M.Sc. in Computational Science

Fundamentals of Atmospheric Modelling

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Lecture 4

The Shallow Water Equations

Physical Laws of the Atmosphere

NEWTON'S LAWS OF MOTION
Describe how the change of velocity is determined by the pressure gradient, Coriolis force and friction
GAS LAW, or EQUATION OF STATE
Relates the pressure, temperature and density
CONSERVATION OF MASS
Continuity Equation: air neither created nor distroyed
CONSERVATION OF WATER
Continuity Equation for water (liquid, solid and gas)
CONSERVATION OF ENERGY
Thermodynamic Equation determines changes of temperature due to heating, compression, etc.

Seven equations; seven variables (u, v, w, ρ, p, T, q).

The Primitive Equations

$$\frac{du}{dt} - \left(f + \frac{u\tan\phi}{a}\right)v + \frac{1}{\rho}\frac{\partial p}{\partial x} + F_x = 0$$
$$\frac{dv}{dt} + \left(f + \frac{u\tan\phi}{a}\right)u + \frac{1}{\rho}\frac{\partial p}{\partial y} + F_y = 0$$
$$\frac{\partial p}{\partial z} + g\rho = 0$$
$$p = R\rho T$$
$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0$$
$$\frac{\partial \rho_w}{\partial t} + \nabla\cdot\rho_w\mathbf{V} = [\text{Sources} - \text{Sinks}]$$
$$\frac{dT}{dt} + (\gamma-1)T\nabla\cdot\mathbf{V} = \frac{\dot{Q}}{c_p}$$

These equations are suitable for a forecast or climate model. For *understanding the dynamics*, we need to simplify them.

Check: Look at the above equations. How many have we got so far?

Equations in Component Form

The equations of motion in the rotating frame are

$$\frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega} \times \mathbf{V} + \frac{1}{\rho}\nabla p - \mathbf{g} = 0 .$$

Next we split the equation of motion into components. We introduce *local cartesian coordinates* (x, y, z).

$$\begin{aligned}
\mathbf{V} &= (u, v, w) \\
d\mathbf{V}/dt &= (du/dt, dv/dt, dw/dt) \\
\mathbf{g} &= (0, 0, -g) \\
\nabla p &= (\partial p/\partial x, \partial p/\partial y, \partial p/\partial z) \\
\mathbf{\Omega} &= (0, \Omega\cos\phi, \Omega\sin\phi) \\
2\mathbf{\Omega} \times \mathbf{V} &= (2w\Omega\cos\phi - 2v\Omega\sin\phi, 2u\Omega\sin\phi, -2u\Omega\cos\phi)
\end{aligned}$$

Note: Certain trigonometric terms have been omitted from the acceleration.

We assume w is much smaller than u and v, and we can neglect the term $2w\Omega\cos\phi$ in the Coriolis force.

The variables x and y are *distances eastward and northward on the globe.* We will ignore the effects of sphericity except in the Coriolis term (see below). Then (x, y, z) are equivalent to *Cartesian coordinates.*

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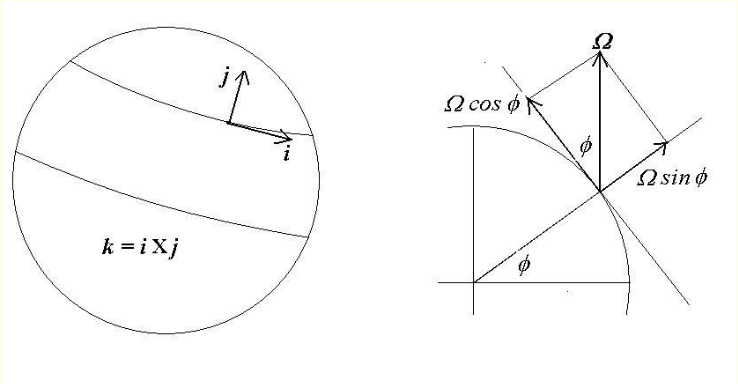

Coordinate System Components of $\mathbf{\Omega}$

The horizontal components of the equation of motion may now be written:

$$\frac{du}{dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{dv}{dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$

where $f = 2\Omega\sin\phi$ is called the Coriolis parameter.

