Kinematics of 2D waves

Up to now we have been considering 1D waves having the form

$$\psi = Re\left[\hat{\varphi}\, e^{ik(x-ct)}\right]$$
$$= Re\left[\hat{\varphi}\, e^{i(kx-\nu t)}\right]$$

If

$$\hat{\varphi} = |\hat{\varphi}| e^{i\phi}$$

the above can be written

$$\psi = Re\left[|\hat{\varphi}| e^{i(kx-\nu t+\phi)}\right]$$
$$= |\hat{\varphi}| \cos(kx-\nu t+\phi)$$

We have chosen to regard $k\ (= 2\pi/L)$ as a positive quantity by definition. ν may then turn out to be +ve or −ve, depending on the problem being examined, and the sign of ν determines the direction of propagation along the x-axis.

We now come to a consideration of 2D waves having the form

$$\psi = Re\left[\hat{\varphi}\, e^{i(kx+ly-\nu t)}\right]$$
$$= |\hat{\varphi}| \cos(kx+ly-\nu t+\phi)$$

The above can be written

$$\psi = Re\left[\hat{\varphi}\, e^{i(\underset{\sim}{K}\cdot\underset{\sim}{r}-\nu t)}\right]$$

where

$$\underset{\sim}{K} = k\underset{\sim}{i} + l\underset{\sim}{j} \quad , \quad \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$$

$\hat{\psi}$ = Complex amplitude

$|\hat{\psi}|$ = amplitude

ϕ = reference phase

$\underset{\sim}{K} \cdot \underset{\sim}{r} - \nu t$ = phase

$\underset{\sim}{K}$ = wave number vector

k = wave number in x

l = wave number in y

ν = frequency

In the 2D case it is preferable to take ν to be positive by definition. k and l may then turn out to be +ve or −ve, and their sign will determine the direction of propagation in the 2D plane.

