M.Sc. in Computational Science

Fundamentals of Atmospheric Modelling

Peter Lynch, Met Éireann

Mathematical Computation Laboratory (Opp. Room 30)
Dept. of Maths. Physics, UCD, Belfield.
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Lecture 10

Rossby Wave Packets

Introduction

- ***First, we consider wave interactions, and introduce the concept of group velocity.***
- ***Then we define Rossby wave packets and study their behaviour.***
- ***Finally, we illustrate the importance of group velocity for Rossby waves, using real atmospheric data.***

Interference of Two Waves

The simplest case to study is the superposition of two waves. We assume the two components have equal amplitudes and *approximately the same wavenumbers and frequencies:*

$$\psi(x,t) = \cos(k_1 x - \omega_1 t) + \cos(k_2 x - \omega_2 t)\,.$$

The components move with respective phase speeds

$$c_1 = \omega_1/k_1 \qquad \text{and} \qquad c_2 = \omega_2/k_2\,.$$

By elementary trigonometry, ψ may be written

$$\psi(x,t) = 2\cos\left(\frac{k_1-k_2}{2}x - \frac{\omega_1-\omega_2}{2}t\right)\cdot\cos\left(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t\right)\,.$$

We write the mean values and the differences

$$\bar{k} = (k_1+k_2)/2 \qquad \text{and} \qquad \bar{\omega} = (\omega_1+\omega_2)/2$$
$$\Delta k = (k_1-k_2)/2 \qquad \text{and} \qquad \Delta\omega = (\omega_1-\omega_2)/2\,.$$

Then the wave combination is

$$\psi(x,t) = 2\cos(\Delta k\cdot x - \Delta\omega\cdot t)\cdot\cos\left(\bar{k}x - \bar{\omega}t\right)\,.$$

Again:

$$\begin{aligned}\psi(x,t) &= 2\cos(\Delta k\cdot x - \Delta\omega\cdot t)\cdot\cos\left(\bar{k}x - \bar{\omega}t\right)\\ &= 2\cos\left[\Delta k\left(x - \frac{\Delta\omega}{\Delta k}\cdot t\right)\right]\cdot\cos\left[\bar{k}\left(x - \frac{\bar{\omega}}{\bar{k}}t\right)\right].\end{aligned}$$

The second term here represents a wave with wavenumber $\bar{k}$ moving with phase speed

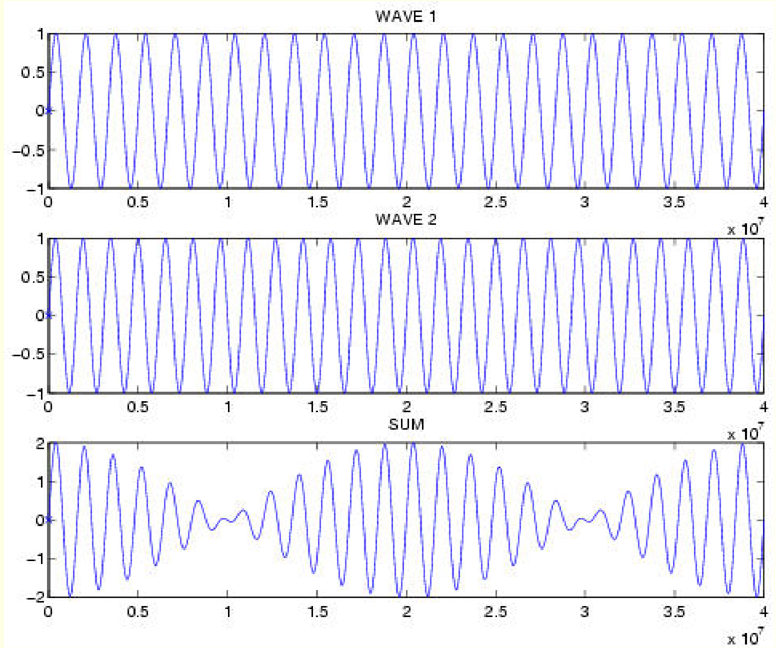
$$\bar{c} = \bar{\omega}/\bar{k}$$

which is close to the phase speeds of the two components.

