Fundamentals of Atmospheric Modelling

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Lecture 3

The Equations of Motion

The Thin Atmosphere

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90% of its mass lies within 10 km of the earth's surface.

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A typical grid-box of a numerical model might have dimensions $10\,\mathrm{km} \times 10\,\mathrm{km} \times 100\,\mathrm{m}$, which has an aspect ratio of one hundred to one, comparable to that of a credit card!

Digression

Did you know?

The ratio of the length to the breadth of a credit card is equal to the ratio of the *Golden Section*:

$$\begin{pmatrix}\textbf{Aspect}\\ \textbf{Ratio}\end{pmatrix} = \left[\frac{\textbf{Length}}{\textbf{Breadth}}\right] = \frac{1+\sqrt{5}}{2} \approx 1.618\,.$$

It is allegedly the most aesthetically pleasing rectangular shape, and is found in numerous classical works of art.

This ratio is ubiquitous throughout nature. It is closely associated with the Fibonacci sequence of numbers

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}\,,$$

where each term is the sum of the preceding two terms.

Hydrostatic Balance

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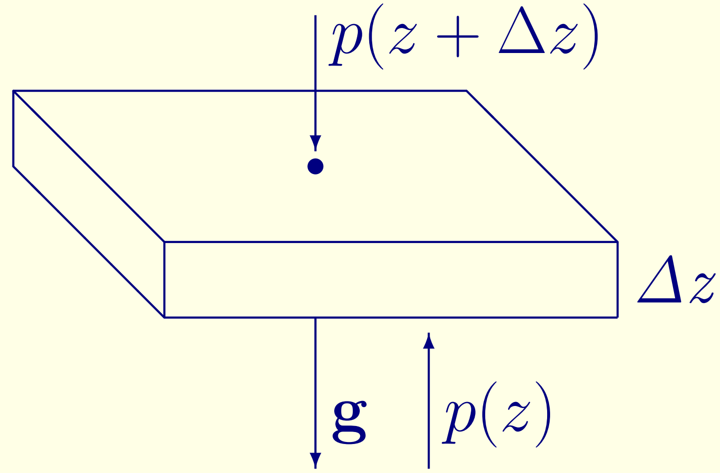

$$\textbf{Force Upward on Box}: \quad +[p(z) \cdot \Delta x \Delta y]$$

$$\textbf{Force Downward on Box}: \quad -[p(z + \Delta z) \cdot \Delta x \Delta y + mg]$$

For equilibrium, the net force must be zero:

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This is the Hydrostatic balance equation. It implies an exact balance between the vertical pressure gradient and gravity.

For an atmosphere at rest, hydrostatic balance holds exactly.

Exercise: Vertical Pressure Gradient

Suppose the atmosphere is in a state of hydrostatic balance.

Calculate approximately the pressure drop over a vertical distance of 100 m, assuming the density is constant at $\rho = 1.2\,\mathrm{kg\,m^{-3}}$ and $g = 9.8\,\mathrm{m\,s^{-2}}$.

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The hydrostatic equation gives

$$\frac{\Delta p}{\Delta z} + \rho g = 0\,.$$

Substituting the numerical values gives

$$\Delta p = -\Delta z \rho g = -100\,\mathrm{m} \times 1.2\,\mathrm{kg\,m^{-3}} \times 9.8\,\mathrm{m\,s^{-2}}$$

(negative, since pressure decreases upwards). Evaluating this gives

$$|\Delta p| = 1176\,\mathrm{kg\,m^{-1}s^{-2}} = 1176\,\mathrm{Pa} = 11.76\,\mathrm{hPa}\,.$$

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The hectoPascal, numerically equal to the millibar, is the pressure unit most commonly used in practice.

Note that the assumption of constant density is unrealistic over large vertical distances. We will relax this assumption presently.

Hydrostatic Approximation

The hydrostatic approximation consists of assuming that balance between the vertical pressure gradient and gravity holds <u>even when the fluid is in motion</u>. For the large scale motions of the atmosphere and ocean, hydrostatic balance holds to a high degree of accuracy.

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***Non-hydrostatic models* are now growing in popularity, particularly where spatial grids of a few kilometres are used. For these models, the vertical velocity is a *prognostic variable*, predicted in the same way as the other dependent variables.**

The Equation of State

We are familiar from elementary physics with Boyle's Law and Charles' Law of gases. They are special cases of the *Equation of State* for a perfect gas:

$$pV = nR^*T$$

where $R^* = 8314\,\mathbf{J\,K^{-1}\,kmol^{-1}}$ is the universal gas constant and n is the number of kilomoles of gas (a *kmole* is the molecular weight in kg).