The vertical component of the equation of motion is

$$\frac{dw}{dt} - 2\Omega u\cos\phi + \frac{1}{\rho}\frac{\partial p}{\partial z} + g = 0 .$$

In the absence of motion, this reduces to the hydrostatic equation,

$$\frac{\partial p}{\partial z} + \rho g = 0 ,$$

expressing a balance between the vertical pressure gradient and gravity.

Note that horizontal component of $\mathbf{\Omega}$ no longer enters the equations, and we write $2\mathbf{\Omega} = f\mathbf{k}$.

For large-scale motions, the *hydrostatic equation* is an excellent approximation to the full vertical equation, and we adopt it from now on.

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As already remarked, the majority of numerical models assume hydrostatic balance. However, as the grid-scales are refined below about 5 km, this assumption becomes les justified. Thus, non-hydrostatic models have been gaining popularity in recent years.

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The equations will now be further simplified, and we will derive the system known as the *Shallow Water Equations.* For a review, read Pedlosky, §§3.1, 3.2 and 3.3.

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As a consequence of spherical geometry, there are additional small terms involving trigonometric functions. These will be omitted, as the resulting errors are small.

The Beta-plane approximation

We restate the momentum and continuity equations:

$$\frac{du}{dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$
$$\frac{dv}{dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial z} + \rho g = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

These are four equations for four dependent variables.

The β-plane approximation: We neglect sphericity except in the Coriolis parameter $f = 2\Omega\sin\phi$.

Thus, the *geometric terms* arising from the sphericity of the earth are omitted. Only the *dynamical* effect, the variation of the vertical component of $\mathbf{\Omega}$, is included.

Eliminating the Vertical Velocity

We will now eliminate the vertical velocity w, thereby reducing the system to three equations for three variables.

Let $h(x, y)$ be the height of the free surface at point (x, y). We integrate the hydrostatic equation between z and h:

$$\int_z^h \frac{\partial p}{\partial z}\,dz + \int_z^h \rho g\,dz = 0 \qquad \text{or} \qquad p(z) - p(h) = \rho g(h - z)$$

(recall that density ρ is assumed to be constant). Thus, the pressure is given by the ***weight of fluid above a point.***

We assume that the pressure $p_0 = p(h)$ at the top of the fluid layer is a constant. Then p_0 does not enter the dynamics:

$$p(z) = p_0 + \rho g(h - z)\,, \ \ \textit{for each point}\ (x, y)\,.$$

The gradient of pressure may now be related to the slope of the free surface:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = g\frac{\partial h}{\partial x}\,; \qquad \frac{1}{\rho}\frac{\partial p}{\partial y} = g\frac{\partial h}{\partial y}\,.$$

In vector notation, this is

$$\frac{1}{\rho}\nabla p = g\nabla h$$

We can now write the (horizontal) equations of motion as

$$\frac{d\mathbf{V}}{dt} + 2\mathbf{\Omega}\times\mathbf{V} + \nabla\Phi = 0\,.$$

where $\Phi = gh$ is the ***geopotential.***

N.B. From now on, $\mathbf{V}$ ***denotes the horizontal velocity*** $(u, v, 0)$.

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- We next assume that, ***at some initial time***, the velocity (u, v) is ***independent of depth***, z.
- Examining the equations for u and v, we note that the accelerations do not vary with depth z.
- Therefore, the ***velocity will remain independent of depth*** for all time.

Integrated Continuity Equation

Next, we integrate the continuity equation through the full depth of the fluid. Since u and v are constant with z,

$$\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz = h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\,.$$

The third term of the continuity equation integrates to

$$\int_0^h \left(\frac{\partial w}{\partial z}\right) dz = w(h) - w(0) = \frac{dh}{dt}\,.$$

Here we have assumed that the bottom boundary is flat, so that the vertical velocity there vanishes: $w(0) = 0$.

