SEMESTER I EXAMINATION 2010/2011

ACM 40460
Dynamic Meteorology

Extern examiner: Professor Peter A Clark
Head of School: Professor Mícheál Ó Searcoid
Lecturer: Professor Ray Bates

Time Allowed: 2 hours

Instructions for Candidates
Answer three (3) questions.
All questions carry equal marks. Total: 75 marks.
Please do not use red pen on the answer books.
Please use separate answer book for each question.

Instructions for Invigilators
Non-programmable calculators may be used during this examination.
Question 1. [Marks: (a): 5, (b): 10, (c): 10]

(a) Define potential temperature ($\Theta$). Show that when $\Theta$ is constant with height $(z)$ the lapse rate of temperature is given by

$$\Gamma_d = -\frac{dT}{dz} = \frac{g}{c_p}$$

(1.1)

where $g$ is the acceleration of gravity and $c_p$ is the specific heat of dry air at constant pressure. (Assume the atmosphere is hydrostatically balanced.)

(b) Given the thermodynamic energy equation in the form

$$c_v \frac{DT}{Dt} + p \frac{D\alpha}{Dt} = J$$

(1.2)

where $c_v$ is the specific heat of dry air at constant volume, $\alpha$ is the specific volume and $J$ the diabatic heating rate per unit mass, show that this equation can be written in terms of potential temperature as

$$\frac{D\Theta}{Dt} = \frac{\Theta}{c_p} J$$

(1.3)

(Use the standard thermodynamic relationship $R = c_p - c_v$).

Hint: You may work backwards from (1.3) using the definition of $\Theta$ if you wish.

(c) Show that in the isobaric coordinate system eq. (1.2) can be written in the form

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p}$$

(1.4)

where

$$S_p = \frac{\alpha}{c_p} \frac{\partial T}{\partial p}$$

(1.5)

Note: $\omega \equiv Dp/Dt$.

Question 2. [Marks: (a): 5, (b): 13, (c): 7]

(a) State the assumptions used in constructing the shallow water model with a free surface of height $h_T(x,y,t)$, bottom topography $h_s(x,y)$ and fluid depth $h(x,y,t)$ on a $\beta$-plane.

(b) Starting from the primitive equations of motion on a $\beta$-plane and using the above assumptions, show that the governing equations for the model in question are

$$\frac{Du}{Dt} = f v - \frac{\partial \Phi_T}{\partial x}$$

(2.1)

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y}$$

(2.2)

$$\frac{Dh}{Dt} = -h \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)$$

(2.3)

where $\Phi_T = gh_T$. 
c) Derive the linearized form of the above equations for small perturbations about a state of rest for the case where \( h_c = 0 \) and \( \beta = 0 \). Hence show the existence of 1D gravity-inertia wave solutions with phase speed

\[
c = \pm \left( gH + \frac{f_0^2}{k^2} \right)^{1/2}
\]  \( \text{(2.4)} \)

where \( H \) is the mean depth of the fluid and \( k \) is the wavenumber in \( x \).

**Question 3.** [Marks. (a): 11. (b): 14]

a) The quasi-geostrophic potential vorticity equation for the free-surface shallow water model on a \( \beta \)-plane \( (f = f_0 + \beta y) \) is given by

\[
\frac{D_{g}}{Dt} \left[ \nabla^2 \psi' + f - \frac{\psi'}{\lambda^2} \right] = -\frac{f_0}{\Phi_0} \frac{D_{g} \Phi'_z}{Dt}
\]  \( \text{(3.1)} \)

where \( D_{g} / Dt \) is the material derivative following the geostrophic motion, \( \psi \) is the geostrophic streamfunction, \( \nabla^2 \) is the horizontal Laplacian, \( \lambda = \sqrt{\Phi_0 / f_0} \), \( \Phi_0 \) is the mean geopotential of the free surface and \( \Phi'_z \) is the geopotential of the orography (assumed small by comparison with \( \Phi_0 \)).