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The mean molecular weight of air is $\mu \approx 29$. Thus, $m = \mu n$.

Dividing by the volume, we get the *equation of state*

$$p = R\rho T$$

where $R = R^*/\mu = 287\,\mathbf{J\,K^{-1}\,kmol^{-1}}$ is the gas constant for dry air.

Table 1: **Main Constitutents of the Atmosphere**

Gas		Percentage	Mol. Wt.
Nitrogen	N_2	80%	28
Oxygen	O_2	20%	32
Air		100%	29

The atmosphere is composed primarily of nitrogen (80%) and oxygen (20%), so the mean molecular weight of air is about 29.

Other constituents, such as carbon dioxide and methane, are vitally important for radiative balance, but their concentrations are quite small.

Water **occurs in all three phases, and is enormously important. However, we will be concentrating on the large-scale dynamics of the atmosphere and will largely ignore water, as it introduces great complexity.**

Vertical Variation of Pressure

Let's consider an *isothermal* atmosphere at rest. Let the constant temperature be T_0. The hydrostatic equation and the equation of state are

$$\frac{\partial p}{\partial z} + g\rho = 0\,, \qquad p = R\rho T_0\,.$$

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Combining these we have

$$\frac{\partial p}{\partial z} = -g\frac{p}{RT_0} \,, \qquad \textbf{so} \qquad \frac{dp}{p} = -\frac{g\,dz}{RT_0} = -\frac{dz}{H} \,,$$

where we define the scale-height by $H = RT_0/g$.

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We integrate over the range $p_0 = p(0)$ to $p = p(z)$ to get

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or

$$p(z) = p_0 \exp(-z/H)\,.$$

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Thus, *pressure decreases exponentially with height*.

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The scale height is easily computed. Suppose $T_0 = 265\,\mathbf{K}$. Then

$$H = \frac{RT_0}{g} = \frac{287 \times 265}{9.8066} = 7755\,\mathrm{m} = 7.755\,\mathrm{km}\,,$$

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Exercise

Relax the assumption of an *isothermal atmosphere*: Assume that temperature decreases linearly with height

$$T = T_0 + \gamma z$$

where the lapse rate, $\gamma = \partial T/\partial z < 0$, is constant.

Calculate the dependence of p on height.

Exercise: Constant Lapse-rate

Answer: **Assume that temperature decreases linearly with height**

$$T = T_0 + \gamma z$$

where the lapse rate $\gamma = \partial T/\partial z < 0$ **is constant.**

Combining the hydrostatic equation and the equation of state as before, we get

$$\frac{dp}{p} = -\frac{g\,dz}{RT} = -\frac{g}{RT_0}\frac{dz}{1+\gamma z/T_0} .$$

Integrating this yields

$$\log\left(\frac{p}{p_0}\right) = -\frac{T_0}{\gamma H}\log\left(1+\frac{\gamma z}{T_0}\right) = \log\left(1+\frac{\gamma z}{T_0}\right)^{-T_0/\gamma H}$$

so that

$$p = p_0\left(1+\frac{\gamma z}{T_0}\right)^{-T_0/\gamma H}$$

[**More Work:** **Show that this reduces to the previous result when** $\gamma \to 0$**. Use** $\lim_{n\to\infty}(1+x/n)^{-n} = \exp(-x)$**.**]

Exercise: Mass of Air Column

Consider a vertical column of air, of cross-section A and infinite height. Suppose the pressure at the bottom of the column is p_0.

What is the mass M of the column?

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However, there is a simpler way: the *force* of the column on the surface below it is given by two quantities, which must be equal:

$$p_0 \times A = M \times g \,, \qquad \text{so that} \qquad M = p_0 A / g \,.$$

For a unit cross-section, $A = 1\,\mathbf{m}^2$, the total column mass is $M = p_0/g \approx 10^4\,$kg or ten tonnes!

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More Work: Recall the TV screen. The force was 12 kN for an area $A = 0.12\,\mathbf{m}^2$. Show that this result is consistent with the above.

