Dynmet1-5 wpd

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2D Rossby waves on an unbounded β-plane

We have seen earlier that the governing equation for the streamfunction ψ in the case of horizontally nondivergent flow (the shallow water model with a rigid lid and a flat bottom) is

$$\frac{\partial}{\partial t}\nabla^2\psi + J(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0 \qquad \text{---}(1)$$

If we assume small amplitude perturbations about a state of rest, the above equation can be linearized to give

$$\frac{\partial}{\partial t}\nabla^2\psi' + \beta\frac{\partial\psi'}{\partial x} = 0 \qquad \text{---}(2)$$

We seek 2D plane wave solutions of the form

$$\psi' = Re\left[\hat{\psi}\, e^{i(kx+ly-\nu t)}\right] \qquad \text{---}(3)$$
$$= Re\left[\hat{\psi}\, e^{i(\underset{\sim}{K}\cdot\underset{\sim}{x}-\nu t)}\right]$$

We can write

$$(k, l) = |K|\,(\cos\alpha, \sin\alpha)$$

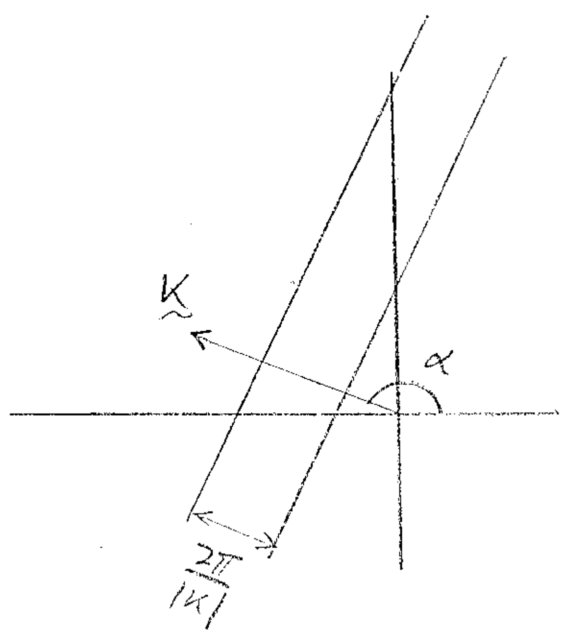
where α is the angle between the x-axis and the wavenumber vector **K**.

Substituting (3) into (2) gives

$$i\nu K^2\hat{\psi} + ik\beta\hat{\psi} = 0$$

i.e. $$\nu = -\frac{\beta k}{K^2} \qquad \text{---}(4)$$

Since we are using the convention that ν is positive, we see from (4) that k must be negative



We see that

$$c_x = \frac{\nu}{k} = -\frac{\beta}{K^2} \qquad \cdots (5)$$

$$c_y = \frac{\nu}{l} = -\frac{k}{l}\left(\frac{\beta}{K^2}\right) = \frac{|k|}{l}\left(\frac{\beta}{K^2}\right) \qquad \cdots (6)$$

Eqn. (5) tells us that the velocity of a wavecrest along the x-axis is negative (westward movement) and depends only on the magnitude of the wavenumber **K**, being independent of its orientation.

Eqn. (6) tells us that the velocity of a wavecrest along the y-axis can be of either sign (positive for $l>0$, negative for $l<0$) and its magnitude depends on the orientation of the wave.

The frequency equation (4) can be written in the form

$$k^2 + l^2 + 2\gamma k = 0$$

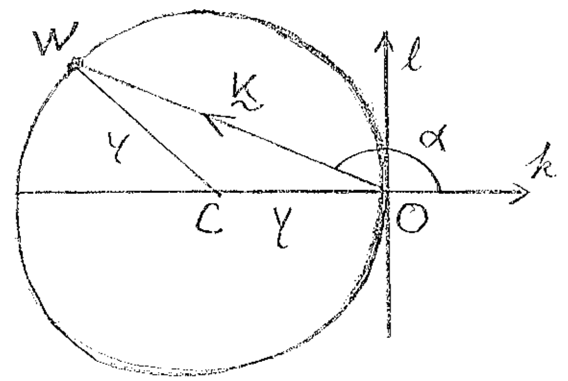
i.e.,

$$(k+\gamma)^2 + l^2 = \gamma^2 \qquad \cdots (7)$$

where

$$\gamma = \beta/2\nu$$

Thus, for fixed values of the frequency ν, the tip of the wave vector **K** (point W in the diagram below) lies on a circle passing through the origin O in the (k,l) plane. The centre of the circle is at $(-\gamma, 0)$ and its radius is equal to γ.



