M.Sc. in Computational Science

Fundamentals of Atmospheric Modelling

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Lecture 6

Vorticity and Divergence

Introduction

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We show that an arbitrary velocity field may be partitioned into curl-free and divergence-free components.

Also, we show that the velocity may be reconstructed from knowledge of vorticity and divergence.

The most important result we derive is the *Conservation of Potential Vorticity.*

Recall the form of the SWE:

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - fv + \frac{\partial \Phi}{\partial x} = 0 \tag{1}$$

$$\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + fu + \frac{\partial \Phi}{\partial y} = 0 \tag{2}$$

$$\left(\frac{\partial \Phi}{\partial t} + u\frac{\partial \Phi}{\partial x} + v\frac{\partial \Phi}{\partial y}\right) + \Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{3}$$

The geopotential is $\Phi = gh$ and the Coriolis parameter is $f = 2\Omega\sin\phi$. Recall that we have neglected all effects of spherical geometry except in the Coriolis term.

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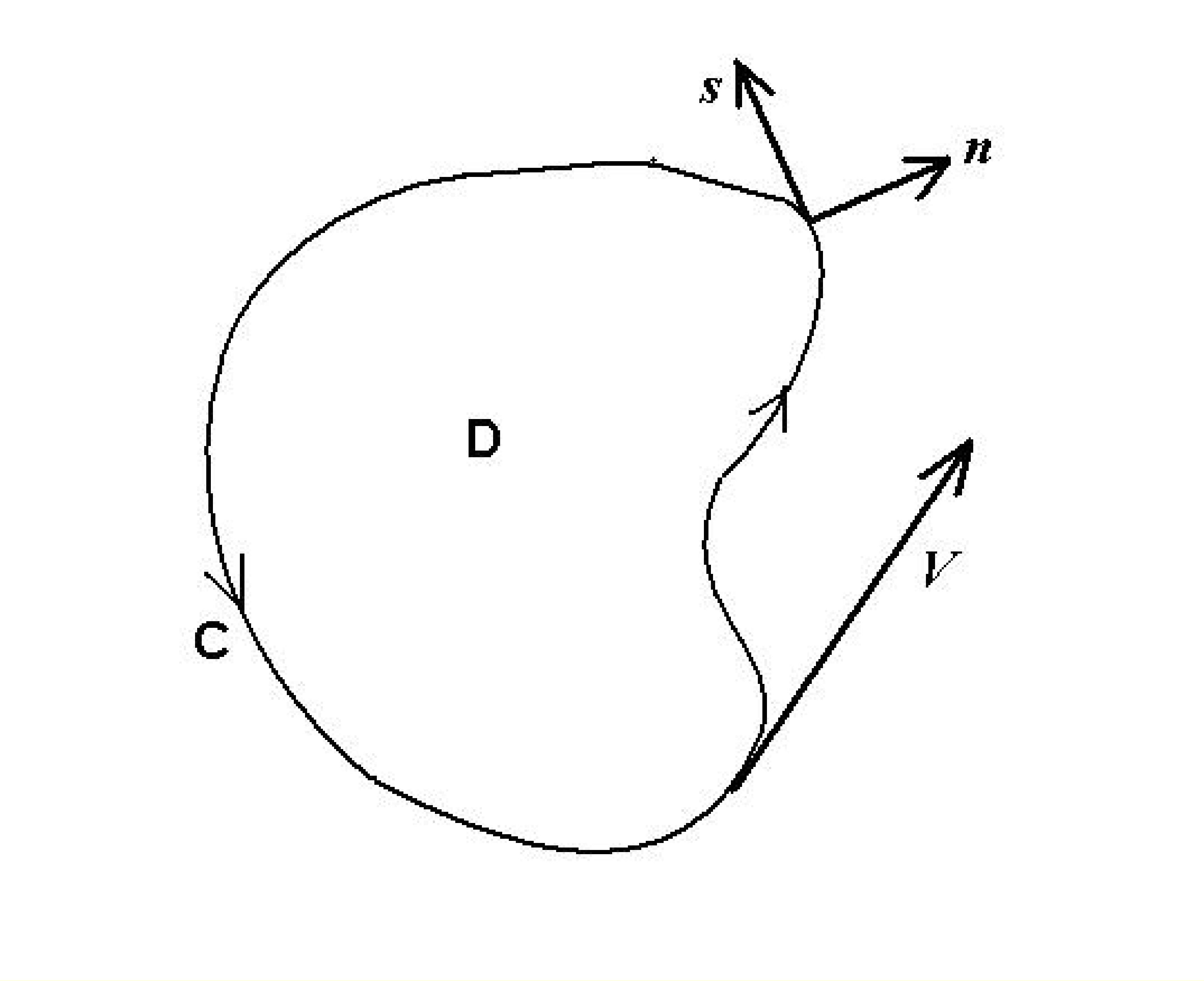
We define the *beta parameter*:

