

Rossby Waves on a β -plane

Rossby waves are quasi-horizontal waves which owe their existence to a gradient of vorticity (or, more generally, potential vorticity) in the basic state. The simplest type of Rossby wave occurs when the basic state is one of rest, the gradient of vorticity in the basic state is due to the poleward variation of the Coriolis parameter (the β -effect) and the motion is horizontally non-divergent. The Rossby wave is to be contrasted to the gravity-inertia wave, which requires the existence of horizontal convergence and divergence and in which the restoring force is gravity.

We look at the simplest case of Rossby waves by taking the shallow water equations with a rigid lid (these are sometimes called the non-divergent barotropic equations) and assuming β -plane geometry. As seen earlier, the rigid lid excludes gravity-inertia waves, thus allowing us to examine Rossby waves in isolation. Later, we shall study the case of the free-surface shallow water equations on a β -plane, in which case gravity-inertia waves and gravity waves occur simultaneously.

The shallow water equations with a rigid lid and a flat horizontal bottom on a β -plane are

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \quad (1)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \quad (2)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3)$$

where $\Phi_T = p_T / \rho$ and $f = f_0 + \beta y$. Taking

$$\left[\frac{D}{Dt} \left(f - \frac{D}{Dt} \right) (1) \right] \text{ and using (3)}$$

gives

$$\frac{D}{Dt} [f + \mathcal{J}] = 0 \quad (4)$$

where

$$\mathcal{J} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Eq. (4) is the vorticity equation. (In the case of the shallow water model with a rigid lid, the vorticity equation and the potential vorticity equation are one and the same.)

The simplest case: 1D Rossby waves propagating in a basic state of rest

Linearizing eqs. (1), (2) and (3) about a basic state of rest gives

$$\frac{\partial u'}{\partial t} = f v' - \frac{\partial \Phi_T'}{\partial x} \quad (5)$$

$$\frac{\partial v'}{\partial t} = -f u' - \frac{\partial \bar{\Phi}'_T}{\partial y} \quad (6)$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \quad (7)$$

We seek a solution

$$(u', v', \bar{\Phi}'_T) = (0, \hat{v}, \hat{\Phi}(y)) e^{ik(x-ct)} \quad (8)$$

where \hat{v} is a constant. Such a solution represents a pure transverse wave, i.e., there is no component of motion in the direction of propagation. Substituting the assumed solution (8) in eqs. (5), (6) and (7) gives

$$0 = f \hat{v} - ik \hat{\Phi} \quad (9)$$

$$-ikc \hat{v} = - \frac{d\hat{\Phi}}{dy} \quad (10)$$

$$0 = 0 \quad (11)$$

Eq. (9) gives

$$\hat{\Phi} = \frac{f}{ik} \hat{v} = \left(\frac{f_0 + \beta y}{ik} \right) \hat{v} \quad (12)$$

Substituting this in (10) gives

$$ikc \hat{v} = \frac{\beta}{ik} \hat{v}$$

i.e.,

$$\boxed{c = - \frac{\beta}{k^2}} \quad (13)$$

This is the Rossby wave phase speed. We see that it owes its existence to the β -effect and that it represents a unidirectional wave propagating in the westward direction.

From (8) and (12) we find, assuming that \hat{v} is real,

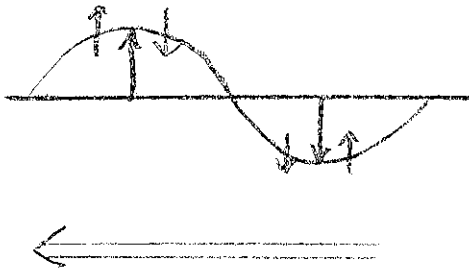
$$u' = 0 \quad (14)$$

$$v' = \hat{v} \cos k(x-ct) \quad (15)$$

$$\bar{\Phi}'_T = \hat{v} \left(\frac{f_0 + \beta y}{k} \right) \sin k(x-ct) \quad (16)$$

The transverse nature of the wave is clearly apparent from the above.

The mechanism of propagation of the Rossby wave



Due to the conservation of absolute vorticity (eq. (4)) and the fact that f varies with y , a particle that is displaced northward acquires a negative relative vorticity (clockwise rotation) while a particle that is displaced southward acquires a positive relative vorticity (anticlockwise rotation). This provides a mechanism whereby a waveform is forced to propagate towards the west.

