Fundamentals of Atmospheric Modelling

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Lecture 7

Potential Vorticity Conservation

Applications of PV Conservation

We consider a few simple applications of Potential Vorticity conservation. The treatment is purely qualitative.

A quantitative treatment will be undertaken in subsequent lectures.

- ***Gravity-Inertia Waves.***
- ***Free Rossby Waves.***
- ***Forced Rossby Waves.***
  Lee-side Trough.

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This may also be put in the following form:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

This is the equation of *conservation of potential vorticity*.

Elementary Applications of PV Conservation

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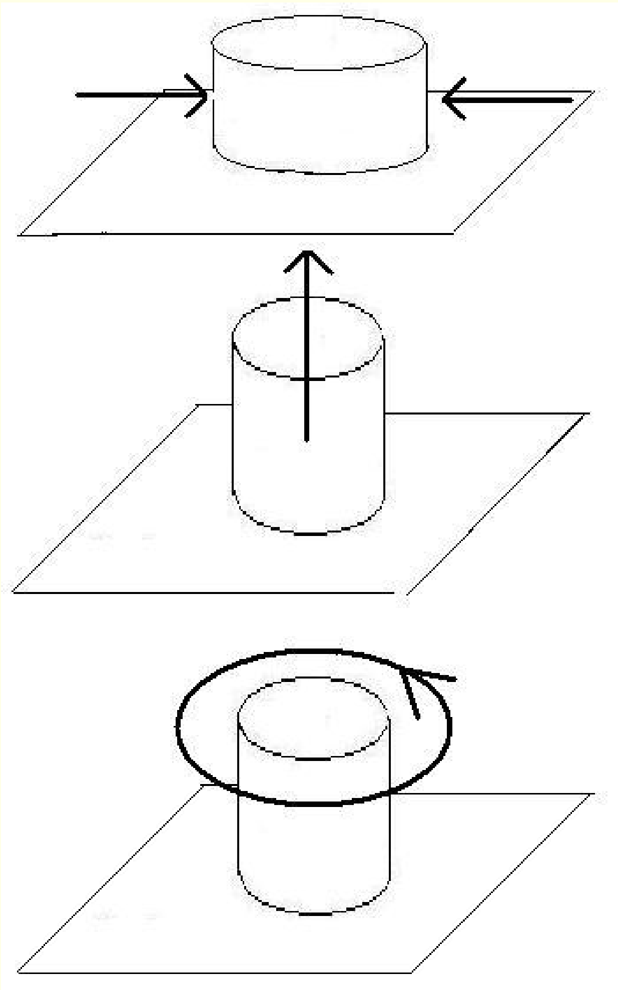

- ***Convergence* induces stretching**
- **Stretching implies increased pressure at the centre**
- **Increasing h also implies increasing ζ**
- **$\zeta > 0$ implies Cyclonic flow**
- **Cyclonic flow around high pressure is *unbalanced***
- **PGF and Coriolis force act outwards**
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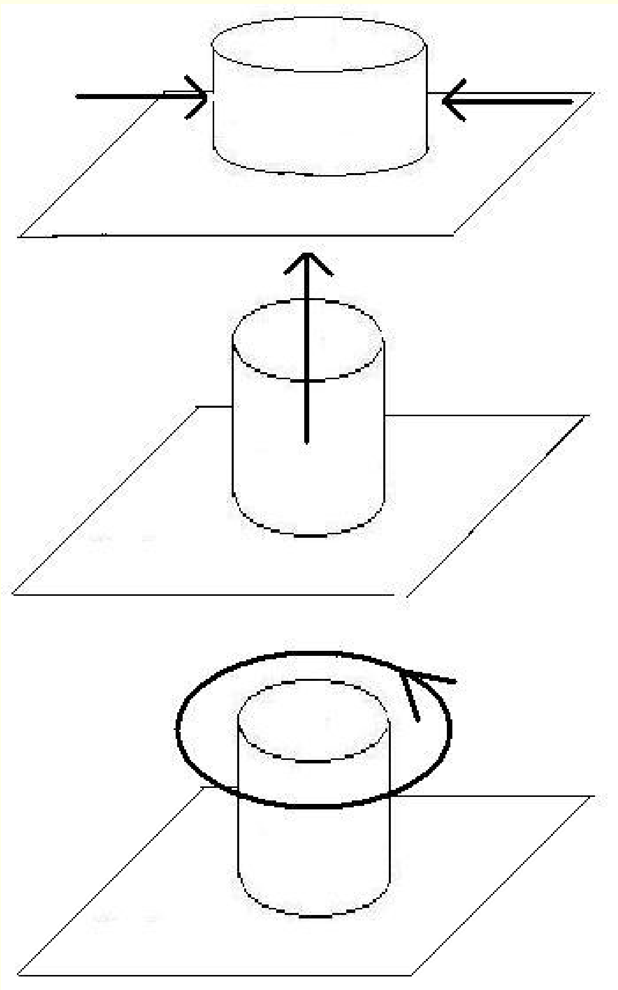