Calculation of the velocity

The velocity vector **V** is given by

$$\underset{\sim}{V} = u\underset{\sim}{i} + v\underset{\sim}{j} = -\frac{\partial\psi}{\partial y}\underset{\sim}{i} + \frac{\partial\psi}{\partial x}\underset{\sim}{j}$$

Assuming (without loss of generality) that $\hat{\varphi}$ is real in (3), we have

$$\psi = \hat{\varphi}\cos(kx+ly-\nu t)$$

$$\therefore \quad \underset{\sim}{V} = (l\underset{\sim}{i} - k\underset{\sim}{j})\hat{\varphi}\sin(kx+ly-\nu t)$$

so that

$$\underset{\sim}{V}\cdot\underset{\sim}{K} = \underset{\sim}{V}\cdot(k\underset{\sim}{i}+l\underset{\sim}{j}) = 0$$

i.e., the velocity is $\perp$ to **K**, so that the 2D wave is transverse, just as in the 1D case.

Group velocity

$$c_g = \underset{\sim}{i}\left(\frac{\partial\nu}{\partial k}\right) + \underset{\sim}{j}\left(\frac{\partial\nu}{\partial l}\right)$$

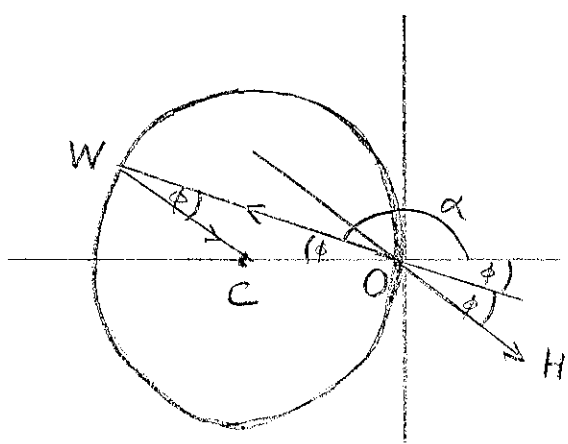
$$= \underset{\sim}{i}\left(\frac{\beta[k^2-l^2]}{(k^2+l^2)^2}\right) + \underset{\sim}{j}\left(\frac{\beta(2kl)}{(k^2+l^2)^2}\right)$$

$$= i\left(\frac{\beta}{K^2}[\cos^2\alpha - \sin^2\alpha]\right) + \underset{\sim}{j}\left(\frac{\beta}{K^2}2\cos\alpha\sin\alpha\right)$$

$$= \underset{\sim}{i}\left(\frac{\beta}{K^2}\cos 2\alpha\right) + \underset{\sim}{j}\left(\frac{\beta}{K^2}\sin 2\alpha\right)$$

$$\alpha = \pi - \phi$$

$$2\alpha = 2\pi - 2\phi$$



Clearly, the vector whose angle with the x-axis is 2α lies along **OH**, which is parallel to **WC**.

Therefore

$$\underset{\sim}{c_g} = \frac{\beta}{K^2} \frac{\overrightarrow{WC}}{WC}$$

$$= \frac{\beta}{K^2} \frac{\overrightarrow{WC}}{Y}$$

$$= \frac{2\nu}{K^2} \overrightarrow{WC}$$

Thus, a wave whose wavenumber **K** is directed towards the northwest will have a group velocity directed towards the southeast, and a wave whose wavenumber **K** is directed towards the west will have a group velocity directed towards the east.