The first term is slowly varying in space: it has wavenumber Δk and frequency $\Delta\omega$, and it moves with a speed c_g, called the group velocity,

$$c_g = \frac{\Delta\omega}{\Delta k}.$$

The group velocity may be radically different from the phase velocity $\bar{c}$, and of the opposite sign!



Two wave components of approximately equal wavelength. The envelope amplitude of the sum is clear.

The MATLAB program `gv1.m` shows the evolution of this waveform in time.

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The group velocity for a pair of waves was defined

$$c_g = \frac{\Delta\omega}{\Delta k}.$$

More generally, there is a *dispersion relation*

$$\omega = \omega(k),$$

and the group velocity is defined by

$$c_g = \frac{\partial\omega}{\partial k}.$$

Group Velocity of Rossby Waves

We consider only a simple case here. A much more detailed discussion may be found in Pedlosky (2003).

For nondivergent quasigeostrophic flow on a beta plane of a wave which is independent of the y-coordinate, the Rossby phase speed is

$$c = \bar{u} - \frac{\beta}{k^2}.$$

Here $\bar{u}$ is the mean zonal flow.

Let us compute the *group velocity*:

$$c_g = \frac{\partial\omega}{\partial k} = \frac{d(kc)}{dk} = \bar{u} + \frac{\beta}{k^2}.$$

We have the surprising result that the group velocity is directed towards the *east* (relative to the mean flow) whereas the phase velocity is towards the *west*.

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More generally, a Rossby wave may be travelling in a direction other than westward. If we assume

$$\psi = \psi_0 \exp[i(kx + \ell y - \omega t)]$$

the dispersion relation is

$$\omega = k\bar{u} - \frac{k\beta}{k^2 + \ell^2}\,.$$

and the phase speed for wavenumber k is thus

$$c(k) = \bar{u} - \frac{\beta}{k^2 + \ell^2}\,.$$

The mean flow $\bar{u}$ simply transports wave patterns eastward (for $\bar{u} > 0$) at a constant speed, so we will ignore this effect by assuming $\bar{u} = 0$.
The components of group velocity in the x and y directions are:

$$\begin{aligned} c_{gx} &= \frac{d\omega}{dk} = + \left(\frac{k^2 - \ell^2}{k^2 + \ell^2}\right)\frac{\beta}{k^2 + \ell^2} \\ c_{gy} &= \frac{d\omega}{d\ell} = + \left(\frac{2k\ell}{k^2 + \ell^2}\right)\frac{\beta}{k^2 + \ell^2} \end{aligned}$$

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The group velocity in the x-direction may be eastward or westward, depending on the sign of $k^2 - \ell^2$: for waves which are large-scale in x (small k) c_{gx} is negative; for waves which are small-scale (large k) it is positive.
The group speed in the y-direction depends on the sign of $k\ell$. However, the phase speed is $c_y = \omega/\ell$, so the ratio is

$$\frac{c_{gy}}{c_y} = -\frac{2\ell^2}{k^2 + \ell^2} < 0\,.$$

Thus the group velocity in the y-direction is in the opposite sense to the phase velocity.
Exercise: Plot the phase and group speeds as functions of the wavenumbers k and ℓ.

Extraction of the Envelope

The envelope of a wave packet may be extracted using ideas based on the Hilbert transform. For full details, see Bracewell (1978, pp. 267–272). Several appications of this technique are presented in Zimin, *et al.*, (2003).
Let $\psi(\lambda)$ be a function on a periodic domain $0 \le \lambda < 2\pi$. We perform the following operations in sequence:

- Compute the Fourier coefficients: $\hat{\psi}_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\lambda}\psi(\lambda)\,d\lambda$.
- Set the coefficients to zero for negative index: $\tilde{\psi}_k = H_k\hat{\psi}_k$ where H_k is the Heaviside sequence.
- Compute the inverse transform: $\Psi(\lambda) = \sum_{k=-\infty}^{k=\infty}\tilde{\psi}_k e^{ik\lambda}$.
- Double and take the absolute value: $A(\lambda) = 2|\Psi(\lambda)|$.

In words, we calculate the Fourier series, throw away the negative frequencies, invert, double and take the absolute value.