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The restoring forces give rise to *Inertia-gravity waves*.

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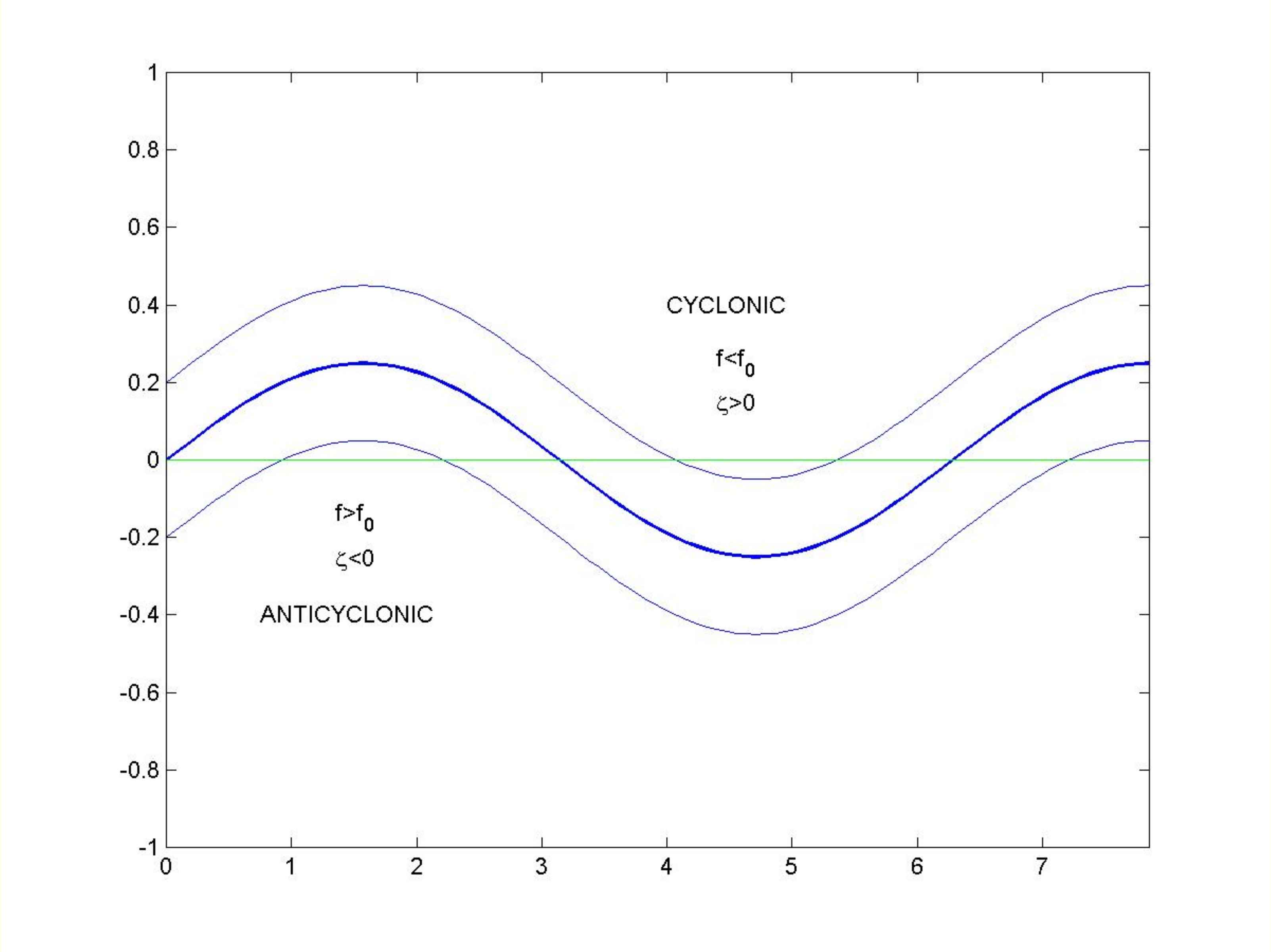
- **Increasing f means decreasing ζ**
- **Negative ζ corresponds to anticyclonic flow**
- **Flow curves back towards $y = y_0$**
- **Then $f = f_0$ and $\zeta = 0$ again**
- **Parcel continues SE and opposite half-cycle occurs.**