$$\beta = \frac{df}{dy} = \frac{2\Omega\cos\phi}{a}\,.$$

For latitudes ϕ not too far from a central value ϕ_0, we may assume that

$$f = 2\Omega\sin\phi \approx 2\Omega\sin\phi_0 \qquad \textbf{and} \qquad \beta = \frac{2\Omega\cos\phi}{a} \approx \frac{2\Omega\cos\phi_0}{a}$$

are both constant, <u>unless differentiated</u> w.r.t. y.



Tangent and normal unit vectors **s** *and* **n**.

“Spin” and “Spread”

The extent to which the fluid is *rotating* may be measured by calculating the *circulation* around a small circle C and taking the limit as the area A goes to zero:

$$\zeta = \lim_{A \to 0} \frac{1}{A} \oint_C \mathbf{V} \cdot \mathbf{s} \, ds \,.$$

We may call this the *Spin* or, more usually, the *Vorticity*.

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Using Stokes’ and Gauss’s Theorems, we will obtain differential forms of the vorticity and divergence.

First, consider Stokes' Theorem:

$$\oint_C \mathbf{V} \cdot \mathbf{s}\, ds = \iint_A \mathbf{k} \cdot \nabla \times \mathbf{V}\, da\,.$$

Assuming the area A of the circle is small, we get

$$\frac{1}{A} \oint_C \mathbf{V} \cdot \mathbf{s}\, ds \approx \mathbf{k} \cdot \nabla \times \mathbf{V}\,.$$

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Now recall Gauss's Theorem

$$\oint_C \mathbf{V}\cdot\mathbf{n}\,ds = \iint_A \nabla\cdot\mathbf{V}\,da\,.$$

Assuming the area A of the circle is small, we get

$$\frac{1}{A}\oint_C \mathbf{V}\cdot\mathbf{n}\,ds \approx \nabla\cdot\mathbf{V}\,.$$

Taking the limit $A \longrightarrow 0$, we define the divergence as

$$\delta = \nabla\cdot\mathbf{V}$$

We define the *vorticity* and *divergence* as follows:

$$\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

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We will derive equations for the vorticity and divergence by differentiating and combining the momentum equations.

Exercise:

Show that the ratio of the vertical to horizontal component of the (3-D) vorticity is of the order w/V so that, with the assumptions we have made, the vertical component dominates.
If we relax the assumption $\partial \mathbf{V}/\partial z = 0$, how does this affect the conclusion?

Exercise: Geostrophic Divergence

Suppose the wind is geostrophic. Derive expressions for vorticity and divergence in terms of geopotential.

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The geostrophic velocity is given by

$$u = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \qquad v = +\frac{1}{f}\frac{\partial \Phi}{\partial x},$$

Calculating vorticity directly, we get

$$\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(+\frac{1}{f}\frac{\partial \Phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{1}{f}\frac{\partial \Phi}{\partial y}\right) = \frac{1}{f}\nabla^2\Phi + \frac{\beta}{f}u\,.$$

For constant f the stream function is $\psi = \Phi/f$.

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Calculating divergence directly, we get

$$\delta = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{f}\frac{\partial\Phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(+\frac{1}{f}\frac{\partial\Phi}{\partial x}\right) = -\frac{\beta}{f}v\,.$$

For f constant, it follows immediately that $\delta = 0$. Therefore, the geostrophic divergence depends on the variation of f, the beta-effect.