Combining terms, the integrated continuity equation is:

$$\frac{dh}{dt} + h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

We are now in a position to write down the full set of ***Shallow Water Equations:***

$$\frac{du}{dt} - fv + \frac{\partial \Phi}{\partial x} = 0 \qquad (1)$$

$$\frac{dv}{dt} + fu + \frac{\partial \Phi}{\partial y} = 0 \qquad (2)$$

$$\frac{d\Phi}{dt} + \Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \qquad (3)$$

This is a set of three equations for (u, v, Φ). The independent variables are (x, y, t). *The vertical velocity does not appear.*

The total time derivative is now given by:

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}.$$

This is a *nonlinear operator*. The Shallow Water Equations are, in general, impossible to solve analytically.

Exercise: Vertical Velocity

Show that the vertical velocity is a linear function of depth.

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Solution:

We define the *horizontal divergence:*

$$\nabla_{\mathrm{H}} \cdot \mathbf{V} = \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

we note that $\nabla_{\mathrm{H}} \cdot \mathbf{V}$ is independent of z.

The continuity equation may be written

$$\nabla_{\mathrm{H}} \cdot \mathbf{V} + \frac{\partial w}{\partial z} = 0$$

We integrate this between 0 and z, noting that the first term is independent of z:

$$(\nabla_{\mathrm{H}} \cdot \mathbf{V})z + w(z) - w(0) = 0.$$

But we assume a flat bottom, so $w(0) = 0$. Therefore,

$$w(z) = (\nabla_{\mathrm{H}} \cdot \mathbf{V})z$$

which increases linearly with z.

Scale Analysis.

We now introduce characteristic scales for the independent and dependent variables, and non-dimensionalize the equations. This enables us to examine the *relative sizes of the terms.*

Let L and V be typical length and velocity scales. For example, we replace u by $\mathsf{V}u^*$, so that u^* is of order unity. Thus,

$$\frac{\partial u}{\partial x} = \left(\frac{\mathsf{V}}{\mathsf{L}}\right)\frac{\partial u^*}{\partial x^*}, \qquad \text{with} \qquad \frac{\partial u^*}{\partial x^*} = O(1),$$

and similarly for the other terms.

We assume an advective time scale T:

$$\frac{\partial}{\partial t} \sim \mathbf{V} \cdot \nabla; \qquad \frac{1}{\mathsf{T}} = \frac{\mathsf{V}}{\mathsf{L}}; \qquad \mathsf{T} = \frac{\mathsf{L}}{\mathsf{V}}.$$

Also, $f = 2\Omega \sin\phi \sim 2\Omega$, provided we are not too close to the equator.

Typical values for the atmosphere are $\mathsf{L} = 1000\,\mathrm{km}$ for the horizontal scale of synoptic weather systems, and $\mathsf{V} = 10\,\mathrm{m\,s^{-1}}$ for the wind speed.

The *mean depth*, H, over an area A is:

$$\mathsf{H} = \frac{1}{A}\iint_A h(x, y)\,dxdy.$$

This is assumed to be equal to the scale-height of the atmosphere. Thus, we choose $\mathsf{H} = 10\,\mathrm{km}$, about the depth of the troposphere.

Just as it is not the absolute pressure which determines the dynamics, but pressure gradients, similarly the dynamically important quantity is the *deviation* of depth from the mean: $h' = h - \mathsf{H}$.

We denote this vertical scale by D. Thus

$$h = \mathsf{H} + h' = \mathsf{H} + \mathsf{D}h'^* \quad \text{with} \quad h'^* = O(1).$$

But what value should we choose for D?

What value should we choose for D?

The typical value of surface pressure is $p_0 = 10^5$ Pa.

However, it is the *deviation* from p_0 that is important. The characteristic variation of surface pressure is about 10 hPa.

