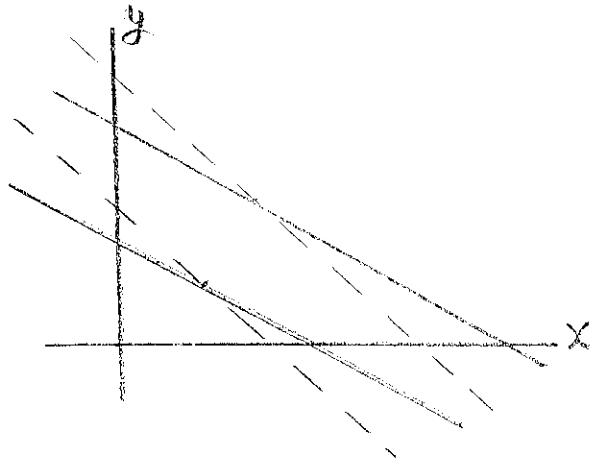
Group Velocity in 2D

SB

In obtaining the expression for the group velocity in one dimension, we considered two waves moving in the x-direction with the same amplitude but slightly different wavenumbers and frequencies. The 1D case is rather special in that the wave crests are parallel.

We now consider two waves in two dimensions. The waves are again regarded as having the same amplitude, but they have slightly different wavenumbers in both the x and y-directions (i.e., they are not parallel) as well as having slightly different frequencies.



The combination of the two waves can be expressed mathematically as follows:

$$\psi = Re\left[e^{i(\{k+\delta k\}x + \{l+\delta l\}y - \{\nu+\delta\nu\}t)} + e^{i(\{k-\delta k\}x + \{l-\delta l\}y - \{\nu-\delta\nu\}t)} \right]$$

$$= Re\left[e^{i(kx+ly-\nu t)} \left\{ e^{i(\delta k x + \delta l y - \delta\nu t)} + e^{-i(\delta k x + \delta l y - \delta\nu t)} \right\} \right]$$

$$= Re\left[e^{i(kx+ly-\nu t)} \left\{ 2\cos(\delta k x + \delta l y - \delta \nu t) \right\} \right]$$

$$= \underbrace{2\cos(kx+ly-\nu t)}_{\text{Carrier Wave}} \underbrace{\cos(\delta k x + \delta l y - \delta\nu t)}_{\text{Envelope}} \quad \text{--(1)}$$

If we assume

$$\nu = \nu(k, l)$$

we have

$$\delta\nu = \frac{\partial\nu}{\partial k}\delta k + \frac{\partial\nu}{\partial l}\delta l$$

Therefore (1) can be written

$$\begin{aligned}\psi &= 2\cos(kx + ly - \nu t)\cos\left(\delta k\left\{x - \frac{\partial\nu}{\partial k}t\right\} + \delta l\left\{y - \frac{\partial\nu}{\partial l}t\right\}\right) \\ &= 2\cos(\underset{\sim}{K}\cdot\underset{\sim}{r} - \nu t)\cos(\delta\underset{\sim}{K}\cdot\{\underset{\sim}{r} - \underset{\sim}{C}t\}) \qquad (2)\end{aligned}$$

where

$$\underset{\sim}{K} = k\underset{\sim}{i} + l\underset{\sim}{j}$$

$$\delta\underset{\sim}{K} = \delta k\,\underset{\sim}{i} + \delta l\,\underset{\sim}{j}$$

$$\underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

and **C** is defined as

$$\underset{\sim}{C} \equiv \frac{\partial\nu}{\partial k}\underset{\sim}{i} + \frac{\partial\nu}{\partial l}\underset{\sim}{j} \qquad (3)$$

From (2), the phase of the carrier wave is constant if

$$\mathbf{K.r} - \nu t = \text{constant}$$

i.e.,

$$KR - \nu t = \text{constant} \qquad K = \sqrt{k^2 + l^2}$$

where $R = r\cos\theta$, where θ is the angle between **K** and **r**. Hence the phase speed is

$$c \equiv (dR/dt)_{\text{phase of carrier constant}} = \nu/K \qquad (4)$$

The phase of the envelope is constant if

$$\mathbf{r} - \mathbf{C}\,t = \text{constant vector}$$

Hence the group velocity is

$$\mathbf{c}_G \equiv (\mathbf{dr}/\mathbf{d}t)_{\text{phase of envelope constant}} = \mathbf{C} = (\partial\nu/\partial k)\,\mathbf{i} + (\partial\nu/\partial l)\,\mathbf{j} \qquad (5)$$

Note that the group velocity is a vector quantity.