M.Sc. in Computational Science

Fundamentals of Atmospheric Modelling

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Lecture 2

The Continuity Equation

Geophysical Fluid Dynamics

Geophysical Fluid Dynamics (GFD) is the study of the dynamics of the fluid systems of the earth and planets. The principal fluid systems in which we are interested are the atmosphere and the oceans.

Inland waters such as lakes and rivers, and glaciers and lava systems as well as ground water and the molten outer core of the earth could technically be included in GFD, but will not be considered further.

The basis of GFD lies in the principles of conservation of momentum, mass and energy. These are expressed mathematically in Newton's equations of motion for a continuous medium, the equation of continuity and the thermodynamic energy equation.

We will first express the equations of motion in an inertial framework

$$\mathbf{F} = m\mathbf{a}\,.$$

Then they will be transformed to a non-Newtonian coordinate system which is fixed with respect to the earth, and rotating with it.

But first we must consider some preliminary issues.

Two Ways to Describe Fluid Flow

- ***Eulerian*: Stay put and watch the flow**
- ***Lagrangian*: Drift along, see where you go.**

The independent variables are the space and time coordinates, $\mathbf{r} = (x, y, z)$ and t.

The dependent variables are the velocity, pressure, density and temperature, $\mathbf{V} = (u, v, w)$, p, ρ and T.

Further variables are needed for a fuller treatment, e.g. humidity q in the atmosphere and salinity s in the ocean.

Each variable is a function of both position and time.

For example,

$$p = p(x, y, z, t)$$

We must consider variations with respect to space and time.

$$p = p(x, y, z, t)\,.$$

- ***Eulerian*: Stay put and watch the flow**

We denote the change of pressure with time at a fixed point by the Eulerian (or partial) derivative:

$$\frac{\partial p}{\partial t} \qquad x, y \text{ and } z \text{ fixed.}$$

- ***Lagrangian*: Drift along, see where you go.**

We denote the change of pressure with time following the flow by the Lagrangian (or material or total) derivative:

$$\frac{dp}{dt} \qquad \text{parcel of fluid fixed.}$$

Digression Partial Derivatives

Partial derivatives were first introduced by the French mathematician Jean Le Rond d'Alembert (1717–1783) in connection with his meteorological studies.

Euler and Lagrange Derivatives

- ***Eulerian or Local Change***

Stand on a bridge, hang a thermometer into the stream. The temperature you measure is at a fixed location. The change in temperature is local, given by the partial time derivative

$$\frac{\partial T}{\partial t} \qquad \text{Change at a fixed location.}$$

- ***Lagrangian or Material Change***

Float on a raft, hang a thermometer into the ocean. The temperature you measure is at a point moving with the current. The change in temperature is given by the total time derivative

$$\frac{dT}{dt} \qquad \text{Change for a material parcel.}$$

Connection: $\partial p/\partial t \Longleftrightarrow dp/dt$

The pressure is a function of both space and time:

$$p = p(x(t), y(t), z(t), t)\,.$$

The total variation, following the flow, is given by the chain rule:

$$\begin{aligned}\frac{dp}{dt} &= \frac{\partial p}{\partial t} + \frac{\partial p}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial p}{\partial z}\cdot\frac{dz}{dt} \\ &= \frac{\partial p}{\partial t} + u\cdot\frac{\partial p}{\partial x} + v\cdot\frac{\partial p}{\partial y} + w\cdot\frac{\partial p}{\partial z} \\ &= \frac{\partial p}{\partial t} + \mathbf{V}\cdot\nabla p\,.\end{aligned}$$

This is true for all variables, so we have

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{V}\cdot\nabla(\ )\,.$$

Exercise: Local *v.* Material Change

Suppose the flow is purely in the x-direction, given by

$$u = a\sin(kx - \omega t)$$

where the amplitude a, wavenumber k and frequency ω are constants. We can also write u as

$$u = a\sin k(x - ct)$$

where $c = \omega/k$ is the phase speed of the wave.

Calculate the local and total time derivative of u.

How do $\partial u/\partial t$ and du/dt change if $u \longrightarrow 2u$?