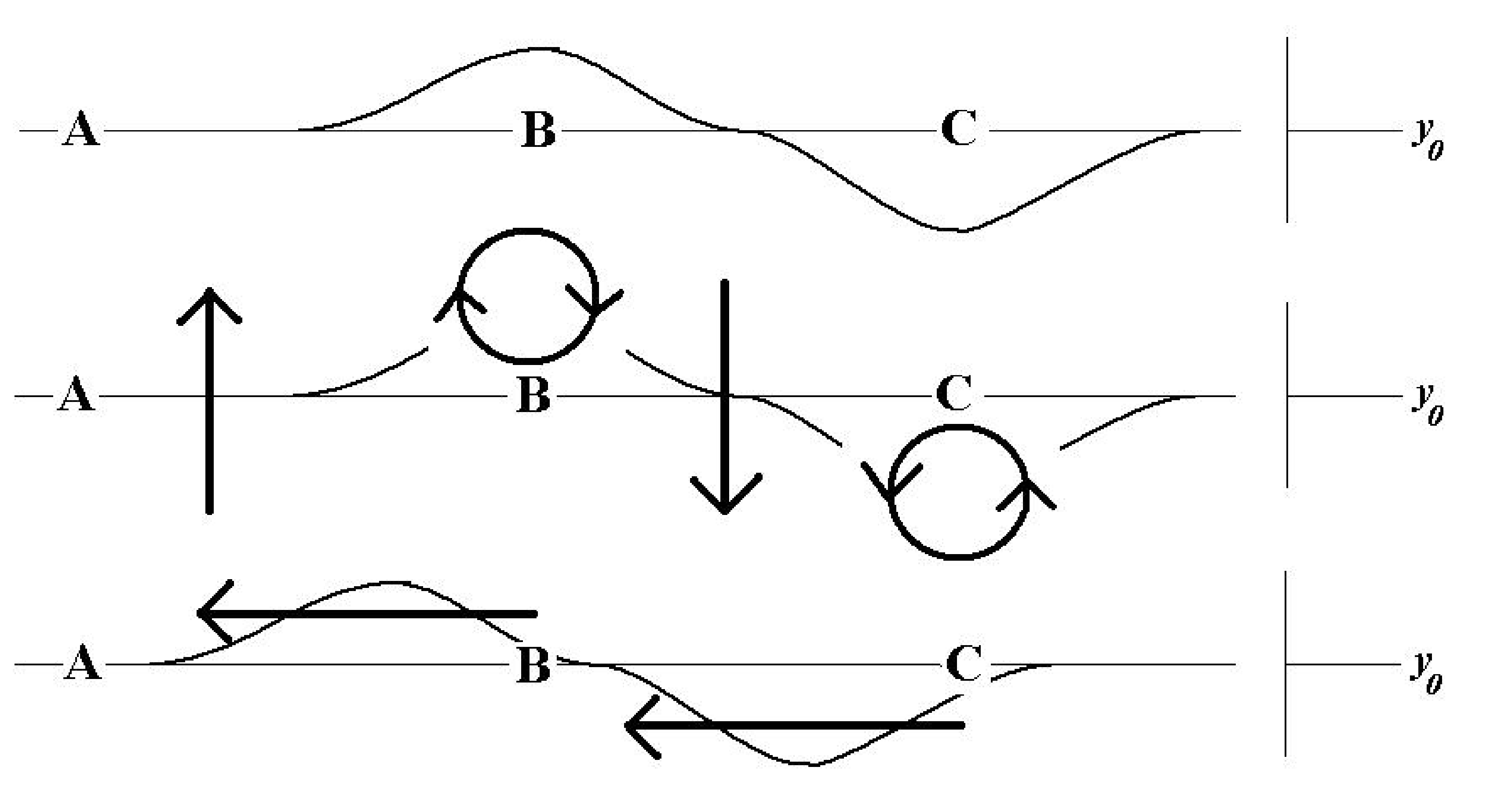
Throughout the motion, $\zeta + f$ keeps the same value.