Using the linearized perturbation form of (3.1) for a resting basic state with \( \Phi'_z = 0 \), derive the phase speed for a 1D Rossby wave propagating in the \( x \)-direction in this model.

b) Again using (3.1) but neglecting \( \beta \), derive the solution for the perturbation streamfunction \( \psi' \) in the case of steady motion forced by a mean current \( \bar{u} \) blowing over orography of the form

\[
\Phi'_z = \text{Re} \left[ \Phi'_z \exp(ikx) \right]
\]  \( \text{(3.2)} \)

Hint: Since the forcing has spatial dependence of the form \( \exp(ikx) \), the solution may also be assumed to have spatial dependence of this form.


(a) Show that in a hydrostatic atmosphere the internal and gravitational potential energies are proportional and that the sum of these two forms of energy (i.e., the total potential energy – TPE) can be written

\[
TPE = \left( \frac{c_p}{c_v} \right) E_I
\]  \( \text{(4.1)} \)
where $c_p$ and $c_v$ are the specific heats of air at constant pressure and constant volume, respectively, and $E_1$ is the internal energy.

(b) Show that the TPE of a unit column of atmosphere of uniform potential temperature $\theta$ extending from the surface ($p=p_0$) to the top of the atmosphere ($p=0$) is

$$TPE = \frac{c_p}{g} \frac{p_0 \theta}{\kappa + 1}$$  \hspace{1cm} (4.2)

where $\kappa = R/c_p$ and $g$ is the acceleration of gravity.

(c) Consider two air masses of uniform potential temperatures $\theta_1$ and $\theta_2$ ($\theta_2 > \theta_1$) which are separated by a vertical partition. Each air mass occupies a horizontal area $A$ and extends from the surface (where $p=p_0$ for each) to the top of the atmosphere. Show that the available potential energy for this system is given by

$$APE = \frac{c_p}{g} \frac{p_0}{\kappa + 1} \left( 1 - \frac{1}{2^\kappa} \right) (\theta_2 - \theta_1)A$$  \hspace{1cm} (4.3)


(a) Given the following governing equations for the two-layer model of baroclinic instability

$$\left( \frac{\partial}{\partial t} + U_1 \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_1}{\partial x^2} + \beta \frac{\partial \psi_1}{\partial x} = \frac{f_0}{\sigma} \omega'$$  \hspace{1cm} (5.1)

$$\left( \frac{\partial}{\partial t} + U_3 \frac{\partial}{\partial x} \right) \frac{\partial^2 \psi_3}{\partial x^2} + \beta \frac{\partial \psi_3}{\partial x} = \frac{f_0}{\sigma} \omega'$$  \hspace{1cm} (5.2)

$$\left( \frac{\partial}{\partial t} - U_m \frac{\partial}{\partial x} \right) (\psi_1' - \psi_3') - U_r \frac{\partial}{\partial x} (\psi_1' + \psi_3') = \frac{\sigma f_0}{\sigma f_0} \omega'$$  \hspace{1cm} (5.3)

where $U_m = (U_1 + U_3)/2$ and $U_r = (U_1 - U_3)/2$, show that the rate of change of the sum of the kinetic and available potential energies is

$$\frac{d}{dt} (K' + P') = 4\lambda^2 U_r \frac{\partial \psi_m}{\partial x}$$  \hspace{1cm} (5.4)
where \( \psi_m = \left( \psi_1' + \psi_3' \right) / 2 \), \( \psi_r = \left( \psi_1' - \psi_3' \right) / 2 \), \( \lambda^2 = \left( f_0 \right)^2 / \sigma(\delta \psi)^2 \) and \( \overline{\cdots} \) denotes an average over the wavelength of the disturbance.

(b) Discuss the energetics of baroclinic waves qualitatively using a box diagram involving \( \bar{P} \) (the mean available potential energy), \( P' \) and \( K' \).

(c) Discuss the physical mechanism of baroclinic instability in terms of the slope of particle trajectories ('the wedge of instability').