We set the scale of pressure variation as $\mathsf{P} = 10^3$ Pa. This gives a scale for D:

$$\frac{1}{\rho}\nabla p = g\nabla h \qquad \Longrightarrow \qquad \frac{\mathsf{P}}{\rho\mathsf{L}} = \frac{g\mathsf{D}}{\mathsf{L}}$$

Thus, the scale for depth variation is

$$\mathsf{D} = \frac{\mathsf{P}}{\rho g} = \frac{10^3}{1 \cdot 10} = 10^2\,\mathrm{m}\,.$$

When we examine the sizes of the terms in the momentum equation, this will be seen to be appropriate.

We now define the scale values:

$$\mathsf{L} = 10^6\,\mathrm{m}\,; \quad \mathsf{V} = 10\,\mathrm{m\,s^{-1}}\,; \quad \mathsf{T} = (\mathsf{L}/\mathsf{V}) = 10^5\,\mathrm{s} \approx 1\,\mathrm{day}$$

$$\mathsf{H} = 10^4\,\mathrm{m}\,; \quad \mathsf{D} = 10^2\,\mathrm{m}\,; \quad f \approx 10^{-4}\,\mathrm{s^{-1}}\,; \quad g = 10\,\mathrm{m\,s^{-2}}\,.$$

The momentum equations may be written in vector form:

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0\,.$$

The magnitudes of the three terms are as follows:

$$\frac{d\mathbf{V}}{dt} \sim \frac{\mathsf{V}^2}{\mathsf{L}}\,; \qquad f\mathbf{k}\times\mathbf{V} \sim 2\Omega\mathsf{V}\,; \qquad \nabla\Phi \sim \frac{g\mathsf{D}}{\mathsf{L}}\,.$$

$$10^{-4} \qquad\qquad 10^{-3} \qquad\qquad 10^{-3}\,.$$

The size of each term (in units $\mathrm{m\,s^{-2}}$) is indicated.

We note that the acceleration is an order of magnitude smaller than the remaining terms. The Coriolis term and the pressure gradient term are of *the same order of magnitude*. This is called ***Geostrophic Balance***.

The Rossby Number.

The ratio of the acceleration to the Coriolis term is

$$\frac{\text{Acceleration}}{\text{Coriolis term}} = \left|\frac{d\mathbf{V}/dt}{f\mathbf{k}\times\mathbf{V}}\right| \sim \frac{\mathsf{V}^2/\mathsf{L}}{2\Omega\mathsf{V}} = \frac{\mathsf{V}}{2\Omega\mathsf{L}}\,.$$

This ratio is called the ***Rossby Number***, denoted Ro:

$$\mathsf{Ro} \equiv \frac{\mathsf{V}}{2\Omega\mathsf{L}}\,.$$

It is a *fundamental number in geophysical fluid dynamics.*

Substituting the chosen values for V, f and L, we get

$$\mathsf{Ro} = \frac{10}{10^{-4}\cdot 10^6} = 10^{-1} \ll 1\,.$$

The smallness of this non-dimensional parameter allows us to make various approximations and perturbation analyses.

Aside: The Froude Number

There is another non-dimensional number which depends on the depth scale D but not on the Coriolis parameter. The *Froude Number* is the ratio of the fluid flow to the speed of gravity waves:

$$\begin{bmatrix}\text{Froude}\\ \text{Number}\end{bmatrix} = \frac{\text{Flow Velocity}}{\text{Gravity Wave Speed}}$$

We will show later that the characteristic speed of gravity waves is $\sqrt{g\mathsf{H}}$, so the Froude number is

$$\mathsf{Fr} = \frac{\mathsf{V}}{\sqrt{g\mathsf{H}}}$$

With the characteristic scale values already chosen, we have

$$\mathsf{Fr} = \frac{10\,\mathrm{m\,s^{-1}}}{\sqrt{10\,\mathrm{m\,s^{-2}}\cdot 10^4\,\mathrm{m}}} \approx \frac{1}{30}$$

Thus, for large-scale geophysical flows, both the Rossby number and the Froude number are *small*:

$$\mathsf{Ro} \ll 1 \qquad \mathsf{Fr} \ll 1\,.$$

The Geostrophic Wind

The momentum equation

$$\frac{d\mathbf{V}}{dt} + f\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0$$

is of the form $A + B + C = 0$. If assume that one term is smaller than the other two, we get various special cases. The most important of these is *geostrophic balance.*

