The quasi-geostrophic form of the free-surface shallow water equations

The primitive form of the free-surface shallow water equations is

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \qquad (1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \qquad (2)$$

$$\frac{D\Phi}{Dt} = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3)$$

where

$$f = f_0 + \beta y$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\Phi = \Phi_T - \Phi_S$$

We have seen earlier that the above equations allow both gravity-inertia wave and Rossby wave solutions.

Proceeding as in the derivation of the 3D quasi-geostrophic equations, we define the geostrophic velocity

$$\underset{\sim}{V}_g \equiv \frac{1}{f_0}\underset{\sim}{k} \times \nabla\Phi_T \quad \left(u_g = -\frac{1}{f_0}\frac{\partial \Phi_T}{\partial y},\ v_g = \frac{1}{f_0}\frac{\partial \Phi_T}{\partial x}\right)$$

and the ageostrophic velocity

$$\underset{\sim}{V}_a \equiv \underset{\sim}{V} - \underset{\sim}{V}_g$$

We assume

$$|V_a| \ll |V_g|$$

Thus, we can make the approximations

$$\frac{Du}{Dt} \rightarrow \frac{D_g u_g}{Dt}$$

$$\frac{Dv}{Dt} \rightarrow \frac{D_g v_g}{Dt}$$

$$\frac{D\Phi}{Dt} \rightarrow \frac{D_g}{Dt}\left(\Phi_T - \Phi_S\right)$$

where

$$\frac{D_g}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{V}_g \cdot \nabla$$

Thus,

Eqn. (1) →
$$\frac{D_g u_g}{Dt} = (f_0 + \beta y)(v_g + v_a) - \frac{\partial \Phi_T}{\partial x}$$
$$\simeq f_0 v_a + \beta y v_g \quad (\text{neglecting } \beta y v_a) \cdots (4)$$

Eqn. (2) →
$$\frac{D_g v_g}{Dt} = -(f_0 + \beta y)(u_g + u_a) - \frac{\partial \Phi_T}{\partial y}$$
$$\simeq -f_0 u_a - \beta y u_g \quad (\text{neglecting } \beta y u_a) \cdots (5)$$

Eqn. (3) →
$$\frac{D_g \Phi_T}{Dt} = -(\Phi_T - \Phi_S)\left[\cancel{\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)} + \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)\right] + \frac{D_g \Phi_S}{Dt} \quad \cdots (6)$$

If we set

$$\Phi_T = \Phi_0 + \Phi_T'(x, y, t) \quad [\Phi_0 = \text{constant}]$$

$$\Phi_S = \Phi_S'(x, y)$$

eqn. (6) can be written

$$\frac{D_g \Phi_T'}{Dt} = -\Phi_0\left(1 + \frac{\Phi_T'}{\Phi_0} - \frac{\Phi_S'}{\Phi_0}\right)\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) + \frac{D_g \Phi_S'}{Dt} \quad \cdots (7)$$

If we assume

$$\left|\Phi'_T\right| \ll \Phi_0, \quad \left|\Phi'_s\right| \ll \Phi_0$$

eqn. (7) can be approximated as

$$\frac{D_g \Phi'_T}{Dt} = -\Phi_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) + \frac{D\Phi'_s}{Dt} \qquad \cdots (8)$$

Eqns (4), (5) and (8) are the quasi-geostropic free-surface shallow water equations.

We form the QG vorticity equation by taking $[\partial(5)/\partial x - \partial(4)/\partial y]$. Hence, using the fact that

$$\frac{\partial}{\partial x}\left[-\underset{\sim}{V}_g \cdot \underset{\sim}{\nabla} v_g\right] + \frac{\partial}{\partial y}\left[\underset{\sim}{V}_g \cdot \underset{\sim}{\nabla} u_g\right] = -\underset{\sim}{V}_g \cdot \nabla \zeta_g$$

we have

$$\frac{\partial \zeta_g}{\partial t} = -\underset{\sim}{V}_g \cdot \nabla \zeta_g - f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) - \beta y \cancel{\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)} - \beta v_g$$

i.e.,

$$\frac{D_g}{Dt}\left(\zeta_g + f\right) = -f_0\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) \qquad \cdots (9)$$

Eliminating $(\partial u_a/\partial x + \partial v_a/\partial y)$ between (8) and (9) we have

$$\frac{D_g}{Dt}\left(\zeta_g + f\right) = \frac{f_0}{\Phi_0}\left[\frac{D_g \Phi'_T}{Dt} - \frac{D_g \Phi'_s}{Dt}\right]$$

i.e.,

$$\frac{D_g}{Dt}\left[\zeta_g + f - \frac{f_0}{\Phi_0}\left(\Phi'_T - \Phi'_s\right)\right] = 0 \qquad \cdots (10)$$

This is the QG potential vorticity equation for the free-surface shallow water model.