Constant Absolute Vorticity (CAV) Trajectories.



Fluid parcels follow trajectories on which $\zeta + f$ remains constant.

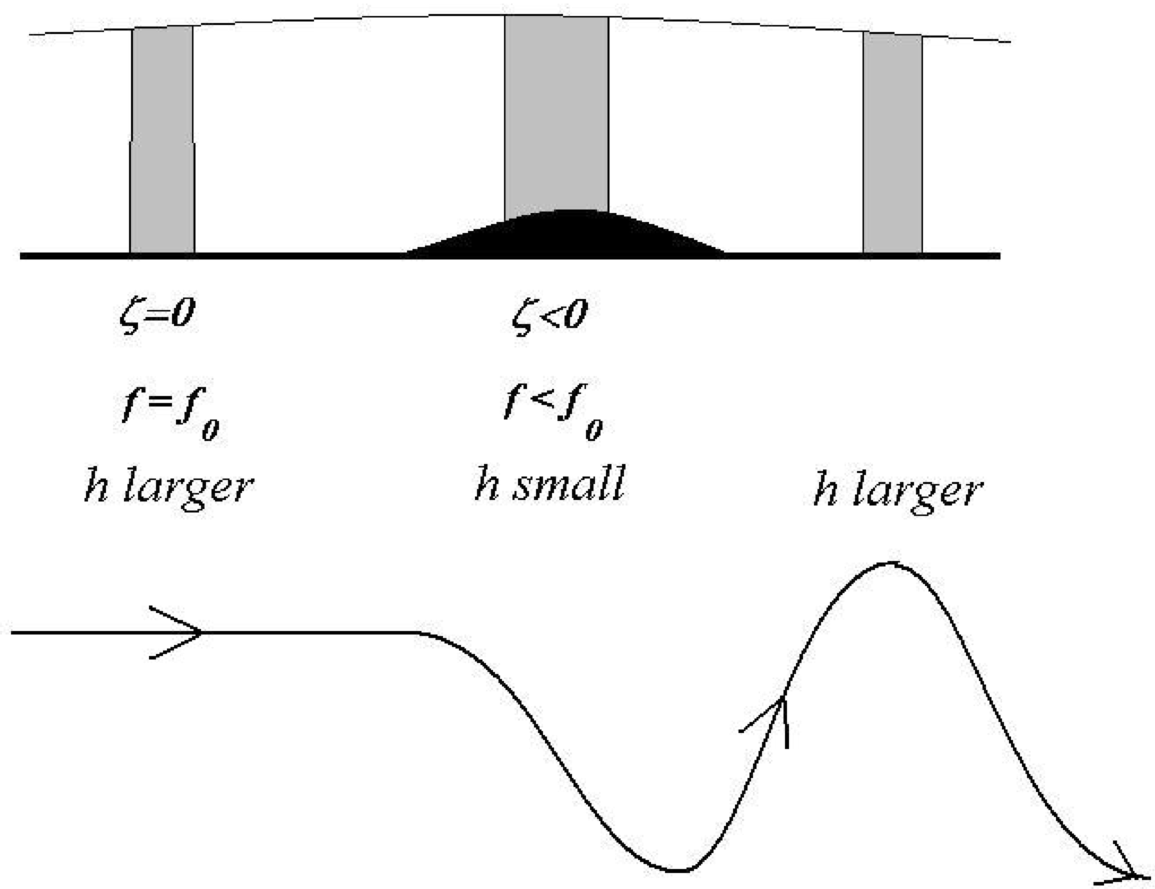
Naïve argument concerning movement of Rossby waves



Handwaving explanation of Rossby Wave Propagation.