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The Eulerian time derivative is a linear function of u:

$$\frac{\partial u}{\partial t} = \left(\frac{\partial a\sin(kx-\omega t)}{\partial t}\right)_{x\ constant} = -\omega\cdot a\cos(kx-\omega t)\,.$$

The Lagrangian time derivative is

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\omega\cdot a\cos(kx-\omega t) + u[k\cdot a\cos(kx-\omega t)]\,.$$

Note that du/dt is a *nonlinear* function of the amplitude a:

$$\frac{du}{dt} = \big[k\sin(kx-\omega t)\cos(kx-\omega t)\big]a^2 - \big[\omega\cos(kx-\omega t)\big]a\,.$$

Conservation of Mass

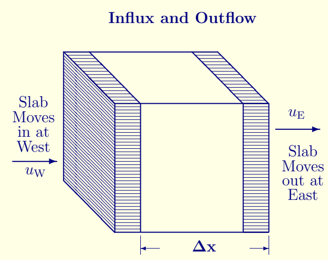
Air is neither created nor destroyed. Therefore, the total mass must remain constant. Moreover, the mass of an identifiable parcel of air must remain unchanged with time.

The mathematical expression of mass conservation is the Continuity Equation.

To illustrate the two methods of describing fluid flow, we will derive the continuity equation in both Eulerian and Lagrangian forms.

We must then show that the two forms are equivalent.

Eulerian Formulation



Consider a cubic region of dimensions $\Delta x = \Delta y = \Delta z$, fixed in space.

Air flows freely through the region.

The change of mass of the air in the cube must equal the net flux of mass into or out of the region.

For simplicity, consider flow in the x-direction. Let u_W be the x-component of velocity at the western face, and u_E be the x-component of velocity at the eastern face.

Total mass of air in the box (density $\times$ volume):

$$M = \rho \cdot \Delta x \Delta y \Delta z = \rho \mathcal{V}$$

Change of mass in time Δt (volume is fixed):

$$\Delta M = \frac{\partial M}{\partial t}\Delta t = \frac{\partial \rho}{\partial t}\Delta t \cdot \mathcal{V}\,.$$

Influx at western face (density $\times$ slab volume):

$$\rho_{\mathrm{W}}(u_{\mathrm{W}}\Delta t)\Delta y \Delta z = (\rho u)_{\mathrm{W}}\Delta t \cdot \Delta y \Delta z$$

Outflow at eastern face (density $\times$ slab volume):

$$\rho_{\mathrm{E}}(u_{\mathrm{E}}\Delta t)\Delta y \Delta z = (\rho u)_{\mathrm{E}}\Delta t \cdot \Delta y \Delta z$$

Net flow $\mathcal{F}$ into the box (influx $-$ outflow) in time Δt:

$$\mathcal{F} = [(\rho u)_{\mathrm{W}} - (\rho u)_{\mathrm{E}}]\Delta t \cdot \Delta y \Delta z = -\frac{(\rho u)_{\mathrm{E}} - (\rho u)_{\mathrm{W}}}{\Delta x}\Delta t \cdot \Delta x \Delta y \Delta z$$

But $\Delta M = \mathcal{F}$, so the quantities in red must be equal:

$$\frac{\partial \rho}{\partial t} = -\frac{(\rho u)_{\mathrm{E}} - (\rho u)_{\mathrm{W}}}{\Delta x} \approx -\frac{\partial(\rho u)}{\partial x}$$

Thus, for flow only in the x-direction we have

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u)}{\partial x}$$

However, there is also flow through the front and back faces, and through the top and bottom of the box.

Symmetry arguments lead us immediately to the result

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)$$

This may be written using the divergence operator as

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0\,.$$

This is the Eulerian form of the continuity equation. It is one of the fundamental equations of atmospheric dynamics.

Lagrangian Formulation

We consider a parcel of air, of mass M, contained in a cube.[1] But we allow the cube to move with the flow. The mass of the parcel does not change with time.

Total mass of air in the box (density $\times$ volume):

$$M = \rho \cdot \Delta x \Delta y \Delta z = \rho \mathcal{V}$$

Change of mass in time Δt must be zero:

$$\Delta M = \frac{dM}{dt}\Delta t = 0 \qquad \text{so} \qquad \frac{dM}{dt} = 0 \qquad \text{so} \qquad \frac{d\log M}{dt} = 0\,.$$

Since $\log M = \log\rho + \log\Delta x + \log\Delta y + \log\Delta z$, this means that

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left(\frac{1}{\Delta x}\frac{d\Delta x}{dt} + \frac{1}{\Delta y}\frac{d\Delta y}{dt} + \frac{1}{\Delta z}\frac{d\Delta z}{dt}\right) = 0 \qquad (*)$$

But now notice that $\Delta x = x_{\mathrm{E}} - x_{\mathrm{W}}$ so that

$$\frac{1}{\Delta x}\frac{d\Delta x}{dt} = \frac{1}{\Delta x}\frac{d(x_{\mathrm{E}} - x_{\mathrm{W}})}{dt} = \frac{1}{\Delta x}\left(\frac{dx_{\mathrm{E}}}{dt} - \frac{dx_{\mathrm{W}}}{dt}\right) = \frac{u_{\mathrm{E}} - u_{\mathrm{W}}}{\Delta x} \approx \frac{\partial u}{\partial x}$$

[1] The assumption that the parcel is initially cubic is purely for mathematical simplicity. We can relax it and consider an arbitrary parcel of mass M.