We saw that the acceleration term is relatively small. Omitting it, we get a diagnostic relationship called geostrophic balance:

$$f\mathbf{k}\times\mathbf{V} + \nabla\Phi = 0\,; \qquad \mathbf{V} = \frac{1}{f}\mathbf{k}\times\nabla\Phi\,.$$

In terms of the pressure field, the geostrophic wind is

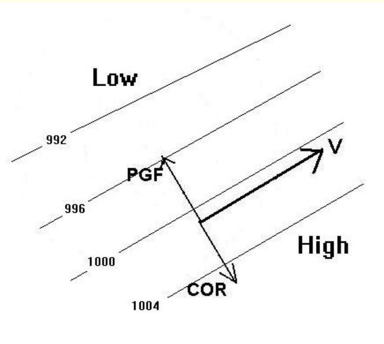
$$f\mathbf{k}\times\mathbf{V} + \frac{1}{\rho}\nabla p = 0\,; \qquad \mathbf{V} = \frac{1}{\rho f}\mathbf{k}\times\nabla p\,.$$

So, *the wind field is determined by the pressure field.*

In terms of coordinates, the geostrophic wind is

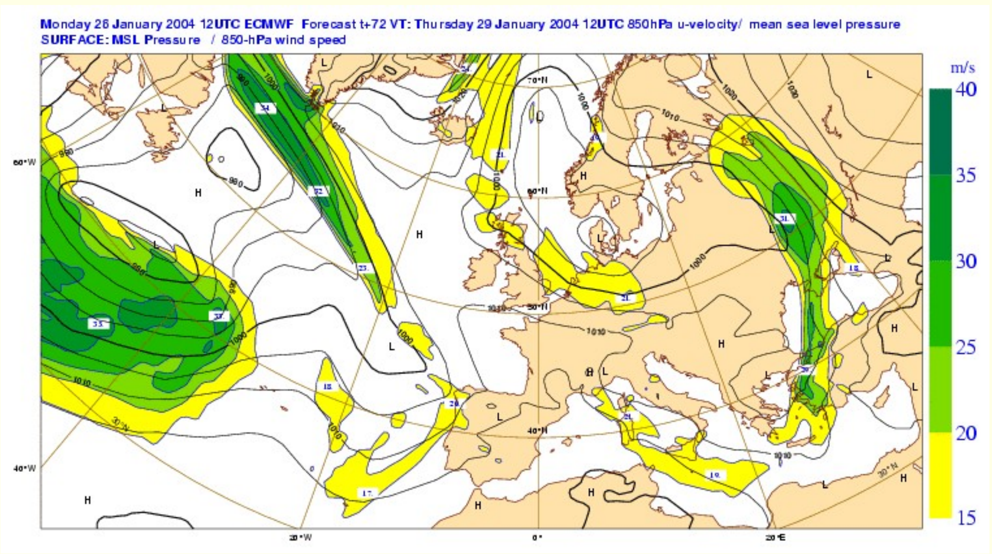
$$u = -\frac{1}{f\rho}\frac{\partial p}{\partial y}\,, \qquad v = +\frac{1}{f\rho}\frac{\partial p}{\partial x}\,.$$

For geostrophic balance, the flow is perpendicular to the gradient of presure. The existence of a fluid flow along the isobars, rather than towards areas of low pressure, is characteristic of geophysical flows, and in *dramatic contrast* to the situation for fluid flow in a non-rotating framework.



ECMWF Forecast Chart

72 hour forecast of sea level pressure and 850 hPa wind



Web Exercise

Download and study a selection of weather charts. Find charts with both pressure and winds. Study the relationship between the wind and pressure fields.

Use stuff from Met Eireann web-site
`http://www.met.ie`

Use stuff from ECMWF web-site
`http://www.ecmwf.int`

Search for other sites
(There are hundreds)