This qualitative argument indicates *westward propagation.*

Formation of a Lee-side Trough



Formation of lee-side trough.

A mountain chain may produce a train of forced Rossby waves.

More Conservation Properties

The Circulation Theorem

The momentum equation in vector form is:

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We easily prove the following vector identity:

$$\mathbf{V} \cdot \nabla\mathbf{V} = \nabla(\tfrac{1}{2}\mathbf{V} \cdot \mathbf{V}) + \zeta\mathbf{k} \times \mathbf{V}$$

Using this, the momentum equation may be written:

$$\frac{\partial \mathbf{V}}{\partial t} + \nabla(\tfrac{1}{2}\mathbf{V} \cdot \mathbf{V}) + (f + \zeta)\mathbf{k} \times \mathbf{V} + \nabla\Phi = 0 \qquad (8)$$

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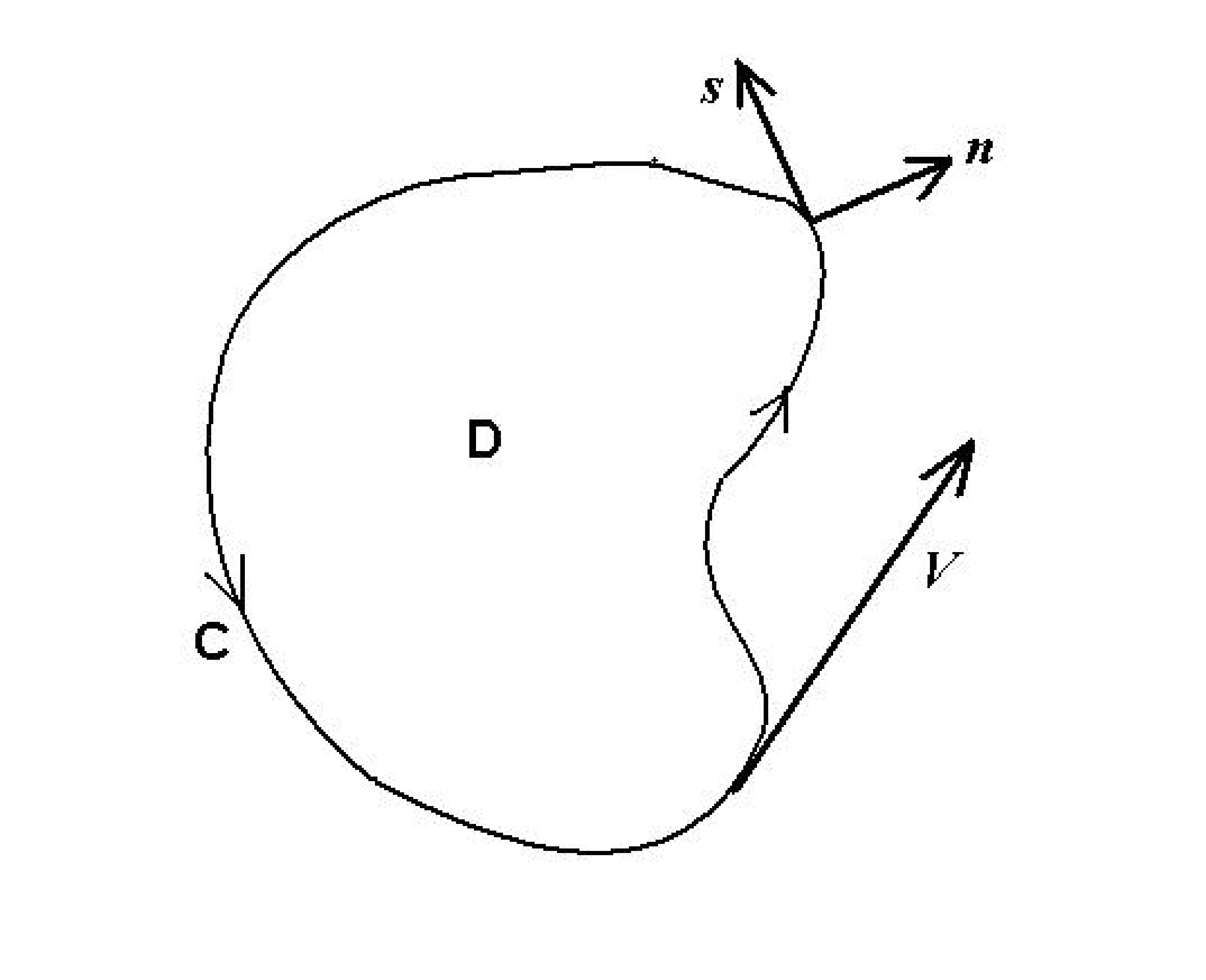
The gradient terms integrate to zero, because the contour is closed.