Substituting in (*) we get

$$\frac{1}{M}\frac{dM}{dt} = 0 = \frac{1}{\rho}\frac{d\rho}{dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

or, rearranging terms,

$$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0\,.$$

Using vector operators, this is

$$\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0\,.$$

This is the Lagrangian form of the continuity equation.

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We recall the Eulerian form, derived above:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0\,.$$

The two forms look different, but must be equivalent.

$$\underbrace{\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} = 0}_{\text{Lagrangian Form}} \qquad \underbrace{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0}_{\text{Eulerian Form}}$$

We recall the relationship between the time derivatives:

$$\frac{d(\ )}{dt} = \frac{\partial(\ )}{\partial t} + \mathbf{V}\cdot\nabla(\ )\,.$$

We also note the vector identity

$$\nabla\cdot\rho\mathbf{V} = \mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V}\,.$$

Substituting in the Lagrangian form, we get:

$$\begin{aligned}\frac{d\rho}{dt} + \rho\nabla\cdot\mathbf{V} &= \left(\frac{\partial\rho}{\partial t} + \mathbf{V}\cdot\nabla\rho\right) + \rho\nabla\cdot\mathbf{V}\\ &= \frac{\partial\rho}{\partial t} + \Big(\mathbf{V}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{V}\Big)\\ &= \frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0\,.\end{aligned}$$

Thus the equivalence of the two forms is established. QED

Incompressibility

For an incompressible fluid, the volume of a parcel remains unchanged. Thus, the material density is constant following the flow: $d\rho/dt = 0$. Thus, the continuity equation reduces to

$$\nabla\cdot\mathbf{V} = 0\,.$$

The assumption of incompressibility is a natural one for the ocean. For the atmosphere, it is less obviously reasonable. Indeed, many atmospheric phenomena depend on compressibility. However, the essential large scale dynamics can be successfully modelled by an incompressible fluid.

The benefit of assuming incompressibility is that we get a closed system without having to consider the thermodynamics explicitly. For compressible flow, we would have to have another equation for ρ, the thermodynamic equation. But this introduces the temperature T, and yet another equation, the equation of state, is required.

Exercise Open Windows

Consider a square frame of dimensions $1\,\mathrm{m}\times 1\,\mathrm{m}$. Suppose the wind blows through the frame with speed $10\,\mathrm{m\,s^{-1}}$. Compute the volume of air which flows through the frame in ten seconds.

Now imagine the frame is that of an open window in an otherwise closed room of dimensions $5\,\mathrm{m}\times 5\,\mathrm{m}\times 4\,\mathrm{m}$. If initially the pressure in the room equals the external pressure, by what proportion will the pressure increase in ten seconds, assuming that air continues to flow in at a constant rate?

Is the result physically reasonable? If not, discuss what important physical factors may have been neglected. Can you deduce a more reasonable value of the pressure increase.

Answer:

Area of the frame (window):

$$A = 1\,\mathrm{m}\times 1\,\mathrm{m} = 1\,\mathrm{m}^2\,.$$

Distance along wind in 10 seconds

$$d = V\times t = 10\,\mathrm{m\,s^{-1}}\times 10\,\mathrm{s} = 100\,\mathrm{m}\,.$$

Volume of air flowing through the frame:

$$d\times A = 100\,\mathrm{m}\times 1\,\mathrm{m}^2 = 100\,\mathrm{m}^3\,.$$

Thus, 100 cubic metres of air flow through the frame in ten seconds.

The volume of the room is

$$\mathcal{V} = 5\,\mathrm{m}\times 5\,\mathrm{m}\times 4\,\mathrm{m} = 100\,\mathrm{m}^3\,.$$

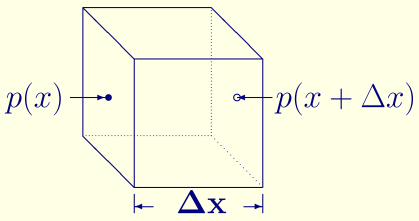
Thus, the air flowing in through the window in ten seconds equals the volume initially within the room. Since the volume of the room is fixed, the mass of air within it must double. Thus, the density must also double. If the temperature remains constant, the pressure will double too!