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$$M = A \int_0^h \rho_0 \, dz = A\rho_0 h \, .$$

The force exerted by the column on the surface below is Mg. Since pressure is force-per-unit-area, the pressure is

$$p_0 = g\rho_0 h$$

so, for given pressure p_0, the depth is

$$h = \frac{p_0}{g\rho_0} \, .$$

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But the scale height of the atmosphere is

$$H = \frac{RT_0}{g} = \frac{p_0}{g\rho_0} \, ,$$

so we see that the depth h equals the scale height H.

The Equations of Motion

Forces Acting on a Parcel of Air

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$$\mathbf{g}^* = -g\mathbf{k}\,.$$

The star on $\mathbf{g}^*$ will be explained below.

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(3) **The force of friction acts in a direction opposite to the velocity of the flow. We could model it as**

$$\mathbf{F}_\mathrm{f} = -\nu\nabla^2\mathbf{V}, \qquad \textbf{or} \qquad \mathbf{F}_\mathrm{f} = -\kappa\mathbf{V}\,.$$

The friction coefficient κ will depend on position and, perhaps, on velocity.

Equations in an Inertial Frame

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The basic equations of motion ($\mathbf{a} = \mathbf{F}/m$) are:

$$\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^{\star} + \mathbf{F}_{\mathrm{f}} \tag{1}$$

where the total, material or Lagrangian derivative is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$$

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$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{V} = 0\,.$$

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If the fluid is *incompressible*, it is especially simple:

$$\nabla \cdot \mathbf{V} = 0 \tag{2}$$

A Complete System

If we assume incompressible, inviscid flow, equations (1) and (2) comprise a system of four equations for the four variables $(u, v, w; p)$**:**

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Written out in *(local) cartesian coordinates*, they are

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) u &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\
\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) v &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) w &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0\,.
\end{aligned}$$

Rotating Coordinate Frames

Theorem: **Consider a vector $\mathbf{A}$ fixed in a frame which is rotating with constant angular velocity $\boldsymbol{\Omega}$. Then the rate of change of $\mathbf{A}$ is**

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\Omega} \times \mathbf{A} \,.$$

(see, e.g., Synge and Griffith, pg. 278.)

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The vector rotates through an angle $\Omega\Delta t$ in time Δt.

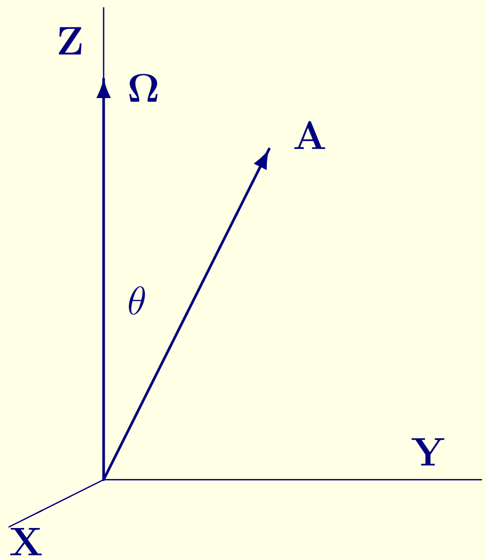
The projection of A on the $\boldsymbol{\Omega}$-axis does not change

The projection of A in the X-Y-plane is $A\sin\theta$. It does not change in magnitude, but its direction changes (see Figure). We have

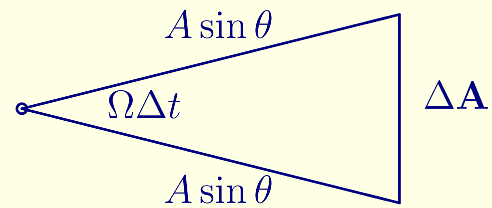
$$\Delta\mathbf{A} = (\Omega A \sin\theta \hat{\mathbf{n}}) \cdot \Delta t$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\boldsymbol{\Omega}$ and A.
Thus

$$\frac{d\mathbf{A}}{dt} = \boldsymbol{\Omega} \times \mathbf{A} \,.$$



Vector in Rotating Coordinates



Horizontal Projection

Consider a vector

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}\,.$$

The rate of change of A in the rotating frame is

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}\,.$$

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$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}\right) + \left(A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt}\right)\,.$$