A *simple example* illustrates the technique. Suppose $\psi(\lambda) = A\cos n\lambda$. There are just two nonvanishing terms in the fourier series: $A\cos n\lambda = \frac{1}{2}A[\exp(n\lambda) + \exp(-n\lambda)]$. Elimination of the negative frequency part leaves $\frac{1}{2}A\exp(n\lambda)$ and twice the absolute value of this is A, as expected.
The generalization for a function $\psi(x)$ which is not periodic is straightforward: the Fourier series is replaced by the Fourier transform:

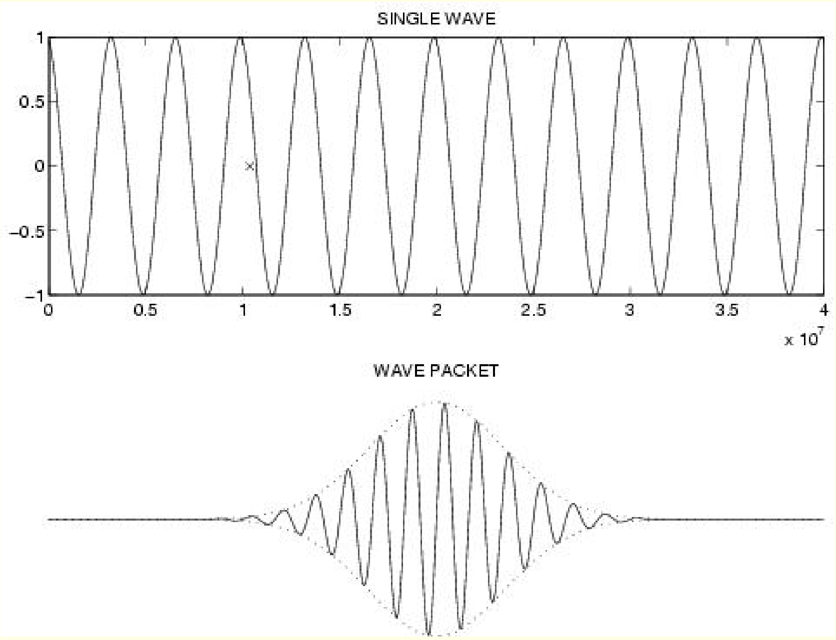
- Compute the Fourier transform: $\hat{\psi}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega x}\psi(x)\,dx$.
- Set the transform to zero for negative ω: $\tilde{\psi}(\omega) = H(\omega)\hat{\psi}(\omega)$ where $H(\omega)$ is the Heaviside function.
- Compute the inverse transform: $\Psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega x}\tilde{\psi}(\omega)\,d\omega$.
- Double and take the absolute value: $A(x) = 2|\Psi(x)|$.

The theoretical explanation of the envelope extraction method is given in Bracewell (*loc. cit.*).
The envelope extraction may be combined with low-pass or band-pass filtering by replacing the Heaviside function by a suitable masking function, for example

$$M(\omega) = \begin{cases} 1, & \omega_L \le \omega \le \omega_H \\ 0, & \text{otherwise} \end{cases}$$

which eliminates all components except in the frequency band $[\omega_L, \omega_H]$.

Gaussian Wave-packet



Suppose that we may express the streamfunction at the initial time $t = 0$ as

$$\psi(x,0) = A\exp[-\tfrac{1}{2}x^2/\sigma_0^2]\exp(ik_0x)\,,$$

that is, as a rapidly varying wave function whose amplitude envelope varies slowly with x.
A straightforward application of Fourier's Theorem allows us to write this as

$$\psi(x,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)\hat{\psi}(k,0)\,dk$$

where the spectral transform is given by

$$\hat{\psi}(k,0) = \int_{-\infty}^{\infty}\exp(-ikx)\psi(k,0)\,dk = \sqrt{2\pi}\sigma_0 A\exp[-\tfrac{1}{2}\sigma_0^2(k-k_0)^2]\,.$$

Assume that the mode with wavenumber k has frequency $\omega(k)$, given by the Rossby wave dispersion relation. Then the phase velocity is

$$c = \bar{u} - \frac{\beta}{k^2}\,. \qquad (1)$$

The group velocity is

$$c_g = \frac{d(kc)}{dk} = \bar{u} + \frac{\beta}{k^2}\,. \qquad (2)$$

We suppose that the governing equation for $\psi(x,t)$ is linear. Then each Fourier component will evolve independently of the others. So the solution may be written

