik}] & +f & 0 \\ -\dfrac{f\bar{u}}{ik} & \bar{\Phi} & ik(\bar{u}-c) \end{pmatrix} \begin{pmatrix} \zeta \\ \delta \\ \Phi \end{pmatrix} = \mathbf{0}\,.$$

Since this is a homogeneous system, there is a solution *iff* the determinant of the coeficient matrix vanishes. Expanding out this gives us the dispersion equation:

$$\left[\alpha(\bar{u}-c) - \frac{\beta}{k^2}\right]^2 (\bar{u}-c) - \left[\frac{f^2}{k^2} + \bar{\Phi}\right](\bar{u}-c) + \left[\frac{f^2}{k^2}\bar{u} + \bar{\Phi}\frac{\beta}{k^2}\right] = 0$$

This is a cubic equation for the phase speed c.
We can solve the cubic analytically or numerically. However, a simpler way is to note the relative sizes of the terms and *estimate the roots approximately*.

High Frequency Roots

First, suppose the magnitude of the phase speed is large. Specifically, let us suppose

$$|\bar{u} - c| \gg |\bar{u}| \qquad \textbf{and} \qquad |\bar{u} - c| \gg |\beta/k^2| \,.$$

High Frequency Roots

First, suppose the magnitude of the phase speed is large. Specifically, let us suppose

$$|\bar{u} - c| \gg |\bar{u}| \qquad \textbf{and} \qquad |\bar{u} - c| \gg |\beta/k^2| \,.$$

Then we may neglect $f^2\bar{u}/k^2$ compared to $(f^2/k^2)(\bar{u} - c)$, and also the terms involving β. The <u>cubic then reduces to</u>

$$\left[\alpha(\bar{u} - c)^2 - \left(\frac{f^2}{k^2} + \bar{\Phi}\right)\right](\bar{u} - c) = 0 \,.$$

High Frequency Roots

First, suppose the magnitude of the phase speed is large. Specifically, let us suppose

$$|\bar{u} - c| \gg |\bar{u}| \qquad \textbf{and} \qquad |\bar{u} - c| \gg |\beta / k^2| \, .$$

Then we may neglect $f^2\bar{u}/k^2$ compared to $(f^2/k^2)(\bar{u} - c)$, and also the terms involving β. The <u>cubic then reduces to</u>

$$\left[\alpha(\bar{u} - c)^2 - \left(\frac{f^2}{k^2} + \bar{\Phi} \right) \right] (\bar{u} - c) = 0 \, .$$

The solution $c = \bar{u}$ cannot be admitted, as we have assumed that c is large. Thus, the quadratic term must vanish, giving the two roots:

$$c = \bar{u} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}} \, .$$

These are the *inertia-gravity wave* solutions.

In the case of no rotation, the inertia-gravity wave phase speed

$$c = \bar{u} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}}$$

reduces to that for *pure gravity wave solutions* with

$$c = \bar{u} \pm \sqrt{\bar{\Phi}}\,.$$

In this case the phase speed is independent of the wavelength (or wavenumber k). More generally, the phase speed is modified by the effect of rotation and depends on k, making the waves *dispersive*.

In the case of no rotation, the inertia-gravity wave phase speed

$$c = \bar{u} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}}$$

reduces to that for *pure gravity wave solutions* with

$$c = \bar{u} \pm \sqrt{\bar{\Phi}}\,.$$

In this case the phase speed is independent of the wavelength (or wavenumber k). More generally, the phase speed is modified by the effect of rotation and depends on k, making the waves *dispersive*.

Taking typical parameter values, we find that the phase speed of these solutions is of the order $c = 300\,$m/s. Thus, our assumptions are justified *a posteriori*.

In the case of no rotation, the inertia-gravity wave phase speed

$$c = \bar{u} \pm \sqrt{\bar{\Phi} + \frac{f^2}{k^2}}$$

reduces to that for ***pure gravity wave solutions*** with

$$c = \bar{u} \pm \sqrt{\bar{\Phi}}\,.$$

In this case the phase speed is independent of the wave-length (or wavenumber k). More generally, the phase speed is modified by the effect of rotation and depends on k, making the waves ***dispersive***.

Taking typical parameter values, we find that the phase speed of these solutions is of the order $c = 300\,\mathrm{m/s}$. Thus, our assumptions are justified ***a posteriori***.

Exercise: Calculate the group velocity of the inertia-gravity waves.

Group velocity is defined two frames below

Low Frequency Roots

Next, suppose the phase speed is small, specifically that it is much smaller than the pure gravity wave speed:

$$c^2 \ll \bar{\Phi} \qquad \textbf{and also} \qquad |\bar{u} - c|^2 \ll \bar{\Phi}\,.$$

Low Frequency Roots

Next, suppose the phase speed is small, specifically that it is much smaller than the pure gravity wave speed:

$$c^2 \ll \bar{\Phi} \qquad \textbf{and also} \qquad |\bar{u} - c|^2 \ll \bar{\Phi} \,.$$