Contour defined by the flow velocity V

Thus, the integral of (8) gives

$$\oint_C \frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{s} = \frac{d}{dt} \oint_C \mathbf{V} \cdot d\mathbf{s} = 0$$

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Alternative view: **the vorticity equation can be written**

$$\frac{\partial \zeta}{\partial t} + \nabla\cdot(\zeta + f)\mathbf{V} = 0$$

Integrating this over D:

$$\frac{d}{dt} \iint_D \zeta \, da = -\iint_D \nabla\cdot(\zeta + f)\mathbf{V} \, da = \oint_C (\zeta + f)\mathbf{V} \cdot \mathbf{n} \, ds = 0$$

Thus, the integral of vorticity over the domain is a constant. Put another way, the *average vorticity* is conserved.

Conservation of Mass.

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Thus we get

$$\iint_D \frac{\partial \rho h}{\partial t}\, da = \frac{d}{dt}\iint_D \rho h\, da = 0$$

But the mass of fluid over an element of area da is $dM = \rho h\, da$. Thus, the equation expresses *conservation of total mass*.

Conservation of Energy

The *potential energy* of a column of fluid is:

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Multiply the continuity equation (9) by Φ to get

$$\left(\frac{g}{\rho}\right)\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}(\tfrac{1}{2}\Phi^2) = -\Phi\nabla\cdot\boldsymbol{\Phi}\mathbf{V} \,.$$

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Next, multiply the momentum equation

$$\frac{\partial \mathbf{V}}{\partial t} + (\zeta + f)\mathbf{k}\times\mathbf{V} + \nabla(\Phi + \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) = 0$$

by $\Phi\mathbf{V}$ to obtain (use $\mathbf{V}\cdot\mathbf{k}\times\mathbf{V} = 0$)

$$\Phi\mathbf{V}\cdot\frac{\partial \mathbf{V}}{\partial t} + \Phi\mathbf{V}\cdot\nabla\Phi + \Phi\mathbf{V}\cdot\nabla(\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}) = 0$$

Add the continuity equation multiplied by $\frac{1}{2}\mathbf{V}\cdot\mathbf{V}$:

$$\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\frac{\partial\Phi}{\partial t}+\tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\nabla\cdot\Phi\mathbf{V}=0$$

to obtain the expression

$$\left(\frac{g}{\rho}\right)\frac{\partial K}{\partial t}+\nabla\cdot[(\tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{V}\cdot\mathbf{V})\Phi\mathbf{V}]+\nabla\cdot\Phi^2\mathbf{V}-\Phi\nabla\cdot\Phi\mathbf{V}=0\,.$$

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Finally, integrate the equations for P and K over the domain:

$$\frac{d}{dt}\iint_D\left(\frac{\rho}{2g}\right)\Phi^2\,da=-\iint_D\left(\frac{\rho}{g}\right)\Phi\nabla\cdot\Phi\mathbf{V}\,da$$

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Adding these gives the energy conservation equation:

$$\frac{d}{dt}\iint_D\left(\frac{\rho}{2g}\right)\left[\Phi\mathbf{V}\cdot\mathbf{V}+\Phi^2\right]da=\frac{d}{dt}[K+P]=0\,. \qquad (10)$$

***This is the energy principle*: the sum of the kinetic plus potential energy of the fluid system remains constant.**

Simplification of the PV Equation

Conservation of potential vorticity implies that the quantity $P = (\zeta + f)/\Phi$, which we call the *potential vorticity*, is conserved following the motion.