We define the quantities (L_x, L_y) according to

$$|k| = \frac{2\pi}{L_x}$$

$$|l| = \frac{2\pi}{L_y}$$

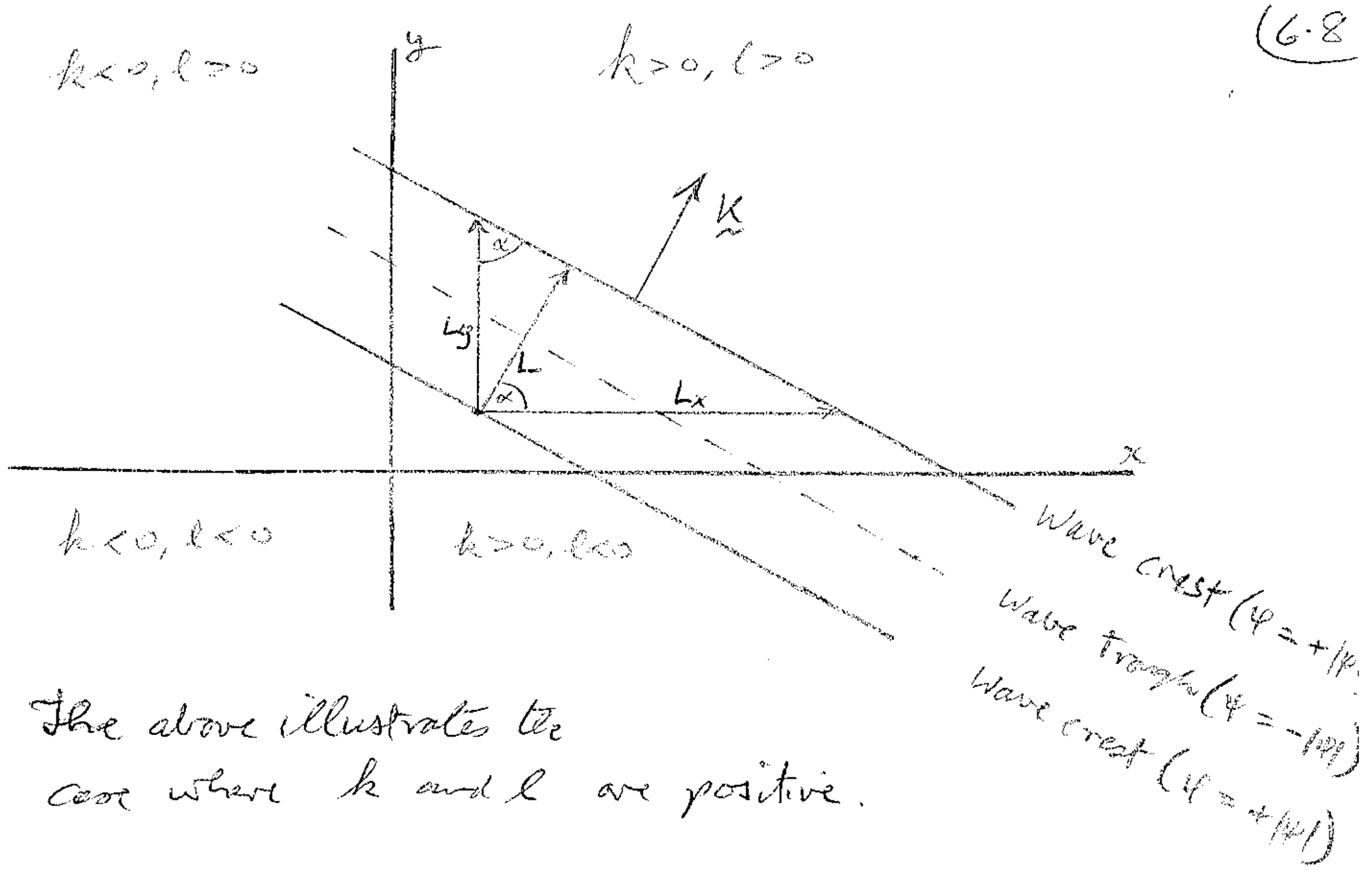
and L according to

$$|K| = \frac{2\pi}{L}$$

L = wavelength (distance from crest to crest along the direction of propagation

L_x = wavelength in x

L_y = wavelength in y



The above illustrates the case where k and l are positive.

If α is the angle between the direction of propagation and the x-axis, we have

$$\sin\alpha = \frac{L}{L_y}, \quad \cos\alpha = \frac{L}{L_x}$$

$$\sin^2\alpha + \cos^2\alpha = 1 \implies \left(\frac{L}{L_x}\right)^2 + \left(\frac{L}{L_y}\right)^2 = 1$$

ie $$\left(\frac{1}{L_x}\right)^2 + \left(\frac{1}{L_y}\right)^2 = \frac{1}{L^2} \qquad \cdots (1)$$

$$\underline{k^2 + l^2 = K^2}$$

Thus we see that, while $\underset{\sim}{K}$ is a vector whose components are (k, l), no such vector relationship holds between L and (L_x, L_y), ie.

$$L^2 \neq L_x^2 + L_y^2$$

The period of the wave (T) is the time that elapses between the passage of two successive crests of the wave past a fixed point. In this time νt increases by 2π, i.e.

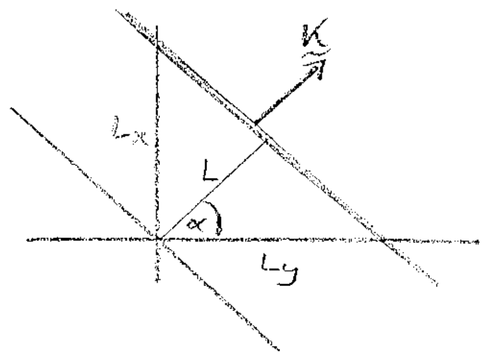
$$\nu T = 2\pi$$

$$T = \frac{2\pi}{\nu}$$

The phase speed c is defined as

$$c = \frac{L}{T} = \frac{\nu}{|K|} \quad \text{(Positive definite)}$$

The phase speeds in the x and y-directions, c_x and c_y, are defined as



$$c_x = \frac{\nu}{k}, \quad c_y = \frac{\nu}{l}$$

These are the speeds with which the wave crests move along the x and y-axes, respectively. They can be of either sign.

We see that

$$c^2 = \frac{\nu^2}{K^2} = \frac{\nu^2}{k^2+l^2}$$

while $$c_x^2 + c_y^2 = \nu^2\left(\frac{1}{k^2}+\frac{1}{l^2}\right)$$

i.e. $$c^2 \neq c_x^2 + c_y^2$$

Thus, c is not a vector whose components are (c_x, c_y)