We also recall that the pure Rossby speed (β/k^2) is very much smaller that the gravity wave speed. Thus, the *cubic term in the dispersion equation can be neglected* compared to the linear term and the equation becomes

$$-\left[\frac{f^2}{k^2} + \bar{\Phi}\right](\bar{u} - c) + \left[\frac{f^2}{k^2}\bar{u} + \bar{\Phi}\frac{\beta}{k^2}\right] = 0 \,.$$

Low Frequency Roots

Next, suppose the phase speed is small, specifically that it is much smaller than the pure gravity wave speed:

$$c^2 \ll \bar{\Phi} \qquad \textbf{and also} \qquad |\bar{u} - c|^2 \ll \bar{\Phi}\,.$$

We also recall that the pure Rossby speed (β/k^2) is very much smaller that the gravity wave speed. Thus, the *cubic term in the dispersion equation can be neglected* compared to the linear term and the equation becomes

$$-\left[\frac{f^2}{k^2} + \bar{\Phi}\right](\bar{u} - c) + \left[\frac{f^2}{k^2}\bar{u} + \bar{\Phi}\frac{\beta}{k^2}\right] = 0\,.$$

This has the following solution

$$c = \bar{u} - \left(\frac{\beta + f^2\bar{u}/\bar{\Phi}}{k^2 + f^2/\bar{\Phi}}\right)\,.$$

For typical values of $\bar{\Phi}$, say $10^5\,\mathbf{m}^2\mathbf{s}^{-2}$, we find that c is quite small, thereby justifying our assumptions.

This solution is, of course, the Rossby wave solution, which *always travels westward relative to the mean flow*:

$$c = \bar{u} - \left(\frac{\beta + f^2 \bar{u}/\bar{\Phi}}{k^2 + f^2/\bar{\Phi}} \right) .$$

For large values of $\bar{\Phi}$, it is approximated by the previously obtained simple formula

$$c = \bar{u} - \frac{\beta}{k^2} .$$

This solution is, of course, the Rossby wave solution, which ***always travels westward relative to the mean flow*****:**

$$c = \bar{u} - \left(\frac{\beta + f^2 \bar{u} / \bar{\Phi}}{k^2 + f^2 / \bar{\Phi}} \right) .$$

For large values of $\bar{\Phi}$, it is approximated by the previously obtained simple formula

$$c = \bar{u} - \frac{\beta}{k^2} .$$

Exercise: Calculate the group velocity of the Rossby waves.

Definition: The group velocity c_g for a wave in the x-direction is defined by

$$c_g = \frac{\partial \omega}{\partial k} = \frac{\partial k c}{\partial k} .$$

Filtering

In the above analysis, we used a tracer α **to mark the term**

$$\frac{d\delta}{dt} = \left(\frac{\partial\delta}{\partial t} + \bar{u}\frac{\partial\delta}{\partial x}\right)$$

in the divergence equation.

Filtering

In the above analysis, we used a tracer α to mark the term

$$\frac{d\delta}{dt} = \left(\frac{\partial\delta}{\partial t} + \bar{u}\frac{\partial\delta}{\partial x}\right)$$

in the divergence equation.

If we now set $\alpha = 0$, the dispersion equation becomes linear and only the Rossby wave solution remains.

Filtering

In the above analysis, we used a tracer α to mark the term

$$\frac{d\delta}{dt} = \left(\frac{\partial\delta}{\partial t} + \bar{u}\frac{\partial\delta}{\partial x}\right)$$

in the divergence equation.

If we now set $\alpha = 0$, the dispersion equation becomes linear and only the Rossby wave solution remains.

To rephrase, omission of the term corresponding to changes in the divergence is sufficient (in the present case) to eliminate solutions corresponding to gravity waves.

Filtering

In the above analysis, we used a tracer α to mark the term

$$\frac{d\delta}{dt} = \left(\frac{\partial\delta}{\partial t} + \bar{u}\frac{\partial\delta}{\partial x}\right)$$

in the divergence equation.

If we now set $\alpha = 0$, the dispersion equation becomes linear and only the Rossby wave solution remains.

To rephrase, omission of the term corresponding to changes in the divergence is sufficient (in the present case) to eliminate solutions corresponding to gravity waves.

This process of modifying the governing equations in such a way as to exclude certain solutions is called *filtering*.

The Quasi-geostrophic Equations

The question arises: How can we generalise this idea? How can the general nonlinear equations be modified so that the high frequency gravity wave solutions are eliminated, while the low frequency Rossby waves, which are the important solutions, are preserved?

The Quasi-geostrophic Equations

The question arises: How can we generalise this idea? How can the general nonlinear equations be modified so that the high frequency gravity wave solutions are eliminated, while the low frequency Rossby waves, which are the important solutions, are preserved?

This question will not be addressed in this lecture series, but an affirmative answer can be obtained. The equations resulting from the filtering procedure are called the *Quasi-geostrophic Equations*.

The Quasi-geostrophic Equations

The question arises: How can we generalise this idea? How can the general nonlinear equations be modified so that the high frequency gravity wave solutions are eliminated, while the low frequency Rossby waves, which are the important solutions, are preserved?

This question will not be addressed in this lecture series, but an affirmative answer can be obtained. The equations resulting from the filtering procedure are called the *Quasi-geostrophic Equations*.

The quasi-geostrophic equations provide a powerful basis for the elucidation of the dynamics of the atmosphere in middle latitudes.