

Mixed gravity-inertia and Rossby waves

We have seen earlier that pure gravity-inertia waves are obtained by choosing a free-surface shallow water model on an f -plane, while pure Rossby waves are obtained by choosing a shallow water model with a rigid lid (i.e., a nondivergent barotropic model) on a β -plane.

We now consider the case of mixed gravity-inertia and Rossby waves by choosing a free-surface shallow water model on a β -plane. Thus, $f = f_0 + \beta y$. We assume a basic state of rest ($\bar{u} = 0$) and no bottom topography ($\Phi_s = 0$).

We use the shallow water equations in vorticity-divergence form. As seen earlier, this form of the equations is

$$\frac{D}{Dt} (f + \zeta) = -(f + \zeta) \bar{D} \quad (1)$$

$$\frac{D\bar{D}}{Dt} + \bar{D}^2 - 2J(u, v) = -\nabla^2 \bar{\Phi}_T + (f\zeta - \beta u) \quad (2)$$

$$\frac{D\bar{\Phi}}{Dt} = -\bar{\Phi} \bar{D} \quad (3)$$

where

$$\bar{D} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \bar{J} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The linearized form of the above equations for a resting basic state with $\Phi_s = 0$ (in which case $\bar{\Phi}_T = \bar{\Phi}$) is

$$\frac{\partial \zeta'}{\partial t} + \beta v' = -\bar{J} \bar{D}' \quad (4)$$

$$\frac{\partial \bar{D}'}{\partial t} = -\nabla^2 \bar{\Phi}' + (\bar{J} \zeta' - \beta u') \quad (5)$$

$$\frac{\partial \bar{\Phi}'}{\partial t} = -\bar{\Phi} \bar{D}' \quad (6)$$

We now make the approximation of taking f as a constant f_0 except where it occurs in differentiated form (this is equivalent to assuming that the motions do not extend over too great a distance in latitude). We also assume that the perturbation quantities are independent of y (i.e., $\partial(\cdot)'/\partial y = 0$). Eqns. [(4), (5), (6)] then become

$$\frac{\partial}{\partial t} \left(\frac{\partial v'}{\partial x} \right) + \beta v' = -\bar{J} \frac{\partial u'}{\partial x} \quad (7)$$

$$\frac{\partial}{\partial t} \left(\frac{\partial u'}{\partial x} \right) = -\frac{\partial^2 \bar{\Phi}'}{\partial x^2} + \left(\bar{J} \frac{\partial v'}{\partial x} - \beta u' \right) \quad (8)$$

$$\frac{\partial \Phi'}{\partial t} = -\bar{\Phi} \frac{\partial u'}{\partial x} \quad (9)$$

Seeking solutions of the form

$$(u', v', \Phi') = (\hat{u}, \hat{v}, \hat{\Phi}) e^{i(-kx - \nu t)}$$

we have

$$(-i\nu)(ik)\hat{v} + \beta\hat{v} = -f_0 ik\hat{u} \quad (10)$$

$$(-i\nu)(ik)\hat{u} = +k^2\hat{\Phi} + f_0 ik\hat{v} - \beta\hat{u} \quad (11)$$

$$(-i\nu)\hat{\Phi} = -\bar{\Phi} ik\hat{u} \quad (12)$$

i.e.,

$$\left(\nu + \frac{\beta}{k}\right)\hat{v} = -if_0\hat{u} \quad (13)$$

$$\left(\nu + \frac{\beta}{k}\right)\hat{u} = k\hat{\Phi} + if_0\hat{v} \quad (14)$$

$$\nu\hat{\Phi} = k\bar{\Phi}\hat{u} \quad (15)$$

We use (13) and (15) to eliminate \hat{u} and \hat{v} from (14).

$$(15) \rightarrow \hat{u} = \left(\frac{\nu}{k\bar{\Phi}}\right)\hat{\Phi} \quad (16)$$

$$\text{Hence, (13)} \rightarrow \hat{v} = -\left(\frac{if_0}{\nu + \beta/k}\right)\hat{u} = -\left(\frac{if_0}{\nu + \beta/k}\right)\left(\frac{\nu}{k\bar{\Phi}}\right)\hat{\Phi} \quad (17)$$

Substituting in (14) we have

$$\left(\nu + \frac{\beta}{k}\right)\left(\frac{\nu}{k\bar{\Phi}}\right)\hat{\Phi} = k\hat{\Phi} + \left(\frac{f_0^2}{\nu + \beta/k}\right)\left(\frac{\nu}{k\bar{\Phi}}\right)\hat{\Phi} \quad (18)$$

i.e.,

$$\left(\nu + \frac{\beta}{k}\right)\left[\nu\left(\nu + \frac{\beta}{k}\right) - k^2\bar{\Phi}\right] = f_0^2\nu \quad (19)$$

This is a cubic equation for ν . We shall see that two of the roots are close approximations to pure gravity-inertia waves while the third root is a close approximation to a pure Rossby wave.