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{\psi}(k,0)\exp[i(kx-\omega t)]\,dk\,. \qquad (3)$$

For large σ, the transform $\hat{\psi}$ is concentrated near $k = k_0$ and we can approximate the frequency ω using the Taylor series

$$\omega(k) \approx \omega(k_0) + \left[\frac{d\omega}{dk}\right]_{k_0}(k-k_0) + \frac{1}{2}\left[\frac{d^2\omega}{dk^2}\right]_{k_0}(k-k_0)^2\,.$$

or, more briefly, with obvious notation,

$$\omega \approx \omega_0 + \omega_0'(k-k_0) + \tfrac{1}{2}\omega_0''(k-k_0)^2\,.$$

Substituting this into (??) and evaluating the integral (about a page of calculus) we get

$$\psi(x,t) = \left(\frac{A\sigma_0}{\sqrt{\sigma_0^2 + i\omega_0'' t}} \right) \exp\left[-\frac{(x-\omega_0' t)^2}{2(\sigma_0^2 + i\omega_0'' t)} \right] \exp[i(k_0 x - \omega_0 t)] \,. \tag{4}$$

This solution has several points of interest. The first term shows that, for large time, the amplitude decreases as $t^{-1/2}$. The second term is the envelope, which we examine presently. The last term represents an oscillation with wavenumber k_0 and frequency ω_0, which travels with phase speed $c_0 = \omega_0/k_0$. The middle term on the right of (??) may be written

$$\exp\left[-\frac{(x-\omega_0' t)^2}{2(\sigma_0^2 + i\omega_0'' t)} \right] = \exp\left[-\frac{(x-\omega_0' t)^2}{2\sigma_0^2(1+\tau^2)} \right] \exp\left[i \left(\frac{\tau(x-\omega_0' t)^2}{2\sigma_0^2(1+\tau^2)} \right) \right] ,$$

where $\tau = (\omega_0''/\sigma_0^2)t$ is re-scaled time.

The first component is a Gaussian envelope, centered at $x = \omega_0' t$, whose width is given by

$$\sigma^2 = \sigma_0^2(1+\tau^2)$$

so it moves with the group velocity $c_g = \omega_0'$ and spreads as time increases. The second term is a chirp-function: its local wavenumber is zero at $x = \omega_0' t$ and increases linearly with distance from this point. The factor $\tau/(1+\tau^2)$ vanishes at $t = 0$ and for large time, reaching its maximum at $\tau = 1$.

The full solution is now written as a product of four components:

$$\psi(x,t) = \underbrace{\left(\frac{A\sigma_0}{\sqrt{\sigma_0^2 + i\omega_0'' t}} \right)}_{Amplitude} \underbrace{\exp\left[-\frac{\xi^2}{2\sigma^2} \right]}_{Gaussian} \underbrace{\exp\left[i \left(\frac{\tau\xi^2}{2\sigma^2} \right) \right]}_{Chirp} \underbrace{\exp[ik_0(x - c_0 t)]}_{Wave} ,$$

where $\xi = x - \omega_0' t$.

The properties of the solution

$$\psi(x,t) = \underbrace{\left(\frac{A\sigma_0}{\sqrt{\sigma_0^2 + i\omega_0'' t}} \right)}_{Amplitude} \underbrace{\exp\left[-\frac{\xi^2}{2\sigma^2} \right]}_{Gaussian} \underbrace{\exp\left[i \left(\frac{\tau\xi^2}{2\sigma^2} \right) \right]}_{Chirp} \underbrace{\exp[ik_0(x - c_0 t)]}_{Wave} ,$$

may be summarised as follows

- Individual wave crests move with the phase velocity c_0.
- The overall amplitude decays as $O(t^{-1/2})$.
- The envelope moves with the group velocity $c_g = \omega_0'$.
- The spread of the envelope grows as $\sigma^2 = \sigma_0^2(1+\tau^2)$.
- What to say about the chirp part?

References

Bracewell, R, 1978: *The Fourier Transform and its Applications.* Second Edn., McGraw-Hill, New York. 444pp.

Zimin, Aleksey V., Szunyogh, Istvan, Patil, D. J., Hunt, Brian R., Ott, Edward. 2003: Extracting Envelopes of Rossby Wave Packets. *Monthly Weather Review*, Vol. 131, 1011–1017.