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The changes in the unit vectors are (by the above theorem)

$$\frac{d\mathbf{i}}{dt} = \mathbf{\Omega}\times\mathbf{i}\,;\quad \frac{d\mathbf{j}}{dt} = \mathbf{\Omega}\times\mathbf{j}\,;\quad \frac{d\mathbf{k}}{dt} = \mathbf{\Omega}\times\mathbf{k}\,.$$

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$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}\right) + \left(A_1\frac{d\mathbf{i}}{dt} + A_2\frac{d\mathbf{j}}{dt} + A_3\frac{d\mathbf{k}}{dt}\right)\,.$$

The changes in the unit vectors are (by the above theorem)

$$\frac{d\mathbf{i}}{dt} = \mathbf{\Omega}\times\mathbf{i}\,; \quad \frac{d\mathbf{j}}{dt} = \mathbf{\Omega}\times\mathbf{j}\,; \quad \frac{d\mathbf{k}}{dt} = \mathbf{\Omega}\times\mathbf{k}\,.$$

Therefore,

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}\right) + \mathbf{\Omega}\times\left(A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}\right)\,.$$

Thus, the relationship between the relative and absolute rates of change of A is:

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} + \mathbf{\Omega} \times \mathbf{A}\,. \qquad (*)$$

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$\star \quad \star \quad \star$

Now let A be the position vector r. Since $(d\mathbf{r}/dt)_{\mathrm{I}} = \mathbf{V}_{\mathrm{I}}$ **and** $(d\mathbf{r}/dt)_{\mathrm{R}} = \mathbf{V}_{\mathrm{R}}$**, we get:**

$$\mathbf{V}_{\mathrm{I}} = \mathbf{V}_{\mathrm{R}} + \mathbf{\Omega} \times \mathbf{r}\,,$$

which relates the relative velocity to that in the inertial frame:

$$\left[\begin{array}{c}\textbf{Inertial}\\ \textbf{Velocity}\end{array}\right] = \left[\begin{array}{c}\textbf{Relative}\\ \textbf{Velocity}\end{array}\right] + \left[\begin{array}{c}\textbf{Velocity}\\ \textbf{of Frame}\end{array}\right]$$

$\star \quad \star \quad \star$

Exercise

If $\mathbf{A} = \mathbf{r}_0$ is a point *fixed* in the rotating frame, the velocity in the absolute frame is

$$\mathbf{V} = \boldsymbol{\Omega} \times \mathbf{r_0} \,.$$

Find the absolute velocity of a point on the earth's surface (i) at the Equator, and (ii) at 60° North.

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The rotation of the earth (assuming solar day ≈ siderial day) is:

$$\boldsymbol{\Omega} = 1\,\mathrm{rev.\ per\ day} = \frac{2\pi}{24 \times 60 \times 60}\,\mathbf{rad/sec} = 7.29 \times 10^{-5}\,\mathrm{s}^{-1}$$

The radius of the earth is

$$a = \frac{2 \times 10^7}{\pi}\,\mathrm{m} \approx 6.37 \times 10^6\,\mathrm{m}\,.$$

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(i) At the equator, $\phi = 0^\circ$ and $\theta = 90^\circ$, so that $\sin\theta = 1$ and

$$\boldsymbol{\Omega} \times \mathbf{r_0} = \Omega a = (7.29 \times 10^{-5}\,\mathrm{s}^{-1}) \times (6.37 \times 10^6\,\mathrm{m}) = 4.64 \times 10^2\,\mathrm{m\,s}^{-1} \sim 1000\,\mathrm{m.p.h.}$$

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(ii) At $\phi = 60^\circ$, we have $\theta = 30^\circ$, so that $\sin\theta = 0.5$, and the value of the velocity due to the earth's rotation is half that at the equator.

Relative Acceleration

Recall from (*) above that

$$\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{A}\,.$$

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Now let A be the absolute velocity $\mathbf{V}_{\mathrm{I}} = \mathbf{V}_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{r}$:

$$\begin{aligned}\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} &= \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + \left(\frac{d\boldsymbol{\Omega} \times \mathbf{r}}{dt}\right)_{\mathrm{R}} + \boldsymbol{\Omega} \times \mathbf{V}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) \\ &= \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \mathbf{V}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\,.\end{aligned}$$

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$$= \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega} \times \mathbf{V}_{\mathrm{R}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\,.$$

The term $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})$ is called the centrifugal acceleration. Since it depends only on position, it can be combined with the gravitational acceleration to give an apparent gravitational attraction

$$\mathbf{g} = \mathbf{g}^{\star} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r})\,.$$

This is a small adjustment to the true gravitational acceleration.