That is, the value of P for a particular parcel of fluid remains constant as that parcel is carried along with the flow.

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Let us assume geostrophic flow:

$$f\mathbf{k} \times \mathbf{V} + \nabla\mathbf{\Phi} = \mathbf{0}\,.$$

The vorticity may then be written in terms of Φ:

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \nabla \cdot \mathbf{V} \times \mathbf{k} = \nabla\cdot(1/f)\nabla\Phi$$

The material time derivative takes the form

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} = \frac{\partial}{\partial t} - \frac{1}{f}\frac{\partial \Phi}{\partial y}\frac{\partial}{\partial x} + \frac{1}{f}\frac{\partial \Phi}{\partial x}\frac{\partial}{\partial y}$$

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Then the potential vorticity equation (6) becomes an equation for a single dependent variable, Φ**:**

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Assume the flow is *quasi-geostrophic* and *quasi-nondivergent*:

$$\mathbf{V} \approx \frac{1}{f}\mathbf{k}\times\nabla\Phi\,, \qquad \mathbf{V} \approx \mathbf{k}\times\nabla\psi\,.$$

To the first order of approximation, we can move the factor $1/f$ inside the differential operator:

$$\frac{1}{f}\mathbf{k} \times \nabla\Phi \approx \mathbf{k} \times \nabla \left(\frac{\Phi}{f}\right) .$$

Thus, the geopotential and stream function are related:

$$\Phi \approx f\psi \, .$$

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Now assume the deviation of geopotential from its mean value is small:

$$\Phi = \bar{\Phi} + \Phi' \qquad \textbf{with} \qquad \Phi' \ll \bar{\Phi} \, .$$

We can equate Φ' with $f\psi$. Then we have

$$\frac{1}{\Phi} = \frac{1}{\bar{\Phi}(1+\Phi'/\bar{\Phi})} \approx \frac{1}{\bar{\Phi}}\left(1 - \frac{\bar{\Phi}}{\bar{\Phi}}\right) \approx \frac{1}{\bar{\Phi}}\left(1 - \frac{f\psi}{\bar{\Phi}}\right)$$

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Thus, the potential vorticity becomes

$$P \equiv \frac{\zeta + f}{\Phi} \approx \frac{1}{\bar{\Phi}}(\zeta + f)\left(1 - \frac{f\psi}{\bar{\Phi}}\right) \approx \frac{f}{\bar{\Phi}} + \frac{\zeta}{\bar{\Phi}} - \frac{f^2\psi}{\bar{\Phi}^2}$$

We can ignore the variation of f in the term containing ψ, so

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Next, we use the nondivergent wind in the Lagrangian derivative:

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{\partial \alpha}{\partial t} + u\frac{\partial \alpha}{\partial x} + v\frac{\partial \alpha}{\partial y} \\ &= \frac{\partial \alpha}{\partial t} + \left\{\frac{\partial \psi}{\partial x}\frac{\partial \alpha}{\partial y} - \frac{\partial \psi}{\partial y}\frac{\partial \alpha}{\partial x}\right\} \\ &= \frac{\partial \alpha}{\partial t} + J(\psi, \alpha)\,.\end{aligned}$$

We can ignore the variation of f in the term containing ψ, so

$$\bar{\Phi} P \approx \zeta + f - F\psi$$

where $F \equiv f_0^2/\bar{\Phi}$ is a constant.

Next, we use the nondivergent wind in the Lagrangian derivative:

$$\begin{aligned} \frac{d\alpha}{dt} &= \frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} \\ &= \frac{\partial\alpha}{\partial t} + \left\{\frac{\partial\psi}{\partial x}\frac{\partial\alpha}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\alpha}{\partial x}\right\} \\ &= \frac{\partial\alpha}{\partial t} + J(\psi,\alpha)\,. \end{aligned}$$

Using this together with the approximation for P derived above, the PV-equation may be written as

$$\frac{\partial}{\partial t}\left(\nabla^2\psi - F\psi\right) + J(\psi,\nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0\,.$$

The Barotropic QGPV Equation

The *barotropic, quasi-geostrophic potential vorticity equation* (the QGPV Equation) is

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We will also study numerical solutions of this equation using a program written in MATLAB and in C.