Introducing the QG streamfunction ψ, defined by

$$\psi = \Phi'_T / f_0$$

we see that

$$\underset{\sim}{V}_g = \underset{\sim}{k} \times \nabla\psi \quad \left(u_g = -\frac{\partial \psi}{\partial y}, \; v_g = \frac{\partial \psi}{\partial x}\right)$$

and (10) can be re-written as

$$\frac{D_g}{Dt}\left[\nabla^2\psi + f - \frac{\psi}{\lambda^2}\right] = -\frac{f_0}{\Phi_0}\frac{D_g \Phi_s'}{Dt} \quad \cdots (11)$$

where

$$\lambda = \sqrt{\Phi_0}/f_0 \quad \text{(Rossby radius of deformation).}$$

Free wave solution of the free-surface quasi-geostrophic shallow water equations

We linearize (11) about a state of rest and set $\Phi_s' = 0$. Hence we have

$$\frac{\partial}{\partial t}\left[\nabla^2\psi' - \frac{\psi'}{\lambda^2}\right] + \beta\frac{\partial \psi'}{\partial x} = 0 \qquad (12)$$

Assuming a solution

$$\psi' = \hat{\psi}\exp\left[ik(x - ct)\right]$$

eqn (12) gives

$$-ikc\left[-k^2 - \frac{1}{\lambda^2}\right]\hat{\psi} + ik\beta\hat{\psi} = 0$$

whence

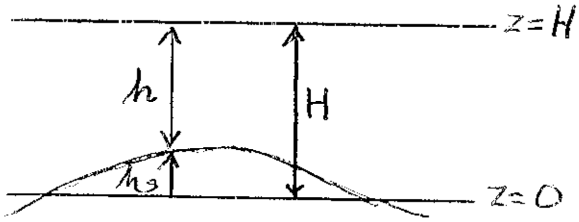
$$c = -\frac{\beta}{k^2 + \frac{1}{\lambda^2}} \qquad (13)$$

This is the Rossby wave solution. The term $1/\lambda^2$, which arises because of the influence of the free surface, shows that the phase speed is influenced by gravity. This influence was not present in the case of the pure Rossby wave solution which we obtained earlier for the case of the shallow water equations with a rigid lid (i.e., the nondivergent case).
The solution (13) is the same as we obtained in the case of the mixed gravity-inertia and Rossby wave solutions (using the primitive form of the free-surface shallow water equations) under the assumption $\nu^2 \ll \Phi k^2$.

Note that when we use the quasi-geostrophic form of the shallow water equations, there are no gravity-inertia wave solutions. <u>The gravity-inertia waves have been filtered out by the use of the quasi-geostrophic approximation and only the Rossby wave solution remains.</u>

The quasi-geostrophic form of the shallow water equations with a rigid horizontal lid and bottom topography

We start from the primitive equations for the shallow water model with a rigid lid and bottom topography derived earlier, i.e.,



$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \quad \cdots (1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \quad \cdots (2)$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) h = \underset{\sim}{V} \cdot \nabla h_s \quad \cdots (3)$$

[see p.(2.1)-(2.2)]

where

$$\Phi_T = p_T/\rho$$

The quasi-geostrophic form of (1) and (2) is as follows:

$$\frac{D_g u_g}{Dt} = (f_0 + \beta y)(v_g + v_a) - \frac{\partial \Phi_T}{\partial x}$$

$$= f_0 v_a + \beta y v_g \;\; (\text{neglecting } \beta y v_a) \quad \cdots (4)$$

$$\frac{D_g v_g}{Dt} = -(f_0 + \beta y)(u_g + u_a) - \frac{\partial \Phi_T}{\partial y}$$

$$= -f_0 u_a - \beta y u_g \;\; (\text{neglecting } \beta y u_a) \quad \cdots (5)$$

where

$$\underset{\sim}{V_g} = \frac{1}{f_0} \underset{\sim}{k} \times \nabla \Phi_T$$

To obtain the quasi-geostrophic form of (3), we assume

$$|h_s| \ll H$$

so that (3) becomes

$$\left[\cancel{\left(\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y}\right)} + \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right)\right] H = \underset{\sim}{V_g} \nabla h_s$$

i.e.,

$$\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) H = \underset{\sim}{V_g} \cdot \nabla h_s \quad \cdots (6)$$

Taking [∂(5)/∂x −∂(4)/∂y] we arrive at the vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -\underset{\sim}{V_g} \cdot \nabla \zeta_g - \beta v_g - f_0 \left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) \qquad (7)$$

Eliminating ($\partial u_a/\partial x + \partial v_a/\partial y$) from (7) using (6) gives

$$\frac{\partial \zeta_g}{\partial t} = -\underset{\sim}{V_g} \cdot \nabla \zeta_g - \beta v_g - \frac{f_0}{H} \underset{\sim}{V_g} \cdot \nabla h_s$$

$$= -\underset{\sim}{V_g} \cdot \nabla \zeta_g - \beta v_g - \frac{f_0}{\Phi_0} \underset{\sim}{V_g} \cdot \nabla \Phi_s \qquad (8)$$

[c.f. Holton's eq. (7.95), but note that he uses h_1 instead of h_s to denote bottom topography]

$(\Phi_0 = gH)$