This result is surprising. We would not expect wind flowing through an open window to cause such huge change in pressure. What has been overlooked?

The Forces on a Parcel of Air

Pressure Force

Pressure on Box



Consider a cubic box of air, of dimension $\Delta x \times \Delta y \times \Delta z = \mathcal{V}$.

The pressure acts *normally* on each face of the cube.

Net force on left-hand face:

$$p(x) \cdot \Delta y \Delta z$$

Net force on right-hand face:

$$-p(x+\Delta x) \cdot \Delta y \Delta z$$

Total pressure force in the x-direction:

$$-\Big[p(x+\Delta x) - p(x)\Big] \cdot \Delta y \Delta z$$

Total pressure force in x-direction:

$$-\Big[p(x+\Delta x) - p(x)\Big] \cdot \Delta y \Delta z = -\left(\frac{p(x+\Delta x) - p(x)}{\Delta x}\right) \cdot \Delta x \Delta y \Delta z$$

But $\Delta x \Delta y \Delta z = \mathcal{V}$, so the force <u>per unit volume</u> is:

$$-\left(\frac{p(x+\Delta x) - p(x)}{\Delta x}\right) \approx -\frac{\partial p}{\partial x}$$

A parcel of mass m has volume $\mathcal{V} = m/\rho$, so a unit mass has volume $1/\rho$. The pressure force <u>per unit mass</u> in the x-direction is thus

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similar arguments apply in the y and z directions. So, the vector force <u>per unit mass</u> due to pressure is

$$\mathbf{F}_p = \left(-\frac{1}{\rho}\frac{\partial p}{\partial x}, -\frac{1}{\rho}\frac{\partial p}{\partial y}, -\frac{1}{\rho}\frac{\partial p}{\partial z}\right) = -\frac{1}{\rho}\nabla p\,.$$

This force acts in the direction of lower pressure.

Exercise Amazing Pressure

This problem is to give you an *impression* of the surprising strength of atmospheric pressure.

Consider a 20" television CRT screen, of width 16" and height 12".
Assume the atmospheric pressure is 10^5 Pa, and that there is a perfect vacuum within the tube.

- What is the force on the screen due to air pressure?
- How does this force compare to that of a fat man (100kg) standing on the screen: (a) much less; (b) similar; (c) much greater?

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Answer:

First, the force due to atmospheric pressure:

$$\begin{aligned} p &= 10^5\,\mathrm{Pa} = 10^5\,\mathrm{N\,m^{-2}} \\ A &= 16\text{''} \times 12\text{''} \approx 40\,\mathrm{cm} \times 30\,\mathrm{cm} = 0.40\,\mathrm{m} \times 0.3\,\mathrm{m} = 0.12\,\mathrm{m}^2 \\ F &= pA = 10^5 \times 0.12 = 12 \times 10^3\,\mathrm{N} = 12\,\mathrm{kN}\,. \end{aligned}$$

Next, the force due to the fat man:

$$\begin{aligned} m &= 10^2\,\mathrm{kg}\,, \quad g = 10\,\mathrm{m\,s^{-2}} \\ F &= mg = 10^2 \times 10 = 10^3\,\mathrm{N} = 1\,\mathrm{kN}\,. \end{aligned}$$

Conclusion:

$$\begin{pmatrix} \text{Air} \\ \text{Pressure} \end{pmatrix} \sim \begin{pmatrix} \text{12 Fat} \\ \text{Men} \end{pmatrix}$$

This should convince you that differences in pressure can result in significant forces. The pressure gradient force is dominant in atmospheric dynamics.

Exercise "The Spring of Air"

This expression was used by Robert Boyle for the tendency of air to resist compression.

Consider a cylinder with a piston which is free to move. Suppose the fat man stands on the piston. ***Assume that the temperature remains constant.*** Derive expressions for

- the pressure increase Δp within the cylinder;
- the change in volume ΔV;
- the work W done by the man in compressing the air;
- show that $W = |\Delta p \Delta V|$.

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Calculate numerical values of Δp, ΔV and W assuming the initial volume is $V = 1\,\mathrm{m}^3$, the initial pressure is $p = 10^5\,\mathrm{Pa}$ and the fat man weighs $m = 10^2\,\mathrm{kg}$. Consider two values of the cross-sectional area of the cylinder: (a) $A = 1\,\mathrm{m}^2$; (b) $A = 10^{-2}\,\mathrm{m}^2$.