Assume that the scale of motion is synoptic, $\mathsf{L} = 10^6\,$m.
Since $a = 6371\,\mathrm{km} \approx 10^7\,\mathrm{m}$, we may assume

$$\frac{\mathsf{L}}{a} \sim \mathsf{Ro} \ll 1\,.$$

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Let us assume that the flow is approximately geostrophic. Then

$$\zeta \sim \frac{\mathsf{V}}{\mathsf{L}} \qquad \delta \sim \frac{\mathsf{V}}{a} \sim \frac{\mathsf{L}}{a}\zeta\,.$$

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The smallness of the divergence is due to approximate cancellation between influx and outflow. The terms $\partial u/\partial x$ and $\partial v/\partial y$ are roughly equal in magnitude but opposite in sign. *This makes accurate calculation of divergence very difficult.*

The Helmholtz Theorem (2D-Form)

A fundamental theorem due to Stokes (1849) states that a velocity field can be decomposed into the sum of an irrotational (curl-free) and a non-divergent part.

$$\mathbf{V} = \mathbf{V}_{\mathrm{D}} + \mathbf{V}_{\mathrm{R}} = \nabla\chi + \nabla \times \mathbf{A}\,,$$

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For two-dimensional flow, the decomposition is as follows:

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Here ψ and χ are the *stream function* and *velocity potential*.

We recall the vector identities ($\mathrm{curl}\,\mathrm{grad}\,\chi = 0$ and $\mathrm{div}\,\mathrm{curl}\,\mathbf{A} = 0$):

$$\nabla \times \nabla\chi = 0 \qquad \nabla \cdot (\nabla \times \mathbf{A}) = 0\,.$$

In 2-D, the latter implies $\nabla \cdot \mathbf{k} \times \nabla\psi = 0$.

Thus,

$$\delta = \nabla \cdot \mathbf{V} = \nabla^2 \chi \qquad \zeta = \mathbf{k} \cdot \nabla \times \mathbf{V} = \nabla^2 \psi \, .$$

Note the useful vector identity: $\mathbf{k} \cdot \nabla \times \mathbf{V} = \nabla \cdot (\mathbf{V} \times \mathbf{k})$**.**

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Procedure:

(1) Solve the two *Poisson equations*

$$\nabla^2 \chi = \delta \,, \qquad \nabla^2 \psi = \zeta \,,$$

for the stream function and velocity potential.

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(2) Calculate the wind from

$$\mathbf{V} = \nabla \chi + \mathbf{k} \times \nabla \psi \,.$$

or, in component form

$$u = \frac{\partial \chi}{\partial x} - \frac{\partial \psi}{\partial y} \,, \qquad v = \frac{\partial \chi}{\partial y} + \frac{\partial \psi}{\partial x} \,.$$

The Vorticity Equation

Recall the momentum equations

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) - fv + \frac{\partial \Phi}{\partial x} = 0 \qquad (1)$$

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Thus the *vorticity equation* may also be written:

$$\frac{d}{dt}(\zeta + f) + (\zeta + f)\delta = 0 \, .$$

The *absolute vorticity* η is defined as the sum of *relative vorticity* and *planetary vorticity*:

$$\underbrace{\eta}_{\substack{\text{Absolute}\\\text{Vorticity}}} = \underbrace{\zeta}_{\substack{\text{Relative}\\\text{Vorticity}}} + \underbrace{f}_{\substack{\text{Planetary}\\\text{Vorticity}}} .$$

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$$\frac{1}{h}\frac{dh}{dt} + \delta = 0 .$$