Characteristic magnitudes

Suppose the β -plane is centered at 45°N . Then

$$f_0 = 2\Omega \sin 45^\circ = \sqrt{2} \Omega, \quad \beta = \frac{2\Omega \cos 45^\circ}{a} = \frac{\sqrt{2} \Omega}{a}$$

and

$$C = -\frac{\beta}{k^2} = -\frac{\beta L^2}{4\pi^2} = -\left(\frac{L}{2\pi}\right)^2 \frac{\sqrt{2} \Omega}{a}$$

Since

$$\Omega = 7.292 \times 10^{-5} \text{ s}^{-1}, \quad a = 6.37 \times 10^6 \text{ m}$$

we have

$$C = -(4.1 \times 10^{-10} \frac{1}{\text{ms}}) L^2$$

$$L = 10^3 \text{ km} \Rightarrow C = -0.41 \frac{\text{m}^2}{\text{s}}, \quad L = 10^4 \text{ km} \Rightarrow C = -4.1 \frac{\text{m}^2}{\text{s}}$$

Therefore, only very long waves have a significant Rossby wave phase speed (short waves do not feel the β -effect).

Group velocity of Rossby waves

We see that

$$\begin{aligned} c_g &= \frac{\partial \omega}{\partial k} = \frac{\partial}{\partial k} (kc) = \frac{\partial}{\partial k} \left(-\frac{c}{k}\right) \\ &= +\beta/k^2 \end{aligned}$$

i.e., the group velocity of the 1D Rossby wave has the same magnitude as the phase speed, but has the opposite sign! Thus, the energy of Rossby wave packets propagates eastward. This is perhaps the most surprising of all the effects of rotation.

The Rossby wave solution satisfies the full nonlinear equations

Even though we have derived the Rossby wave solution by linearizing the governing equations, it is easy to show that the solution satisfies the full nonlinear equations (1), (2) and (3):

$$\text{Eq. (1)} \quad 0 = (\hat{f} + \beta y) \hat{v} \cos k(x-ct) - \hat{v} (\hat{f} + \beta y) \cos k(x-ct) \quad \checkmark$$

$$\text{Eq. (2)} \quad \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) v \stackrel{?}{=} 0 - \hat{v} \frac{\beta}{k} \sin k(x-ct)$$

$$\text{or} \quad k c \hat{v} \sin k(x-ct) \stackrel{?}{=} -\hat{v} \frac{\beta}{k} \sin k(x-ct)$$

Using (13) we see that this is satisfied.

$$\text{Eq. (3)} \quad 0 = 0 \quad \checkmark$$

Solution of the 1D Rossby wave in terms of the streamfunction

When the motion is horizontally nondivergent (i.e., eq. (3) is satisfied) we can define a streamfunction ψ such that

$$\underline{V} = \underline{k} \times \underline{\nabla} \psi$$

i.e.,

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Then (3) is automatically satisfied and

$$f = \nabla^2 \psi$$

The vorticity equation (4) can be written

$$\left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) (\nabla^2 \psi + f) = 0$$

or

$$\left(\frac{\partial}{\partial t} - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \right) (\nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0$$

i.e.,

$$\frac{\partial}{\partial t} \nabla^2 \psi + J(\psi, \nabla^2 \psi) + \beta \frac{\partial \psi}{\partial x} = 0 \quad \text{--- (17)}$$

where J is the Jacobian, defined by

$$J(\psi, \nabla^2 \psi) = \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} (\nabla^2 \psi) - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} (\nabla^2 \psi)$$

Assuming a 1D solution of the form

$$\psi = \hat{\psi} e^{ik(x-ct)} \quad \text{--- (18)}$$

where $\hat{\psi}$ is a constant, we have

$$\frac{\partial \psi}{\partial y} = 0$$

$$\nabla^2 \psi = -k^2 \psi$$

Therefore, eq. (17) becomes

$$(ikc)k^2 \hat{\psi} + \beta ik \hat{\psi} = 0$$

i.e.,

$$c = -\frac{\beta}{k^2} \quad (19)$$

so that we have the same solution as before. Again note that (18) satisfies the full nonlinear equation (17).

The solutions for v and Φ_T follow from (18). By definition,

$$v \equiv \frac{\partial \psi}{\partial x} = ik \hat{\psi} e^{ik(x-ct)}$$

To obtain Φ_T we must go back to the momentum equations:

$$\frac{\partial \Phi_T}{\partial x} = f v - \frac{Dv}{Dt} = (f_0 + \beta y) ik \hat{\psi} e^{ik(x-ct)} \quad (20)$$

$$\frac{\partial \Phi_T}{\partial y} = -\mu - \frac{Dv}{Dt} = -k^2 c \hat{\psi} e^{ik(x-ct)}$$

$$= \beta \hat{\psi} e^{ik(x-ct)} \quad (21)$$

It is obvious that the solution of (21) and (22) is

$$\Phi_T = (f_0 + \beta y) \hat{\psi} e^{ik(x-ct)}$$

i.e., the same solution as before.