Answer:

Without the fat man, the forces on the top and bottom of the piston are

$$F_\uparrow = pA \qquad F_\downarrow = pA\,.$$

Naturally, they are equal. After he hops on board, the forces are:

$$F_\uparrow = (p + \Delta p)A \qquad F_\downarrow = pA + mg\,.$$

These must also be equal, hence

$$\Delta p = \frac{mg}{A}\,.$$

By Boyle's Law, the product of pressure and volume is constant. Thus

$$(p + \Delta p)(V + \Delta V) = pV$$

which, rearranging terms, leads to

$$\Delta V = -\frac{V\Delta p}{p + \Delta p}$$

The work done by the man is *force multiplied by distance.* But the distance is $\Delta h = \Delta V/A$, so

$$W = mg \times \Delta h = \left(\frac{mg}{A}\right)\Delta V = \Delta p \cdot \Delta V$$

which completes the first part of the answer.

The fixed values are $V = 1\,\mathrm{m}^3$, $p = 10^5\,\mathrm{Pa}$ and $m = 10^2\,\mathrm{kg}$.
We consider two values of A.

(a) $A = 1\,\mathrm{m}^2$. Therefore the height of the air column is $h = V/A = 1\,\mathrm{m}$.

$$\Delta p = \frac{mg}{A} = \frac{10^2 \times 10}{1} = 10^3\,\mathrm{Pa}\,.$$

The percentage change in pressure is $100\Delta p/p = 1\%$.

The volume change is

$$\Delta V = -\frac{V\Delta p}{p + \Delta p} = -\frac{1 \times 10^3}{(10^5 + 10^3)} \approx -\frac{1 \times 10^3}{10^5} = -10^{-2}\,\mathrm{m}^3\,.$$

The percentage volume decrease is $100|\Delta V|/V = 1\%$.

The work done is

$$W = \Delta p \cdot \Delta V = 10^3 \times 10^{-2} = 10\,\mathrm{J}\,.$$

(b) $A = 10^{-2}\,\mathrm{m}^2$. Therefore the height of the air column is $h = V/A = 10^2\,\mathrm{m}$.

$$\Delta p = \frac{mg}{A} = \frac{10^2 \times 10}{10^{-2}} = 10^5\,\mathrm{Pa}\,.$$

The percentage change in pressure is $100\Delta p/p = 100\%$.

The volume change is

$$\Delta V = -\frac{V\Delta p}{p + \Delta p} = -\frac{1 \times 10^5}{(10^5 + 10^5)} = -0.5\,\mathrm{m}^3\,.$$

The percentage volume decrease is $100|\Delta V|/V = 50\%$.

The work done is

$$W = \Delta p \cdot \Delta V = 10^5 \times 0.5 = 5 \times 10^4\,\mathrm{J} = 50\,\mathrm{kJ}$$

which is 5000 times greater than in the previous case.

Remarks:

- The above results imply that we can compress air to an arbitrary state by using a cylinder of sufficiently small cross-section. Is this physically reasonable?
- The work done in compressing the gas must go somewhere. Where does it go?
- Compare two cases: (a) ***Isothermal***, where the cylinder is immersed in a bath of water held at a constant temperature; (b) ***Adiabatic***, where the cylinder is insulated, so that no heat enters or leaves.

Force of Gravity

Newton's law of gravity states that two bodies of mass m_1 and m_2 attract each-other with a force given by

$$F = G\frac{m_1 m_2}{d^2}$$

where d the distance between them. The constant G is the universal gravitational constant.

Near the earth's surface, a parcel of air of mass m is attracted towards the earth with a force

$$F = m\frac{GM}{a^2}$$

where M is the mass of the earth and a its radius.

We define the acceleration due to gravity by

$$g = \frac{GM}{a^2}$$

The acceleration due to gravity can be evaluated as follows:

$$G = 6.672\times10^{-11}\,, M = 5.974\times10^{24}\,, a = 6.375\times10^{6} \quad\Longrightarrow\quad g = 9.807$$

(all values are in SI units). So, roughly, $g \approx 10\,\mathbf{m\,s^{-2}}$.

The force due to gravity acts vertically downward, towards the centre of the earth. If k is a unit vector pointing upward, we may write it:

$$\mathbf{F}_g = -mg\mathbf{k}\,.$$

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Digression Definition of Metre

A simple way to remember the earth's radius.

- The *metre* is defined in terms of the size of the earth.
- The distance from the equator to the pole is 10 million metres.
- Thus, the circumference of the earth is 4×10^7 m.
- Thus, the radius of the earth is

$$a = \frac{4 \times 10^7}{2\pi} \approx 6.366 \times 10^6\,\mathrm{m}\,.$$