(a) Suppose $|v| \gg \beta/k$ (i.e., the frequency is much greater than the Rossby wave frequency).

Then (19) can be approximated by

$$v [v^2 - k^2 \bar{\Phi}] = f_0^2 v$$

whence we have

$$v = \pm \sqrt{k^2 \bar{\Phi} + f_0^2}, \text{ i.e. } c = \pm \sqrt{\bar{\Phi} + f_0^2/k^2} \quad (20)$$

i.e., we have the gravity-inertia wave solution.

Note: The assumption $|v| \gg \beta/k$ is justified a posteriori if $v^2 \gg (\beta/k)^2$. Using (20), this becomes

$$f_0^2 + k^2 \bar{\Phi} \gg \left(\frac{\beta}{k}\right)^2$$

i.e.,

$$1 + 4\pi^2 \left(\frac{R}{L}\right)^2 \gg \left(\frac{\beta}{f_0 k}\right)^2$$

where $R = \sqrt{\bar{\Phi}}/f_0$ (Rossby radius of deformation).

But

$$\frac{\beta}{f_0} = \left(\frac{2\Omega \cos\theta}{a}\right) / 2\Omega \sin\theta = \frac{1}{a \tan\theta} \approx \frac{1}{a} \text{ if } \theta \approx 45^\circ$$

Therefore the above inequality holds if

$$1 + 4\pi^2 \left(\frac{R}{L}\right)^2 \gg \frac{1}{4\pi^2} \left(\frac{L}{a}\right)^2$$

This obviously holds if $L/a \ll 1$, which is a reasonable assumption in the context of gravity-inertia waves.

(b) Suppose $v^2 \ll \bar{\Phi} k^2$ (i.e., the frequency is much less than the gravity-inertia wave frequency) and suppose in addition that $\beta/k^2 \ll \sqrt{\bar{\Phi}}$ (i.e., the Rossby wave phase speed is much less than the gravity wave phase speed).

Then (19) can be approximated, making use of the first of these approximations, by

$$\left(v + \frac{\beta}{k}\right) \left[v \frac{\beta}{k} - k^2 \bar{\Phi} \right] = f_0^2 v \quad \left\{ \begin{array}{l} v \frac{\beta}{k} - k^2 \bar{\Phi} = k^2 \left[\frac{v \beta}{k^2} - \bar{\Phi} \right] \\ \approx k^2 [-\bar{\Phi}] \end{array} \right.$$

and this can be approximated, using the first and second approximations, by

$$\left(v + \frac{\beta}{k}\right) \left[-k^2 \bar{\Phi} \right] = f_0^2 v$$

Hence we have

$$v = - \frac{\beta}{k} \left[\frac{k^2 \bar{\Phi}}{f_0^2 + k^2 \bar{\Phi}} \right] \quad (21)$$

which gives

$$C = - \frac{\beta}{k^2} \left[\frac{1}{1 + \frac{L^2}{4\pi^2 R^2}} \right] \quad (22)$$

Thus, we have the Rossby wave solution slightly modified by gravitational effects through the term $L^2 / 4\pi^2 R^2$. For this term to be significant we must have $L > R$. Its effect is to slow down the very long waves.

Note: Using (21), our assumption $v^2 \ll \bar{\Phi} k^2$ is justified a posteriori if

$$\left(\frac{\beta}{k} \right)^2 \left[\frac{1}{1 + \frac{L^2}{4\pi^2 R^2}} \right]^2 \ll \bar{\Phi} k^2$$

i.e., if

$$\frac{\beta}{k^2} \ll \sqrt{\bar{\Phi}} \left(1 + \frac{L^2}{4\pi^2 R^2} \right) \quad (23)$$

This is satisfied a fortiori if our second assumption $\beta/k^2 \ll \sqrt{\bar{\Phi}}$ is satisfied. In the real atmosphere, the Rossby wave phase speed β/k^2 is small, amounting to only tens of metres per second for even the longest waves, while the gravity-inertia wave phase speed $\sqrt{\bar{\Phi}}$ is large, having the value 313m/s if we take $H = 10\text{km}$ (the approximate depth of the troposphere). Thus (23) holds and our approximation is justified.