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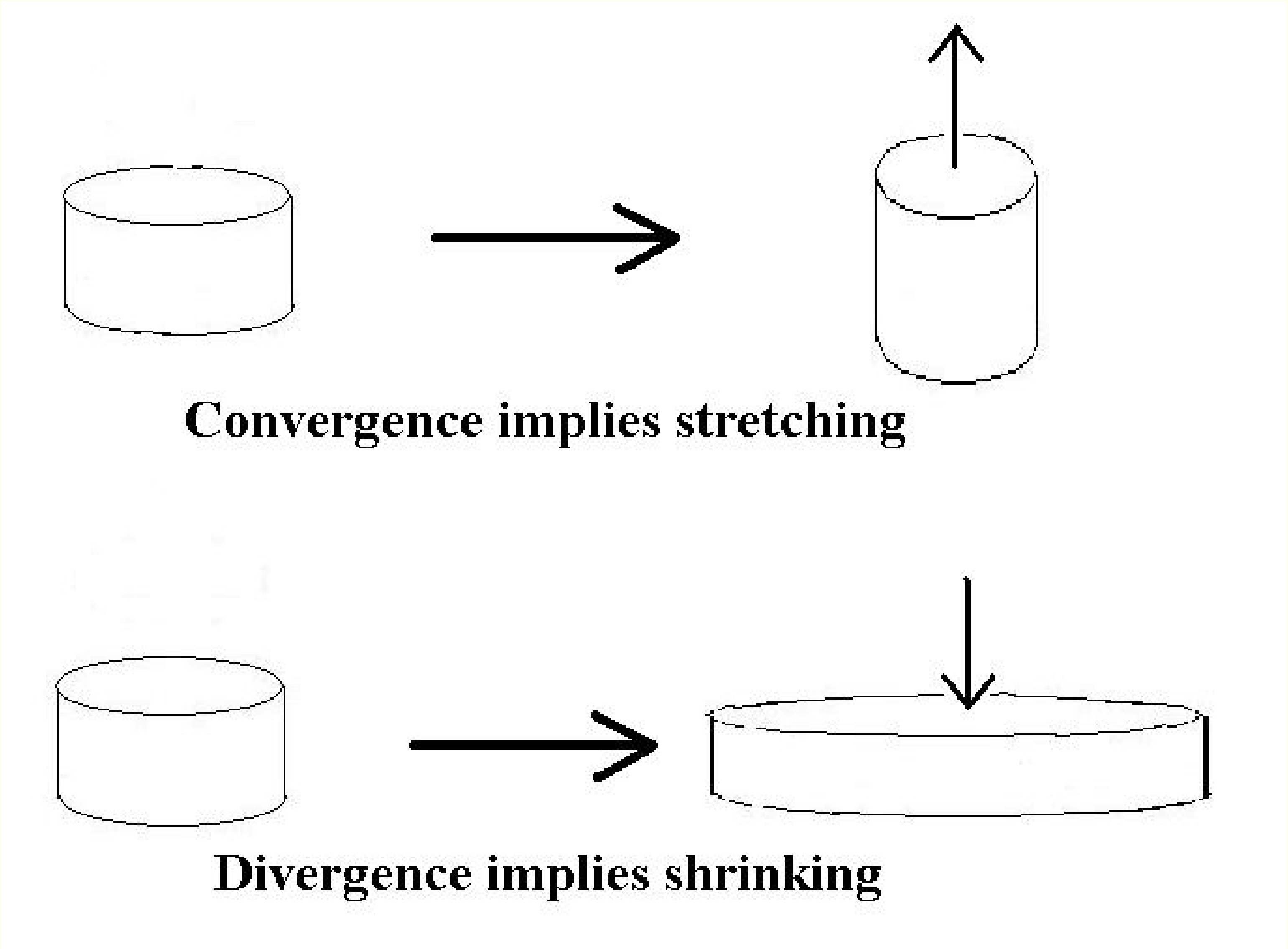
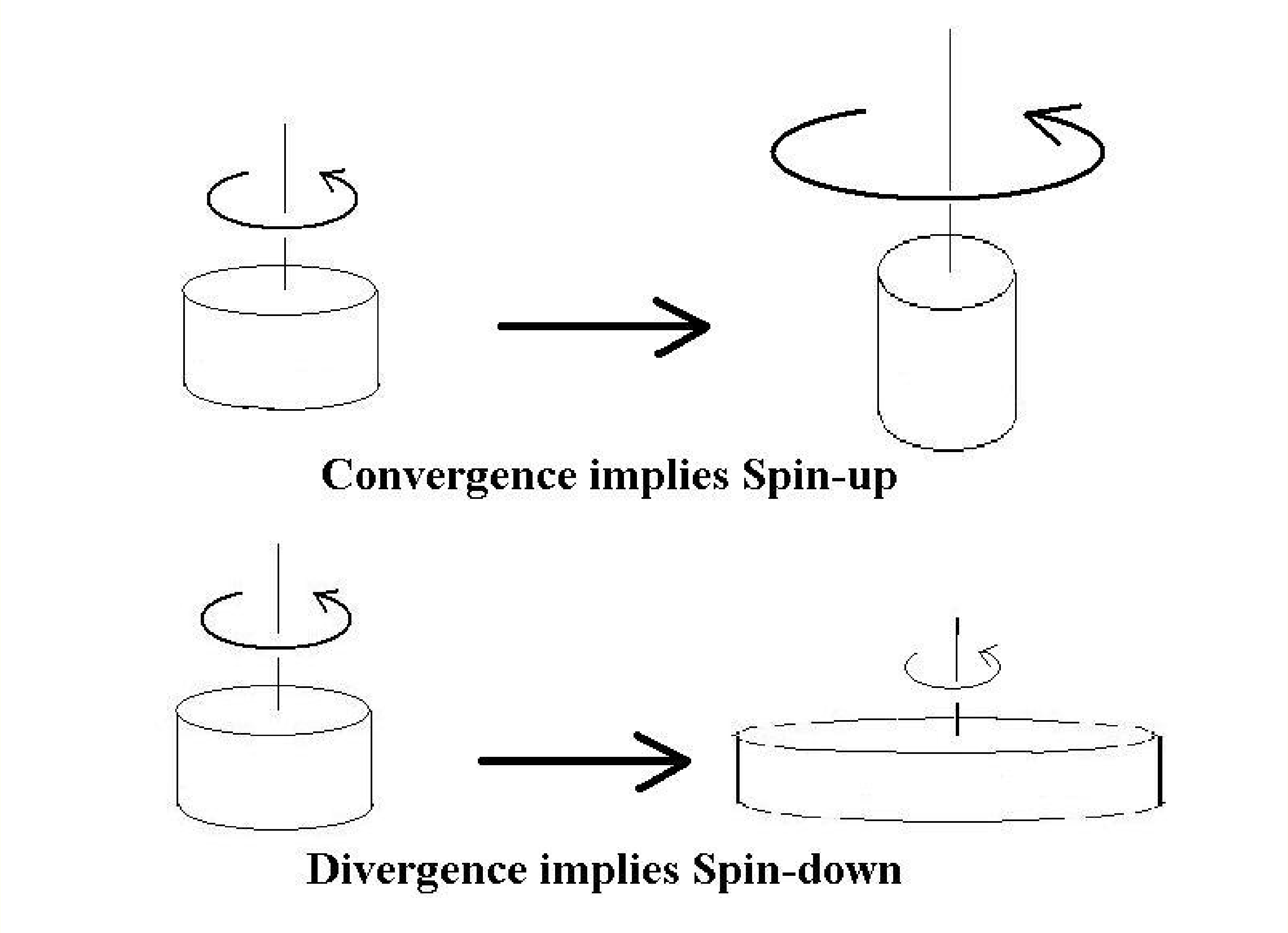
We may illustrate this by considering a column of fluid.