Exercise: Centrifugal Acceleration

Calculate the magnitude of the centrifugal acceleration at the Equator and compare it to the magnitude of the gravitational acceleration.

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First note that

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = -\Omega^2 \mathbf{R} \,,$$

where R is the projection of r on the equatorial plane. Thus, near the earth's surface, the magnitude of the centrifugal acceleration is

$$\Omega^2 a = (7.29 \times 10^{-5}\,\mathrm{s}^{-1})^2 \times (6.37 \times 10^{6}\,\mathrm{m}) = 3.18 \times 10^{-2}\,\mathrm{m\,s}^{-2}$$

Now, comparing with true gravity, the percentage correction is

$$\frac{\Omega^2 a}{g} \times 100 \approx 0.3\% \,.$$

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The centrifugal acceleration is responsible for the *flattened form of the earth*, which assumes an oblate spheroidal shape.

Do you lose weight when you travel to the Tropics? If so, how much?

The Coriolis Acceleration

The term $2\mathbf{\Omega} \times \mathbf{V}$ is called the Coriolis acceleration. It varies linearly with the speed V and is perpendicular to the velocity $\mathbf{V}$

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However, once the air is moving, it is subject to the deflecting effect of this term. This is why the atmospheric flow is *predominantly rotational* in character.

Exercises

(1) **Calculate the deflection of a golf ball travelling for 10 seconds at 10 m/s. Assume a latitude of 60°N, and make reasonable assumptions to simplify the problem.**

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(2) **Suppose the pressure at Cork is 1014 hPa and at Sligo is 1008 hPa. Take the distance between Cork and Sligo to be 330 km. Assume the isobars are east-west, and assume that Cork and Sligo are on the same meridian.**
Calculate the acceleration due to the pressure gradient (assume $\rho = 1.2\,\mathrm{kg\,m^{-3}}$).
What wind speed would give a Coriolis acceleration of the same magnitude (take $2\Omega\sin\phi = 10^{-4}\,\mathrm{s^{-1}}$)?

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[For further problems, see Holton, Chapter 1.]

Alternative Treatment of Rotation

Let (X, Y, Z) be the coordinates in a *fixed frame*, and (x, y, z) be those in a rotating frame. Suppose the rotating frame spins about the Z-axis with angular velocity Ω. The coordinates are related by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} .$$

If we differentiate with respect to time, we get

$$\begin{pmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} - \Omega y \\ \dot{y} + \Omega x \\ \dot{z} \end{pmatrix} .$$

If we differentiate once again, we get

$$\begin{pmatrix} \ddot{X} \\ \ddot{Y} \\ \ddot{Z} \end{pmatrix} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ddot{x} - 2\Omega\dot{y} - \Omega^2 x \\ \ddot{y} + 2\Omega\dot{x} - \Omega^2 y \\ \ddot{z} \end{pmatrix} .$$

The equations of motion in non-rotating coordinates are

$$\ddot{X} = F_X \,, \qquad \ddot{Y} = F_Y \,, \qquad \ddot{Z} = F_Z \,.$$

Substituting from the matrix equation, we get

$$\begin{aligned}
(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\cos\Omega t - (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\sin\Omega t &= F_X \\
(\ddot{x} - 2\Omega\dot{y} - \Omega^2 x)\sin\Omega t + (\ddot{y} + 2\Omega\dot{x} - \Omega^2 y)\cos\Omega t &= F_Y \\
\ddot{z} &= F_Z
\end{aligned}$$

Solving for the terms with $\ddot{x}$ and $\ddot{y}$, we get

$$\begin{aligned}
\ddot{x} - 2\Omega\dot{y} - \Omega^2 x &= \cos\Omega t F_X + \sin\Omega t F_Y \equiv F_x \\
\ddot{y} + 2\Omega\dot{x} - \Omega^2 y &= \cos\Omega t F_Y - \sin\Omega t F_X \equiv F_y \\
\ddot{z} &= F_Z \equiv F_z
\end{aligned}$$

Thus, the rotation introduces additional terms:
The terms $-2\Omega\dot{y}$ and $2\Omega\dot{x}$ are the *Coriolis acceleration*.
The terms $\Omega^2 x$ and $\Omega^2 y$ are the *centrifugal acceleration*.