The *Continuity Equation* may be interpreted pictorially.



Convergence is associated with *stretching* of the column.
Divergence is associated with *shrinking* of the column.

The *Vorticity Equation* may be interpreted pictorially.



Convergence is associated with *spin-up* of the fluid column.
Divergence is associated with *spin-down* of the column.

The Divergence Equation

Recall again the momentum equations

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Taking the x-derivative of (1) and adding it to the y-derivative of (2), we get an equation for δ:

$$\frac{\partial \delta}{\partial t} + u\frac{\partial \delta}{\partial x} + v\frac{\partial \delta}{\partial y} - \zeta f + \delta^2 - 2J(u,v) + \beta u + \nabla^2\Phi = 0\,.$$

The Jacobian term is defined as

$$J(u,v) = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)\,.$$

Note: The derivation of the divergence equation in the above form is elementary, but it requires a page or two of algebraic manipulation.

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Observation: **for large-scale atmospheric flow in middle latitudes, the divergence is much smaller than the vorticity:**

$$|\delta| \ll |\zeta| \, .$$

This allows us to make approximations to the equations.

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New Definitions:

The flow is *cyclonic* if $f\zeta > 0$.
The flow is *anticyclonic* if $f\zeta < 0$.

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The continuity equation may be written:

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Wo eliminate δ between the vorticity and continuity equations to get:

$$\frac{1}{\zeta + f}\frac{d(\zeta + f)}{dt} = \frac{1}{h}\frac{dh}{dt}\,.$$

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$$\frac{1}{\zeta + f}\frac{d(\zeta + f)}{dt} = \frac{1}{h}\frac{dh}{dt}\,.$$

This may also be put in the following form (take logs):

$$\frac{d}{dt}\left(\frac{\zeta + f}{h}\right) = 0\,.$$

This is the equation of *conservation of potential vorticity.*

Exercise: Bottom Orography

We have assumed the bottom surface is flat. Now we will relax this.

Assume the height of the bottom boundary is $h_{\mathrm{B}}(x, y)$.

Show that the Conservation of Potential Vorticity takes the form:

$$\frac{d}{dt}\left(\frac{\zeta + f}{h - h_{\mathrm{B}}}\right) = 0\,.$$

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$\star \qquad \star \qquad \star$

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⋆ ⋆ ⋆

This is an important exercise. **The solution will not be given. The proof is straightforward, requiring only a minor adjustment of the derivation in the case $h_{\rm B}(x,y) \equiv 0$.**